Solving the word problem of the O'Hare monoid far, far away from sweet home Chicago

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Joint work in progress with Robert D. Gray

Oh, sorry, wrong pic...!

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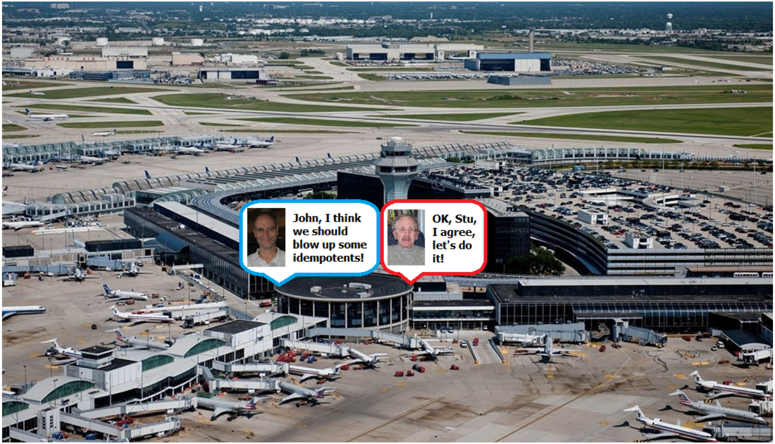
(Or maybe not that terribly wrong... ☺)

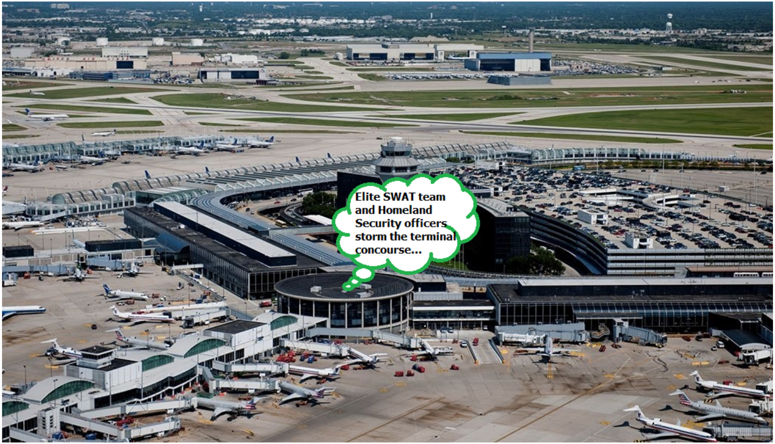
This is the Chicago O'Hare International Airport (IATA code: ORD)

It is the second busiest airport on the planet (after Atlanta).

Two mathematicians engage in a most lovely conversation



Some words have (dire) consequences



Escape

Fortunately, Elwood and Jake show up with their Bluesmobile just in time to save Stu and John from an awkward situation...

She caught the Katy (and left me a mule to ride)

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But: what's the such big fuss about special inverse monoids in the first place?

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Theorem (W. Magnus, 1932)

Every one-relator group has a solvable word problem.

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Lallement (1977) and L. Zhang (1992) provided alternative proofs for the first case (of special monoids $\mathrm{Mon}\langle A \mid u = 1 \rangle$). The proof of Zhang is particularly compact and elegant.

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Adjan and Oganessian (1987): The word problem for one-relator monoids can be reduced to the special case of

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Theorem (Ivanov, Margolis & Meakin, 2001)

If the word problem is decidable for all special inverse monoids $\mathrm{Inv}\langle\, A \mid w = 1 \,\rangle$ *– where* w *is a reduced word over* $A \cup A^{-1}$ *– then the word problem is decidable for every one-relator monoid.*

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This holds basically because $M = \mathrm{Mon}\langle\, A \,|\, asb = atc \,\rangle$ embeds into $I = \mathrm{Inv}\langle\, A \,|\, asbc^{-1}t^{-1}a^{-1} = 1 \,\rangle$.

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Even though this case seems to have zero intersection with the one-relator monoid problem, it is still important to study in order to gain some understanding how the WP works for special one-relator inverse monoids.

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This allows to solve the word problem of M for an array of various types of words $w \in (A \cup A^{-1})^+$.

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E-unitary non-examples:

- $\mathrm{Inv}\langle\, a, b, c, d \mid abc = 1,\ adc = 1 \,\rangle$.
- $\mathrm{Inv}\langle\, A \mid uvu^{-1} = 1 \,\rangle$ provided $u, v \in A^+$ have different terminal letters (so that uvu^{-1} is reduced as written).

Searching for simpler generators of P_w

A factorisation

$$w \equiv \beta_1 \cdots \beta_k$$

is called unital if all β_i represent elements of U_M, where $M = \mathrm{Inv}\langle\, A \mid w = 1 \,\rangle$.

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If $=$ holds, we say that the considered factorisation is conservative.

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Lemma (ID & RDG, 2017)

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$$U_M = \langle \beta_1, \dots, \beta_k \rangle$$

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First I am going to show that it is a) unital, and then that it is b) finest. For each of these statements I am going to show you two proofs: one 'geometric', and one 'combinatorial'.

