# Universal locally finite maximally homogeneous semigroups

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# Isten éltesen, Laci! Boldog évfordulót!

# Joint work with *Monsieur le docteur* Robert D'Gray

Special thanks go to UEA campus bunnies for creating a thoroughly pleasant working environment! ☺

# And so the story begins...

Once upon a time, deep in the magic forest of Al-Jabr, there liveth a beast, both fearsome and beautiful, daunting and enchanting, and its name was

Hall's universal group.

This beast, countably infinite by its size, hath the following properties:

- Universal: It containeth a copy of every finite group as a subgroup.
- Locally finite: Every finitely generated subgroup was finite.
- Homogeneous: Every two isomorphic subgroups $A, B$ were conjugate. In fact, any isomorphism $\phi : A \to B$ was a restriction of some inner automorphism (of the whole beast).

# And so the story begins...

All the people, peasants and nobleman alike, feareth the beast.

For along came Ser Roland, the finest and bravest knight of the proud Fraïssé clan, and told the people: there is only One. There was no other creature in the whole Universe quite like this one.

The beast was ancient, createth from Darkness at the dawn of Time. It was even known to the old and famous sorcerer Euclides...!

Naaah, I'm just kidding you folks ☺, it was constructed by Philip Hall in 1959 in his beautiful paper *Some constructions for locally finite groups*.

# Construction of Hall's universal group $\mathcal{U}$

- Take any finite group $G = G_0$ with at least 3 elements.
- By Cayley's Theorem, $G$ embeds into $\mathbb{S}_G = G_1$ via the right regular representation $\phi : g \mapsto \rho_g$, where $\rho_g$ is a permutation of $G$ defined by $x\rho_g = xg$ for all $x \in G$.
- Repeat. (That is, go to step 1 with $G_1$ in the role of our finite group $G$, etc. etc. etc.)

This results in a chain of embeddings $G_0 \to G_1 \to G_2 \to \dots$

## Theorem (P.Hall, 1959)

*Regardless of the initial group $G_0$, the direct limit of this chain is, up to isomorphism, one and the same countable group $\mathcal{U}$. It is universal (for finite groups), locally finite, and homogeneous; moreover, it is the unique countable group with these properties.*

# How, on Earth, something isomorphic (but not conjugate) can become conjugate?

Well, this is the magic of the Cayley embedding. ☺ Let's take a look at an example.

Let $G = \mathbb{S}_4$, and consider its subgroups:

$$K = \{(), (12)\}, \quad L = \{(), (12)(34)\}.$$

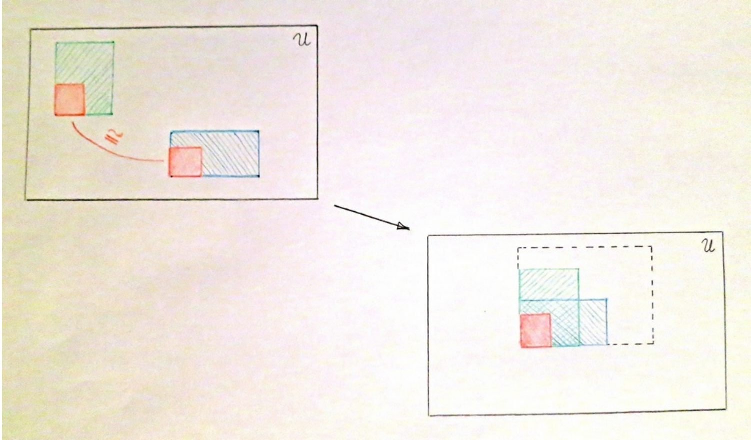
Clearly, $K \cong \mathbb{Z}_2 \cong L$, but they are not conjugate in $G$ because of the different cycle structure.

However, in $\mathbb{S}_{\mathbb{S}_4}$, both $\rho_{(12)}$ and $\rho_{(12)(34)}$ are permutations (of a 24-element set) of order 2 without any fixed points. Therefore, they are both products of 12 disjoint transpositions, and thus it follows that $K\phi$ and $L\phi$ are conjugate (in $\mathbb{S}_G$).

