Variants of semigroups - the case study of finite full transformation monoids

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Prime suspects

Mr. Shady Corleone

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Now seriously... co-authors

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Variants of semigroups

Let $(S,\cdot)$ be a semigroup and $a \in S$. Given these, one can easily define an alternative product $\star_a$ on S, namely

$$x \star_a y = xay.$$

This is the variant $S^a = (S, \star_a)$ of S with respect to a.

First mention of variants (as far as we know): Lyapin's book from 1960 (in Russian).

Magill (1967): Semigroups of functions $X \to Y$ under an operation defined by

$$f \cdot g = f \circ \theta \circ g,$$

where θ is a fixed function $Y \to X$. For $Y = X$, this is exactly a variant of $\mathcal{T}_X$.

History of variants – continued

Hickey (1980s): Variants of general semigroups $\to$ a new characterisation of Nambooripad's order on regular semigroups

Khan & Lawson (2001): Variants of regular semigroups (natural relation to Rees matrix semigroups). In fact, they obtain a natural generalisation of the notion of group of units for non-monoidal regular semigroups.

G. Y. Tsyaputa (2004/5): variants of finite full transformation semigroups $\mathcal{T}_n$

- classification of non-isomorphic variants
- idempotents, Green's relations
- analogous questions for $\mathcal{PT}_n$

A more accessible account of her results may be found in the monograph of Ganyushkin & Mazorchuk *Classical Finite Transformation Semigroups* (Springer, 2009).

Several examples

For a group G and $a \in G$, we always have $G^a \cong G$ via $x \mapsto xa$. The identity element in G^a is a^{-1}.

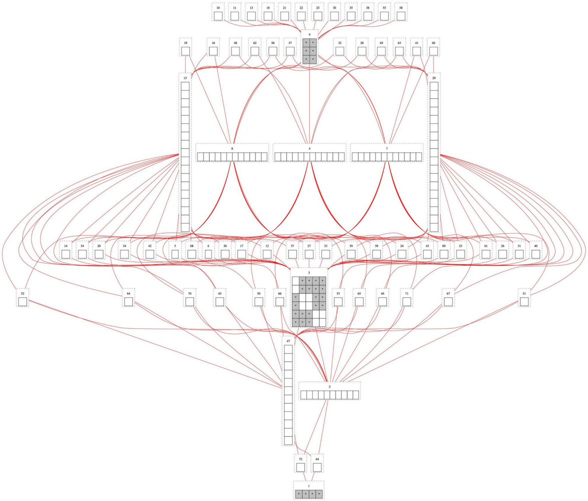
On the other hand, if S the bicyclic monoid, then $a, b \in S$, $a \neq b$ implies $S^a \ncong S^b$.

If S is a monoid, $a, u, v \in S$, and u, v are units, then $S^{uav} \cong S^a$ via $x \mapsto vxu$.

Thus, for any $a \in \mathcal{T}_X$ there exists $e \in E(\mathcal{T}_X)$ such that $\mathcal{T}_X^a \cong \mathcal{T}_X^e$.

A WORD OF CAUTION: If S is a regular semigroup, S^a is not regular in general! However, for regular S and arbitrary $a \in S$, $\mathrm{Reg}(S^a)$ is always a subsemigroup of S^a (Khan & Lawson).

A word of caution, you said...?



Egg-box picture of $\mathcal{T}_4^a$ for $a = [1, 2, 3, 3]$

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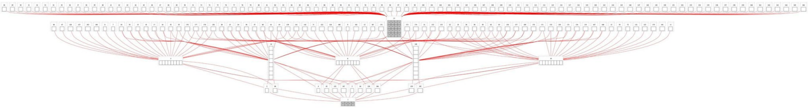
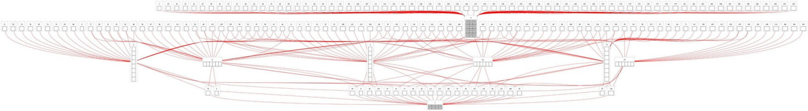


Egg-box picture of $\mathcal{T}_4^a$ for $a = [1, 1, 3, 3]$



Egg-box picture of $\mathcal{T}_4^a$ for $a = [1, 1, 1, 4]$

Three important sets

$$P_1 = \{x \in S : \; xa \,\mathscr{R}\, x\}, \qquad P_2 = \{x \in S : \; ax \,\mathscr{L}\, x\},$$

$$P = P_1 \cap P_2$$

Easy facts:

- $y \in P_1 \;\Leftrightarrow\; L_y \subseteq P_1$,
- $y \in P_2 \;\Leftrightarrow\; R_y \subseteq P_2$,
- $\mathrm{Reg}(S^a) \subseteq P$