Stephen's procedure

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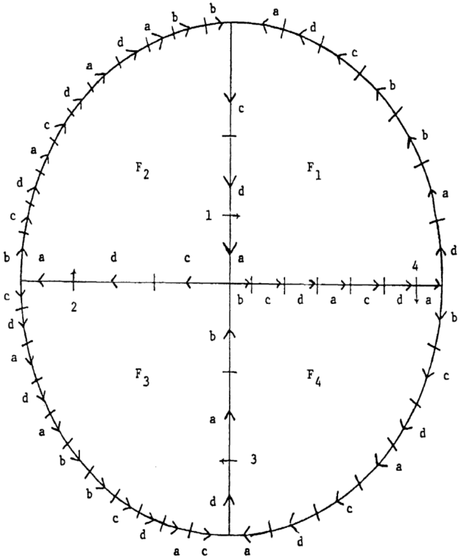
Any graph obtained after a finite number of sewings+foldings is called a finite approximation of the Schützenberger graph in question, and it represents a particular piece of that graph.

Shake your tail feather



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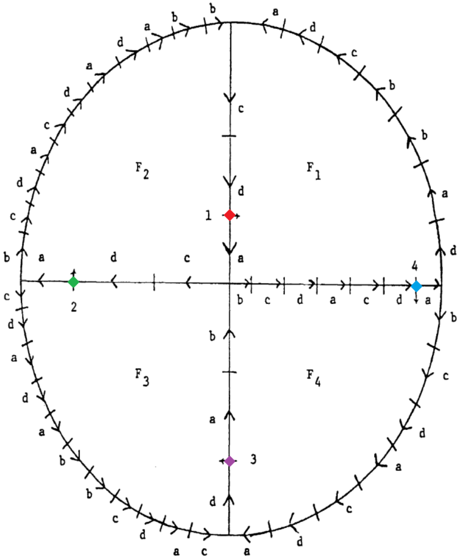

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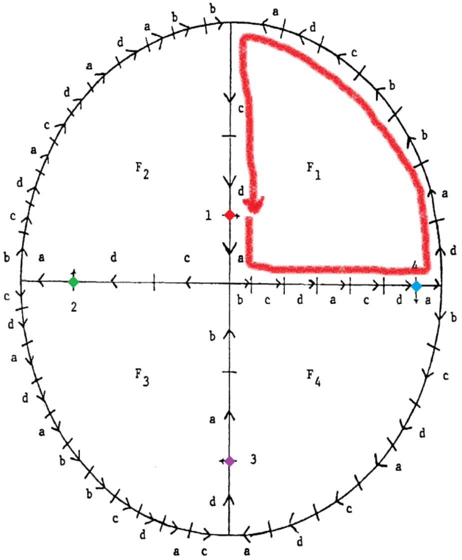

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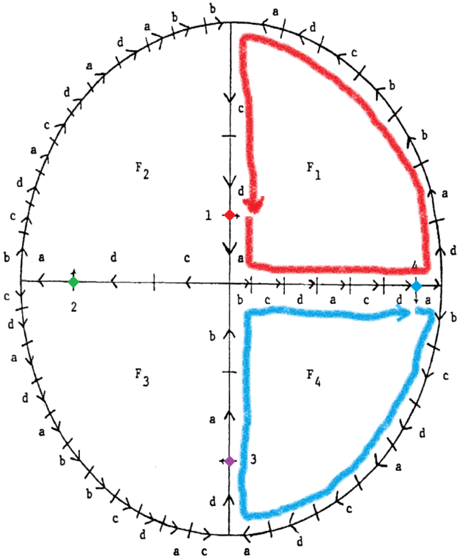

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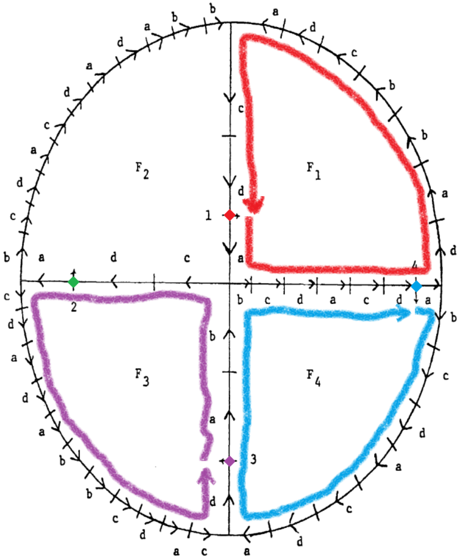

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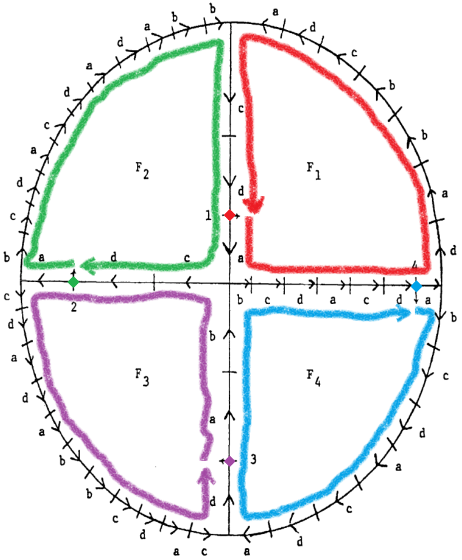

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So, each of $abcd$, acd, ad, $abbcd$ is both right and left invertible.