# Manfred Droste at AAA83, March 2012, Novi Sad

Is there a countable universal
locally finite homogeneous (inverse) semigroup?

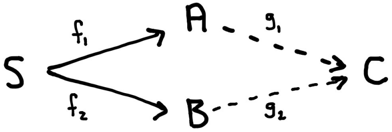
# The keyword: Amalgamation



# Amalgamation and the Fraïssé Theorem

An amalgam in a class $\mathcal{K}$ of first-order structures is an ensemble $(S, A, B, f_1, f_2)$ consisting of structures $S, A, B \in \mathcal{K}$ along with two embeddings $f_1 : S \to A$ and $f_2 : S \to B$.

An amalgam $(S, A, B, f_1, f_2)$ in $\mathcal{K}$ is embeddable into a structure $C$ if there are embeddings $g_1 : A \to C$ and $g_2 : B \to C$ such that $f_1 g_1 = f_2 g_2$, that is:




$\mathcal{K}$ has the amalgamation property (AP) if any amalgam in $\mathcal{K}$ can be embedded into some structure $C \in \mathcal{K}$.

# Amalgamation and the Fraïssé Theorem

## Fact

Finite groups have the AP. Actually, they form an amalgamation class (= AP + countably many isomorphism types + closed for taking substructures + JEP).

## Theorem (R.Fraïssé)

- *For any countably infinite homogeneous structure $A$, the class of all of its finitely generated substructures (the age of $A$) is an amalgamation class.*
- *For any amalgamation class $\mathcal{K}$ of finitely generated structures there exists a countably infinite homogeneous structure $A$ whose age is $\mathcal{K}$. This $A$ is unique up to isomorphism and is called the Fraïssé limit of $\mathcal{K}$.*

So, in model-theoretical terminology, Hall's universal group $\mathcal{U}$ is the Fraïssé limit of the class of all finite groups.

# Houston, we've got a (slllight) problem...!

- Kimura (1957, PhD thesis): The class of finite semigroups does not have the AP.
- T.E.Hall (1975, proof credited to C.J.Ash): The class of finite inverse semigroups does not have the AP.

Conclusion: There is no countable universal locally finite homogeneous semigroup. There is no such inverse semigroup either.

# So...(?)

# Modifying the question

Instead, it might be sensible to ask:

How homogeneous can a countable universal
locally finite (inverse) semigroup be?

How one can 'measure' the degree of (partial) homogeneity (of a semigroup)? Answer: amalgamation bases!

A finite (inverse) semigroup $S$ is an amalgamation base for the class of all finite (inverse) semigroups if every amalgam based on $S$ (i.e. an amalgam of the form $(S, \ldots, \ldots, \ldots, \ldots)$) embeds into some finite (inverse) semigroup.

# Amalgamation bases

- T.E.Hall (1975): A finite inverse semigroup $S$ belongs to $\mathcal{A}$, the class of all amalgamation bases for finite inverse semigroups, if and only if $S$ is $\mathscr{J}$-linear (i.e. $\mathscr{J}$-classes form a chain). Reproved by Okniński and Putcha (1991) using representation theory.
- The class $\mathcal{B}$ of all amalgamation bases for finite semigroups has not been characterised so far. We do know that any semigroup in $\mathcal{B}$ must be $\mathscr{J}$-linear, but the converse is not true.
- Known: $\mathcal{B}$ contains all finite groups, all reducts of inverse semigroups from $\mathcal{A}$, and, most importantly, all full transformation semigroups $\mathcal{T}_n$ (K.Shoji, 2016).

# Maximal homogeneity

$T$ – a countable universal locally finite (inverse) semigroup
$S$ – a finite (inverse) semigroup

We say that $\mathrm{Aut}(T)$ acts homogeneously on copies of $S$ in $T$ if for all $U_1, U_2 \leq T$ such that $U_1 \cong S \cong U_2$, every isomorphism $\phi : U_1 \to U_2$ extends to an automorphism of $T$.