Green's relations: $\mathscr{R}^a, \mathscr{L}^a, \mathscr{H}^a, \mathscr{D}^a$

$$R_x^a = \begin{cases} R_x \cap P_1 & \text{if } x \in P_1 \\ \{x\} & \text{if } x \in S \setminus P_1, \end{cases}$$

$$L_x^a = \begin{cases} L_x \cap P_2 & \text{if } x \in P_2 \\ \{x\} & \text{if } x \in S \setminus P_2, \end{cases}$$

$$H_x^a = \begin{cases} H_x & \text{if } x \in P \\ \{x\} & \text{if } x \in S \setminus P, \end{cases}$$

$$D_x^a = \begin{cases} D_x \cap P & \text{if } x \in P \\ L_x^a & \text{if } x \in P_2 \setminus P_1 \\ R_x^a & \text{if } x \in P_1 \setminus P_2 \\ \{x\} & \text{if } x \in S \setminus (P_1 \cup P_2). \end{cases}$$

Group $\mathcal{H}$-classes vs group $\mathcal{H}^a$-classes (in P)

Let $S = \mathcal{T}_4$ and $a = [1, 2, 3, 3]$.

x	Is H_x a group $\mathscr{H}$-class of $\mathcal{T}_4$?	Is H_x a group $\mathscr{H}^a$-class of $\mathcal{T}_4^a$?
$[1, 1, 3, 3]$	Yes	Yes
$[4, 2, 2, 4]$	Yes	No
$[2, 4, 2, 4]$	No	Yes
$[1, 3, 1, 3]$	No	No

Our goal for today...

...is to conduct a thorough algebraic and combinatorial analysis of $\mathcal{T}_X^a$ where $|X| = n$ and a is a fixed transformation on X.

As we noted, we may assume that a is idempotent with $r = \mathrm{rank}(a) < n$,

$$a = \begin{pmatrix} A_1 & \cdots & A_r \\ a_1 & \cdots & a_r \end{pmatrix},$$

so that $a_i \in A_i$ for all $i \in [1, r]$.

Here $A = \mathrm{im}(a) = \{a_1, \ldots, a_r\}$ and $\alpha = \ker(a) = (A_1|\cdots|A_r)$, with $\lambda_i = |A_i|$. Furthermore, for $I = \{i_1, \ldots, i_m\} \subseteq [1, r]$ we write $\Lambda_I = \lambda_{i_1} \cdots \lambda_{i_m}$ and $\Lambda = \lambda_1 \cdots \lambda_r$.

P_1, P_2, P in $\mathcal{T}_X^a$

Let $B \subseteq X$ and let β be an equivalence relation on X. We say that B saturates β if each β-class contains at least one element of B. Also, we say that β separates B if each β-class contains at most one element of B.

$$\begin{aligned} P_1 &= \{f \in \mathcal{T}_X : \ \mathrm{rank}(fa) = \mathrm{rank}(f)\} \\ &= \{f \in \mathcal{T}_X : \ \alpha \text{ separates } \mathrm{im}(f)\} \end{aligned}$$

$$\begin{aligned} P_2 &= \{f \in \mathcal{T}_X : \ \mathrm{rank}(af) = \mathrm{rank}(f)\} \\ &= \{f \in \mathcal{T}_X : \ A \text{ saturates } \ker(f)\} \end{aligned}$$

$$P = \{f \in \mathcal{T}_X : \ \mathrm{rank}(afa) = \mathrm{rank}(f)\} = \mathrm{Reg}(\mathcal{T}_X^a) \leq \mathcal{T}_X^a$$

Green's relations in $\mathcal{T}_X^a$ (Tsyaputa, 2004)

$$R_f^a = \begin{cases} R_f \cap P_1 & \text{if } f \in P_1 \\ \{f\} & \text{if } f \in \mathcal{T}_X \setminus P_1, \end{cases}$$

$$L_f^a = \begin{cases} L_f \cap P_2 & \text{if } f \in P_2 \\ \{f\} & \text{if } f \in \mathcal{T}_X \setminus P_2, \end{cases}$$

$$H_f^a = \begin{cases} H_f & \text{if } f \in P \\ \{f\} & \text{if } f \in \mathcal{T}_X \setminus P, \end{cases}$$

$$D_f^a = \begin{cases} D_f \cap P & \text{if } f \in P \\ L_f^a & \text{if } f \in P_2 \setminus P_1 \\ R_f^a & \text{if } f \in P_1 \setminus P_2 \\ \{f\} & \text{if } f \in \mathcal{T}_X \setminus (P_1 \cup P_2). \end{cases}$$

