The word problem for one-relator inverse monoids: new developments

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Algebra szeminárium

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Starring

Robert D. Gray (UEA, Norwich)

Lt. Col. Frank Slade (US Army)

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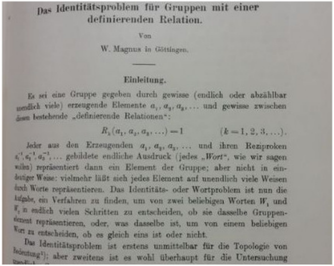
Similarly, one defines the word problem for finitely generated monoids / inverse monoids; in that case the input requires two words u, v and the problem asks if $u\pi = v\pi$ holds in the corresponding monoid.

The beginning of the story: back to the Great Depression



Das Identitätsproblem für Gruppen mit einer definierenden Relation.

Von

W. Magnus in Göttingen.

Einleitung.

Es sei eine Gruppe gegeben durch gewisse (endlich oder abzählbar unendlich viele) erzeugende Elemente $a_1, a_2, a_3, \ldots$ und gewisse zwischen diesen bestehende „definierende Relationen“:

$$R_k(a_1, a_2, a_3, \ldots) = 1 \qquad (k = 1, 2, 3, \ldots).$$

Jeder aus den Erzeugenden $a_1, a_2, a_3, \ldots$ und ihren Reziproken $a_1^{-1}, a_2^{-1}, a_3^{-1}, \ldots$ gebildete endliche Ausdruck (jedes „*Wort*“, wie wir sagen wollen) repräsentiert dann ein Element der Gruppe; aber nicht in eindeutiger Weise; vielmehr läßt sich jedes Element auf unendlich viele Weisen durch Worte repräsentieren. Das Identitäts- oder Wortproblem ist nun die Aufgabe, ein Verfahren zu finden, um von zwei beliebigen Worten W_1 und W_2 in endlich vielen Schritten zu entscheiden, ob sie dasselbe Gruppenelement repräsentieren, oder, was dasselbe ist, um von einem beliebigen Wort zu entscheiden, ob es gleich eins ist oder nicht.

Das Identitätsproblem ist erstens unmittelbar für die Topologie von Bedeutung[1]); aber zweitens ist es wohl überhaupt für die Untersuchung

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If $w \in \overline{X}^*$ is cyclically reduced and $A \subset X$ is such that at least one letter from $X \setminus A$ appears in w, then the subgroup of $\mathrm{Gp}\langle X \mid w = 1 \rangle$ generated by A is free.

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Theorem (Shirshov, 1962)

Every one-relator Lie algebra has decidable word problem.

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Lallement (1977) and L. Zhang (1992) provided alternative proofs for the result about special monoids. The proof of Zhang is particularly compact and elegant.

The (French) connection

Adyan and Oganessyan (1987): The word problem for one-relator monoids can be reduced to the special case of

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where $a, b, c \in X$, $b \neq c$ and $s, t \in X^*$ (and their duals).

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This holds basically because $M = \mathrm{Mon}\langle X \mid asb = atc \rangle$ embeds into $I = \mathrm{Inv}\langle X \mid asbc^{-1}t^{-1}a^{-1} = 1 \rangle$.

The plot thickens

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There exists a one-relator inverse monoid $\mathrm{Inv}\langle X \mid w = 1\rangle$ *with undecidable word problem.*

Inverse monoid basics

Inverse monoid $=$ a monoid M such that for every $a \in M$ there is a unique $a^{-1} \in M$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$.

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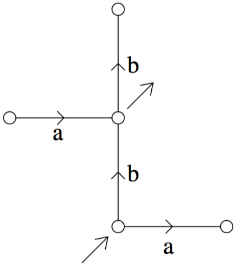

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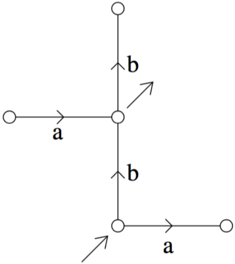

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$$aa^{-1}bb^{-1}ba^{-1}abb^{-1} = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b.$$

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Assume that the prefix membership problem is decidable for $G = \mathrm{Gp}\langle X \mid w = 1\rangle$. If, in addition, $M = \mathrm{Inv}\langle X \mid w = 1\rangle$ is E-unitary then the word problem for M is decidable.

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This allows to solve the word problem of M for an array of various types of words $w \in \overline{X}^+$.

Gray (2019), Inventiones Mathematicae (to appear)

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Theorem B
There exist one-relator groups with undecidable submonoid membership problem.

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- This, in turn, defines an HNN extension $A(\Gamma, \psi)$ which naturally embeds $A(\Gamma)$.

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Theorem (Lohrey & Steinberg, 2008)

There exists a (fixed) finitely generated submonoid T of $A(P_4)$ with undecidable membership problem.

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So, the RAAG $A(P_4)$ embeds into this one-relator group.

Theorem (Lohrey & Steinberg, 2008)

There exists a (fixed) finitely generated submonoid T of $A(P_4)$ with undecidable membership problem.

... $\Longrightarrow$ Theorem B.

A general construction

Let $r, w_1, \ldots, w_k \in \overline{X}^*$ be arbitrary, where $X = \{a_1, \ldots, a_n\}$.