### Proposition

$\mathrm{Aut}(T)$ *acts homogeneously on copies of* $S$ *in* $T \Longrightarrow S \in \mathcal{B}$ *(resp.* $S \in \mathcal{A}$ *in the inverse case).*

$T$ is maximally homogeneous if $\mathrm{Aut}(T)$ acts homogeneously on copies of $S$ in $T$ for all $S \in \mathcal{B}$ (resp. $S \in \mathcal{A}$).

# The maximally homogeneous semigroup $\mathcal{T}$

Recall that Hall's universal group $\mathcal{U}$ was obtained as a direct limit of a chain of embeddings of symmetric groups.

If we have a chain of semigroup embeddings

$$M_0 \to M_1 \to M_2 \to \dots$$

such that $M_i \cong \mathcal{T}_{n_i}$ for all $i$ and some $n_i \geq 1$, then the semigroup arising as the direct limit of this chain is called a full transformation limit semigroup.

Note that every infinite full transformation limit semigroup must be universal and locally finite. So, maybe it makes sense to restrict our quest to semigroups of this type. And indeed...

### Theorem (ID & Gray, 2017)

*There is a unique maximally homogeneous full transformation limit semigroup $\mathcal{T}$.*

# The maximally homogeneous inverse semigroup $\mathcal{I}$

Similarly, if we have a chain of inverse semigroup embeddings

$$K_0 \to K_1 \to K_2 \to \dots$$

such that $K_i \cong \mathcal{I}_{n_i}$ for all $i$ and some $n_i \geq 1$, then the inverse semigroup arising as the direct limit of this chain is called a symmetric inverse limit semigroup.

**Theorem (ID & Gray, 2017)**

*There is a unique maximally homogeneous symmetric inverse limit semigroup $\mathcal{I}$.*

# The methods and non-methods (to construct $\mathcal{T}$ and $\mathcal{I}$)

- $\mathcal{T}$ and $\mathcal{I}$ are not homogeneous, so they cannot be constructed using Fraïssé's Theorem.
- However, one can make use of a well-known generalisation, sometimes called the Hrushovski construction.
  - Recommended reading: D.Evans' Lecture notes from his talks at the Hausdorff Institute of Maths, Bonn, September 2013.
- Of course, it is tempting to try and iterate the Cayley / Vagner-Preston Theorem for semigroups / inverse semigroups and look at the direct limits of chains:

$$\mathcal{T}_n \to \mathcal{T}_{\mathcal{T}_n} \to \mathcal{T}_{\mathcal{T}_{\mathcal{T}_n}} \to \dots \quad \text{and} \quad \mathcal{I}_n \to \mathcal{I}_{\mathcal{I}_n} \to \mathcal{I}_{\mathcal{I}_{\mathcal{I}_n}} \to \dots$$

  However, this will fail: we can prove that (most of) the maximal subgroups of these limits are not isomorphic to $\mathcal{U}$, whereas all the maximal subgroups of both $\mathcal{T}$ and $\mathcal{I}$ are isomorphic to $\mathcal{U}$.

# Structure of $\mathcal{I}$ and $\mathcal{T}$

Even though $\mathcal{I}$ and $\mathcal{T}$ are not homogeneous, they still display a high degree of symmetry in their combinatorial and algebraic structure.

In particular, we hope to stumble upon a number of well-known homogeneous objects in the course of studying the structural features of $\mathcal{I}$ and $\mathcal{T}$.

# The structural properties of $\mathcal{I}$

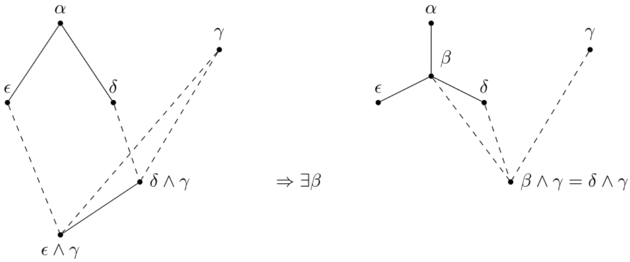
Theorem (ID & RDG, 2017)