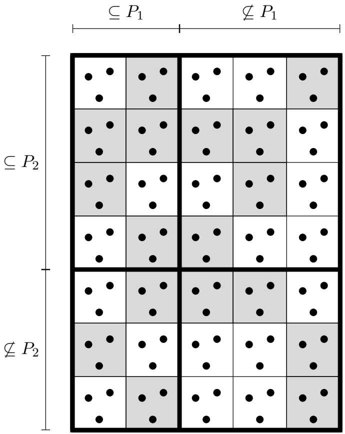
'High-energy semigroup theory'

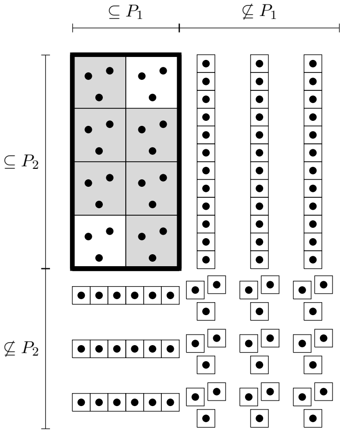
- Recall that in $\mathcal{T}_X$, the $\mathscr{D}$-classes form a chain:

$$D_n > D_{n-1} > \cdots > D_2 > D_1.$$

- Each of the $\mathscr{D}$-classes $D_{r+1}, \ldots, D_n$ is completely 'shattered' into singleton 'shrapnels' / $\mathscr{D}^a$-classes in $\mathcal{T}_X^a$.
- Since all constant maps trivially belong to P, D_1 is preserved, and remains a right zero band.
- For $2 \leq m \leq r$, the class D_r separates into a single regular chunk $D_r \cap P$ and a number of non-regular pieces, as seen on the following picture...

'High-energy semigroup theory'





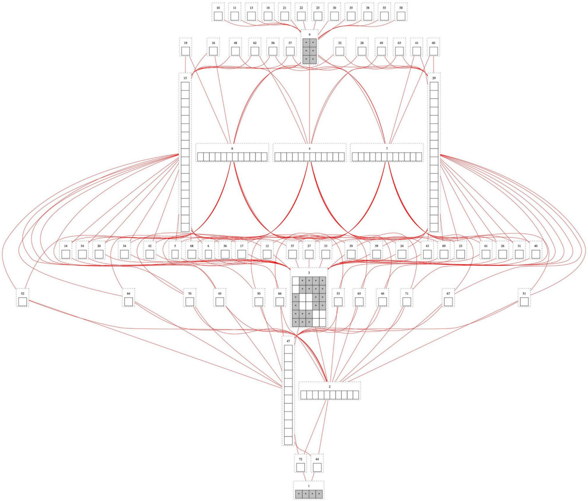
Order of the $\mathscr{D}^a$-classes

Let $f, g \in \mathcal{T}_X$. Then $D_f^a \leq D_g^a$ in $\mathcal{T}_X^a$ if and only if one of the following holds:

- $f = g$,
- $\operatorname{rank}(f) \leq \operatorname{rank}(aga)$,
- $\operatorname{im}(f) \subseteq \operatorname{im}(ag)$,
- $\ker(f) \supseteq \ker(ga)$.

The maximal $\mathscr{D}^a$-classes are those of the form $D_f^a = \{f\}$ where $\operatorname{rank}(f) > r$.

Order of the $\mathscr{D}^a$-classes



The rank of $\mathcal{T}_X^a$

Let $M = \{f \in \mathcal{T}_X : \ \mathrm{rank}(f) > r\}$.

Then $\mathcal{T}_X^a = \langle M \rangle$; furthermore, any generating set for $\mathcal{T}_X^a$ contains M.

Consequently, M is the unique minimal (with respect to containment or size) generating set of $\mathcal{T}_X^a$, and

$$\mathrm{rank}(\mathcal{T}_X^a) = |M| = \sum_{m=r+1}^{n} S(n,m)\binom{n}{m} m!,$$

where $S(n,m)$ denotes the Stirling number of the second kind.