Invertible pieces of w reloaded

Lemma

Let $u \in (A \cup A^{-1})^*$ be any word representing a right invertible element of $M = \mathrm{Inv}\langle\, A \mid w = 1 \,\rangle$, and let $\overline{u}$ be the (free-group-)reduced form of u. Then $u = \overline{u}$ holds in M.

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So, since

$$\beta = \alpha\delta^{-1}\alpha = (\alpha\beta)(\delta\beta)^{-1}\alpha$$

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In a similar fashion we obtain that $\alpha\beta\gamma$, $\alpha\beta$ and α are invertible, and so are γ and δ.

Finest unital factorisation – Take 1

An easy (inductive) analysis of the Stephen procedure for the O'Hare monoid shows that the initial vertex (corresponding to $1 \in M$) is incident with precisely two edges: an outgoing edge labelled a and an incoming edge labelled d.

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It follows immediately that there can be no unital factorisation of the O'Hare word finer than

$$abcd \cdot acd \cdot ad \cdot abbcd \cdot acd.$$

Finest unital factorisation – Take 2

Deductions of the type:

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Deductions of the type:

ab invertible

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$$ab \text{ invertible} \Rightarrow bcd \text{ invertible (because of } abbcd\text{)}$$

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$$\Rightarrow a \text{ invertible (because of } abcd) \Rightarrow d \text{ invertible (because of } ad)$$

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ab invertible $\Rightarrow$ bcd invertible (because of $abbcd$)
$\Rightarrow$ a invertible (because of $abcd$) $\Rightarrow$ d invertible (because of ad)
$\Rightarrow$ c invertible (becuase of acd)

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All possible cases lead to the same conclusion: if there would be a finer unital factorisation $\Rightarrow$ all of a, b, c, d would be invertible and M would be a group.

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However, this is not the case (thank you, Nik!) as M admits a homomorphism onto the bicyclic monoid $B = \mathrm{Inv}\langle\, x, y \mid xy = 1\,\rangle$ via $a \mapsto x$, $b, c \mapsto 1$, $d \mapsto y$ (taking the O'Hare word to $xyxyxyxy$, a relator in B).

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Corollary

$U_M = \langle abcd, acd, ad, abbcd\rangle = \langle aba^{-1}, aca^{-1}, ad\rangle$

(even as a monoid).

(Dancin' to the) Jailhouse rock

$$\begin{aligned} G &= \mathrm{Gp}\langle\, a, b, c, d \mid abcdacdadabbcdacd = 1 \,\rangle \\ &= \mathrm{Gp}\langle\, a, b, c, d, x, y, z \mid x = aba^{-1},\ y = aca^{-1},\ z = ad,\ xyzyzzxxyzyz = 1 \,\rangle \\ &= \mathrm{Gp}\langle\, a, b, c, d, x, y, z \mid b = a^{-1}xa,\ c = a^{-1}za,\ d = a^{-1}z,\ xyzyzzxxyzyz = 1 \,\rangle \\ &= \mathrm{Gp}\langle\, a, x, y, z \mid xyzyzzxxyzyz = 1 \,\rangle \end{aligned}$$

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\end{aligned}
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$$
\begin{aligned}
P_w &= \mathrm{Mon}\langle a, ab, abc, abcd, ac, acd, ad, abb, abbc, abbcd\rangle \\
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So, the prefix monoid P_w of G w.r.t. the O'Hare presentation is in fact the positive part/submonoid of G w.r.t. the new presentation $\langle\, a, x, y, z \mid xyzyzzxxyzyz = 1 \,\rangle$!!!

The band! The band!! I can see the light!!!

Theorem (Blues Brothers, 2017)

Let u be a strictly positive word over A. Then the positive part of $\mathrm{Gp}\langle\, A \,|\, u = 1 \,\rangle$ *has a decidable membership problem.*

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So, a word v over $A \cup A^{-1}$ represents an element from the positive part of G if and only if $\overline{v}$ fails to contain any letter from B^{-1}. □

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This implies that the prefix monoid P_w of the O'Hare group has a decidable membership problem. By the Ivanov-Margolis-Meakin Theorem, the WP of the O'Hare inverse monoid is soluble.

Everybody needs somebody (or some problem) to love

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- Can we at least prove (via the prefix monoid method) that $\mathrm{Inv}\langle\, A \mid w = 1 \,\rangle$ has a solvable WP if w is a positive word (i.e. $\in A^+$)?

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- This points to the old & famous problem: the generalised WP for one-relator groups. In particular, what about the subgroups generated by $\alpha_1, \ldots, \alpha_k$ for an arbitrary factorisation $\alpha_1 \cdots \alpha_k$ of the (positive) relator w?

THANK YOU!

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