1. *$\mathcal{I}/\mathscr{J}$ is a chain order-isomorphic to $\mathbb{Q}$.*
2. *Every maximal subgroup of $\mathcal{I}$ is isomorphic to Hall's group $\mathcal{U}$.*
3. *We have $\mathscr{D} = \mathscr{J}$ in $\mathcal{I}$, and all principal factors are isomorphic to the Brandt semigroup $B(\mathbb{N}, \mathcal{U})$.*
4. *The semilattice of idempotents $E(\mathcal{I})$ is isomorphic to the countable universal homogeneous semilattice $\Omega$.*

# The structural properties of $\mathcal{I}$

To establish (4), we used the characterisation of $\Omega$ by Droste, Kuske & Truss (1999), stating that a countable $\wedge$-semilattice is $\cong \Omega$ if and only if:

- it has no minimal or maximal elements;
- any pair of elements has an upper bound;
- the following axiom $(*)$ holds:



# The structural properties of $\mathcal{I}$

- Given $\alpha, \gamma, \delta, \epsilon \in E(\mathcal{I})$, there is a finite inverse subsemigroup $S \cong \mathcal{I}_n$ (for some $n \geq 1$) that contains them.
- We use the Vagner-Preston embedding $S \to \mathcal{I}_S$ and consider $\mathcal{I}_S$ as an extension of $S$; here we find a suitable idempotent $\beta$.
- Using the fact that $\mathcal{I}_n \in \mathcal{A}$ is an amalgamation base for finite inverse semigroups (because it is $\mathscr{J}$-linear) and the Extension Property from the Hrushovski construction, we 'tuck in' $\beta$ back into $\mathcal{I}$ to conclude that $E(\mathcal{I})$ has $(*)$.

# The structural properties of $\mathcal{T}$

Theorem (ID & RDG, 2017)

1. *$\mathcal{T}/\mathscr{J}$ is a chain order-isomorphic to $\mathbb{Q}$.*
2. *Every maximal subgroup of $\mathcal{T}$ is isomorphic to Hall's group $\mathcal{U}$.*
3. *We have $\mathscr{D} = \mathscr{J}$ in $\mathcal{T}$, and, as $\mathsf{Aut}(\mathcal{T})$ act transitively on the chain of $\mathscr{J}$-classes, all principal factors are isomorphic to each other.*
4. *$\mathcal{T}$ is regular, idempotent generated, and self-dual (isomorphic to its opposite semigroup).*
5. *The Graham-Houghton graph of every $\mathscr{D}$-class of $\mathcal{T}$ is isomorphic to the countable random bipartite graph.*

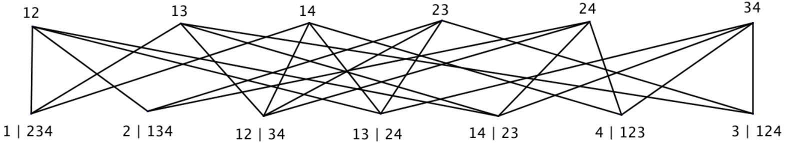
# Graham-Houghton graphs: a reminder

Vertices: $\mathscr{R}$-classes (one part) and $\mathscr{L}$-classes (the other part) of a fixed $\mathscr{D}$-class of a semigroup $S$.

Edges: $(R, L)$ is an edge $\Leftrightarrow$ $R \cap L$ is a group $\Leftrightarrow$ $R \cap L$ contains an idempotent

Example: The rank 2 $\mathscr{D}$-class of $\mathcal{T}_4$.

| | 12 | 13 | 14 | 23 | 24 | 34 |
|---|---|---|---|---|---|---|
| 123\|4 | | | * | | * | * |
| 124\|3 | | * | | * | | * |
| 134\|2 | * | | | * | * | |
| 234\|1 | * | * | * | | | |
| 12\|34 | | * | * | * | * | |
| 13\|24 | * | | * | * | | * |
| 14\|23 | * | * | | | * | * |