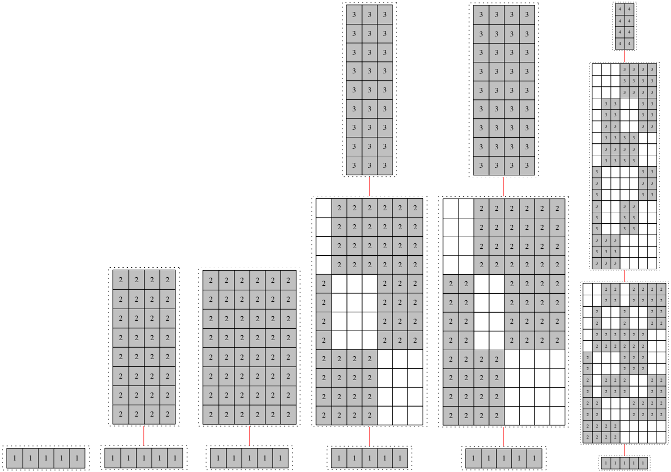
‘Positioning’ with respect to the regular classes

- If $f \in P$, then $D^a_f \leq D^a_g$ if and only if $\mathrm{rank}(f) \leq \mathrm{rank}(aga)$.
- If $g \in P$, then $D^a_f \leq D^a_g$ if and only if $\mathrm{rank}(f) \leq \mathrm{rank}(g)$.

Consequences:

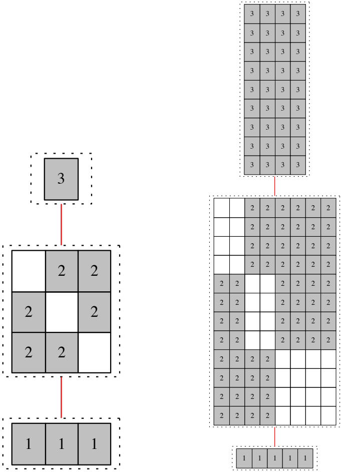
- The regular $\mathscr{D}^a$-classes of $\mathcal{T}^a_X$ form a chain: $D^a_1 < \cdots < D^a_r$ (where $D^a_m = \{f \in P : \ \mathrm{rank}(f) = m\}$ for $m \in [1, r]$).
- ‘Co-ordinatisation’ of the non-regular, ‘fragmented’ $\mathscr{D}^a$-classes: if $\mathrm{rank}(f) = m \leq r$ and $\mathrm{rank}(afa) = p < m$, then D^a_f sits below D^a_m and above D^a_p.
- The ‘crown’: A maximal $\mathscr{D}^a$-class $D^a_f = \{f\}$ sits above D^a_r if and only if $\mathrm{rank}(afa) = r$. The number of such $\mathscr{D}^a$-classes is equal to $(n^{n-r} - r^{n-r})r!\Lambda$.

$\mathrm{Reg}(\mathcal{T}_X^a)$ – examples



Egg-box diagrams of the regular subsemigroups $P = \mathrm{Reg}(\mathcal{T}_5^a)$ in the cases (from left to right): $a = [1, 1, 1, 1, 1]$, $a = [1, 2, 2, 2, 2]$, $a = [1, 1, 2, 2, 2]$, $a = [1, 2, 3, 3, 3]$, $a = [1, 2, 2, 3, 3]$, $a = [1, 2, 3, 4, 4]$.

Do you see what I am seeing???



Egg-box diagrams of $\mathcal{T}_3$ (left) and $\mathrm{Reg}(\mathcal{T}_5^a)$ for $a = [1, 2, 2, 3, 3]$ (right).

No, this is not just a coincidence...!

$$\mathcal{T}(X, A) = \{f \in \mathcal{T}_X : \ \mathrm{im}(f) \subseteq A\}$$

$$\mathcal{T}(X, \alpha) = \{f \in \mathcal{T}_X : \ \ker(f) \supseteq \alpha\}$$

– transformation semigroups with restricted range (Sanwong & Sommanee, 2008), and restricted kernel (Mendes-Gonçalves & Sullivan, 2010).

Fact:

$$\mathrm{Reg}(\mathcal{T}(X, A)) = \mathcal{T}(X, A) \cap P_2$$

$$\mathrm{Reg}(\mathcal{T}(X, \alpha)) = \mathcal{T}(X, \alpha) \cap P_1$$

Structure Theorem – Part 1

$$\psi : f \mapsto (fa, af)$$

is a well-defined embedding of $\mathrm{Reg}(\mathcal{T}_X^a)$ into the direct product $\mathrm{Reg}(\mathcal{T}(X, A)) \times \mathrm{Reg}(\mathcal{T}(X, \alpha))$. Its image consists of all pairs (g, h) such that

$$\mathrm{rank}(g) = \mathrm{rank}(h) \quad \text{and} \quad g|_A = (ha)|_A.$$

Thus $\mathrm{Reg}(\mathcal{T}_X^a)$ is a subdirect product of $\mathrm{Reg}(\mathcal{T}_X^a)$ and $\mathrm{Reg}(\mathcal{T}(X, \alpha))$.