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Proposition (!!!)

Let $T = \mathrm{Mon}\langle w_1, \dots, w_k \rangle \leq G = \mathrm{Gp}\langle X \mid r = 1 \rangle$. *Then for all* $u \in \overline{X}^*$ *we have*

$$u \in T \leq G \Longleftrightarrow (tut^{-1})(tu^{-1}t^{-1}) = 1 \text{ in } M.$$

Touche!

Theorem

If $M = \mathrm{Inv}\langle X \mid er = 1\rangle$ has decidable word problem then the membership problem for T in $G = \mathrm{Gp}\langle X \mid r = 1\rangle$ is decidable.

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  $$e = aa^{-1}zz^{-1}(tw_1t^{-1})(tw_1^{-1}t^{-1})\dots(tw_kt^{-1})(tw_k^{-1}t^{-1})z^{-1}za^{-1}a;$$
- Then by the above theorem
  $$M = \mathrm{Inv}\langle a, z, t \mid eazaz^{-1}a^{-1}za^{-1}z^{-1} = 1\rangle$$
  has undecidable WP. Q.E.D.

Now, let's keep it positive!

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Remember: in the E-unitary case, solving the WP for $\mathrm{Inv}\langle X \mid w = 1\rangle$ is the same as solving the prefix membership problem for $\mathrm{Gp}\langle X \mid w = 1\rangle$.

A sampler of positive results

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And now...

- IgD & RD Gray, 2019+

A glimpse into the toolbox (1)

Theorem (Benois, 1969)

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Let $M = \mathrm{Inv}\langle X \mid w = 1\rangle$. A factorisation

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Lemma

$P_w \leq G = \mathrm{Gp}\langle X \mid w = 1 \rangle$ *is generated by* $\bigcup_{i=1}^{m} \mathrm{pref}(w_i)$.

A glimpse into the toolbox (2)

In fact, for any factorisation $w \equiv w_1 \cdots w_m$ we can consider the submonoid of G

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Theorem

(i) *Any unital factorisation is conservative.*

(ii) *If* $\mathrm{Inv}\langle X \mid w = 1 \rangle$ *is E-unitary then every conservative factorisation if unital.*

Theorem A

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Then the membership problem for M in G is decidable.

Theorem B

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Then the membership problem for M in G is decidable.

Applications (1): Unique marker letter theorem

Theorem

Let $G = \mathrm{Gp}\langle X \mid w = 1 \rangle$ *and assume that* $w \equiv u(w_1, \ldots, w_k)$ *determines a conservative factorisation of* w*.*

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Let $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ *and assume that* $w \equiv u(w_1, \dots, w_k)$ *determines a conservative factorisation of* w*. Furthermore, suppose that for all* $1 \le i \le k$ *there is a letter* $x_i \in X$ *which appears exactly once in* w_i *and does not appear in any* w_j*,* $j \neq i$*.*

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So, if w satisfies these conditions and $M = \mathrm{Inv}\langle X \mid w = 1\rangle$ is E-unitary then M has decidable word problem.

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Example

The group

$$G = \mathrm{Gp}\langle a, b, x, y \mid axbaybaybaxbaybaxb = 1\rangle$$

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has decidable prefix membership problem $\Longrightarrow$ the inverse monoid

$$M = \mathrm{Inv}\langle a, b, x, y \mid axbaybaybaxbaybaxb = 1\rangle$$

has decidable WP.

Applications (2): The (in)famous O'Hare monoid

Constructed by M&M while waiting for a connecting flight at the O'Hare Int. Airport, Chicago, sometime in the 1980s:

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Proposition

Let $M = \mathrm{Inv}\langle Y, a, d \mid (au_{i_1}d)\dots(au_{i_m}d) = 1\rangle$, with a, d not appearing in u_{i_j}'s.

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Then $G = \mathrm{Gp}\langle Y, a, d \mid (au_{i_1}d)\dots(au_{i_m}d) = 1\rangle$ has decidable prefix membership problem, and so M as decidable WP.

Applications (3): Disjoint alphabets theorem

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The group

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has decidable prefix membership problem $\Longrightarrow$ the inverse monoid

$$M = \mathrm{Inv}\langle a, b, x, y \mid ababcdcdababcdcdcdcdabab = 1\rangle$$

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Applications (4): Cyclically pinched presentations

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The prefix membership problem is decidable for one-relator groups defined by cyclically pinched presentations:

$$G = \mathrm{Gp}\langle X \cup Y \mid uv^{-1} = 1 \rangle$$

where u, v are reduced words over disjoint X, Y, respectively.

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Then the membership problem for M in G^* is decidable.

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Then for any finite $W_0, W_1, \ldots, W_d, W_1', \ldots, W_d' \subseteq G$, $d \geq 0$, the membership problem for

$$M = \mathrm{Mon}\langle W_0 \cup W_1 t \cup W_2 t^2 \cup \cdots \cup W_d t^d \cup t W_1' \cup \cdots \cup t^d W_d' \rangle$$

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By modifying slightly the ideas from [Gray, 2019], we obtain

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KÖSZÖNÖM A FIGYELMET!

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