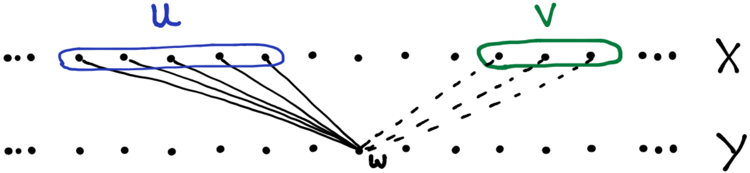


# The random bipartite graph

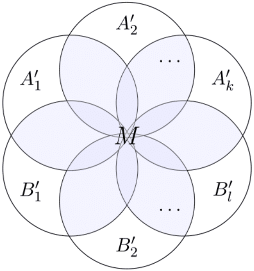
The countable random bipartite graph = the Fraïssé limit of the class of all finite bipartite graphs

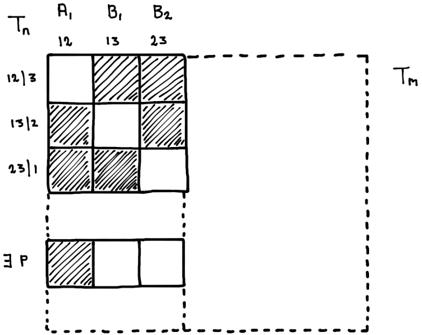
It is uniquely characterised among countably infinite bipartite graph by the condition:

*For any two finite disjoint sets $U, V$ from one part of the bipartition, there is a vertex $w$ in the other part such that $w \sim U$ and $w \not\sim V$.*



# Key ingredient: The Flower Lemma




## Lemma

*Let $A_1, \ldots, A_k, B_1, \ldots, B_\ell$ be t-element subsets of*

$$[m] = \{1, \ldots, m\}.$$

*If $|M| < t$ then there exists a partition $P$ of $[m]$ with $t$ parts such that $P \perp A_i$ and $P \not\perp B_j$.*

## Proposition

*Let $1 < r < n$. Then $\exists \phi : \mathcal{T}_n \to \mathcal{T}_m$ such that $\forall a_1, \ldots, a_k, b_1, \ldots, b_\ell \in J_r \subseteq \mathcal{T}_n$ from distinct $\mathscr{L}$-classes $\exists c \in \mathcal{T}_m$ such that in $\mathcal{T}_m$:*

- $R_c \cap L_{a_i\phi}$ *are groups*
- $R_c \cap L_{b_j\phi}$ *are not groups*

# Doing it the other way round

- A statement dual to the previous proposition also holds. By using the Extension Property once again, this suffices to obtain the random bipartite graph result.
- However, this relies solely on the fact that we can prove $\mathcal{T} \cong \mathcal{T}^{\mathrm{opp}}$. This we establish by showing that $\mathcal{T}^{\mathrm{opp}}$ is maximally homogeneous and full transformation limit, hence, by our earlier uniqueness result, it must be $\cong \mathcal{T}$.
- It is completely unclear what are the full combinatorial ramifications of this dual result.
- It is related to the following interesting combinatorial question: Given a family of distinct partitions $P_1, \dots, P_k$, $Q_1, \dots, Q_\ell$ of $[m]$, each with exactly $t$ non-empty parts, under what conditions can one guarantee that there is a $t$-element subset $A$ of $[m]$ which is a transversal of each of $P_i$, and of none of $Q_j$?

# Speaking of questions...

Problem 1: Describe the (unique) principal factor of $\mathcal{T}$. Is this the same as the unique countable universal $\mathcal{CS}$-homogeneous completely 0-simple semigroup? (Which exists, btw, again by the Hrushovski construction.)

Problem 2: Is $\mathcal{T}$ (resp. $\mathcal{I}$) the only countable universal locally finite maximally homogeneous (inverse) semigroup? (That is, can we drop the limit condition?)

Problem 3: Does every countable locally finite (inverse) semigroup embed into $\mathcal{T}$ (resp. $\mathcal{I}$)?

Problem 4: Does there exist a countable locally finite (inverse) semigroup which embeds every countable locally finite (inverse) semigroup?

## Remark

There exist $2^{\aleph_0}$ non-isomorphic countable locally finite groups, and $\mathcal{U}$ embeds all of them.

# THANK YOU!

Questions and comments to:

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Further information may be found at:

**http://people.dmi.uns.ac.rs/∼dockie**