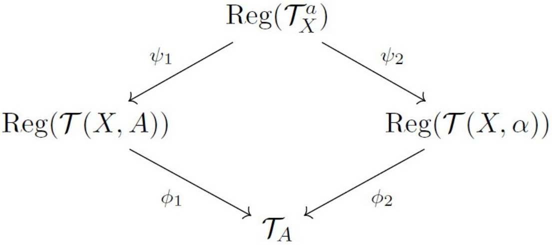
Structure Theorem – Part 2

The maps

$$\phi_1 : \mathrm{Reg}(\mathcal{T}(X,A)) \to \mathcal{T}_A : g \mapsto g|_A$$

$$\phi_2 : \mathrm{Reg}(\mathcal{T}(X,\alpha)) \to \mathcal{T}_A : g \mapsto (ga)|_A$$

are epimorphisms, and the following diagram commutes:



Further, the induced map $\phi = \psi_1\phi_1 = \psi_2\phi_2 = \mathrm{Reg}(\mathcal{T}_X^a) \to \mathcal{T}_A$ is an epimorphism that is ‘group / non-group preserving’.

Size and rank of $P = \mathrm{Reg}(\mathcal{T}_X^a)$

$$|P| = \sum_{m=1}^{r} m! m^{n-r} S(r, m) \sum_{I \in \binom{[1,r]}{m}} \Lambda_I .$$

Let D be the top (rank-r) $\mathscr{D}^a$-class of P.

$$\mathrm{rank}(P) = \mathrm{rank}(D) + \mathrm{rank}(P : D) = r^{n-r} + 1$$

The idempotent generated subsemigroup $\langle E_a(\mathcal{T}_X^a)\rangle_a$

- $E_a(\mathcal{T}_X^a) = \{f \in \mathcal{T}_X : (af)|_{\mathsf{im}(f)} = \mathrm{id}|_{\mathsf{im}(f)}\}$.
- $|E_a(\mathcal{T}_X^a)| = \sum_{m=1}^{r} m^{n-m} \sum_{I \in \binom{[1,r]}{m}} \Lambda_I$.
- We obtain a pleasing generalisation of celebrated Howie's Theorem:

$$\mathcal{E}_X^a = \langle E_a(\mathcal{T}_X^a)\rangle_a = E_a(D) \cup (P \setminus D).$$

The idempotent generated subsemigroup $\langle E_a(\mathcal{T}_X^a)\rangle_a$

- $$\operatorname{rank}(\mathcal{E}_X^a) = \operatorname{idrank}(\mathcal{E}_X^a) = r^{n-r} + \rho_r,$$
  where $\rho_2 = 2$ and $\rho_r = \binom{r}{2}$ if $r \geq 3$.
- The number of idempotent generating sets of $\mathcal{E}_X^a$ of the minimal possible size is
  $$\left[(r-1)^{n-r}\Lambda\right]^{\rho_r}\Lambda! S(r^{n-r}, \Lambda) \sum_{\Gamma \in \mathbb{T}_r} \frac{1}{\lambda_1^{d_\Gamma^+(1)} \cdots \lambda_r^{d_\Gamma^+(r)}}.$$
  where $\mathbb{T}_r$ is the set of all strongly connected tournaments on r vertices.

The ideals of P

- The ideals of P are precisely

$$I^a_m = \{f \in P : \ \mathrm{rank}(f) \leq m\}$$

  for $m \in [1, r]$.
- They are all idempotent generated (by $E_a(D^a_m)$) except $P = I^a_r$ itself.
-

$$\mathrm{rank}(I^a_m) = \mathrm{idrank}(I^a_m) = \begin{cases} m^{n-r} S(r, m) & \text{if } 1 < m < r \\ n & \text{if } m = 1. \end{cases}$$

Future work

- Conduct an analogous study for variants of:
  - full linear (matrix) monoids
  - symmetric inverse semigroups
  - various diagram semigroups (partition, (partial) Brauer, (partial) Jones, wire, Kaufmann,...)
  - ...
- Consider an 'Ehresmann-style' defined small (semi)category (aka partial monoid / semigroup) S. One can turn each hom-set S_{ij} (i - domain, j - codomain) into a semigroup by fixing a 'sandwich' element $a \in S_{ji}$ and defining

  $$x \star y = x \circ a \circ y.$$

  These sandwich semigroups generalise the variants.
  - applicable to functions, matrices, diagrams,...

THANK YOU!

Questions and comments to:

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Further information may be found at:

http://people.dmi.uns.ac.rs/∼dockie