Free idempotent generated semigroups: maximal subgroups and the word problem

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Mathematics Colloquium
MI SANU, Belgrade, 16 December 2016

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I'm free to be sure of what
I am and who I need not be
I'm much freer - like the meaning
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Man is condemned to be free.

Jean-Paul Sartre

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Hence:

What can we say about the structure of the free-est idempotent-generated (IG) semigroup with a fixed structure/configuration of idempotents ???

Biordered sets of idempotents

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Alternatively: Biordered set of a semigroup S = the partial algebra on $E(S)$ obtained by retaining the products of basic pairs (in S).

Free IG semigroups: idea

- To every semigroup S with idempotents E associate the free-est semigroup $\mathrm{IG}(E)$ whose idempotents form the same biordered set as in S.

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- To every regular semigroup S with idempotents E associate the free-est regular semigroup $\mathrm{RIG}(E)$ in whose idempotents form the same biordered set as in S.

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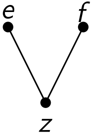
Example 1: Three-element meet semilattice

Let $S =$ 

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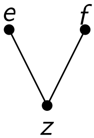
Let $S =$ 

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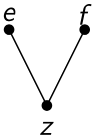
Let $S =$ 

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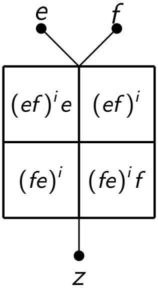

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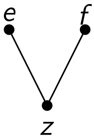
Let $S =$ 

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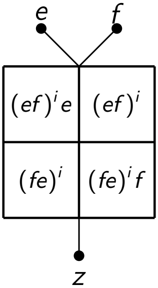

$RIG(S) = \langle e, f, z \mid \mathrm{IG},\ ef = fe = z \rangle = S.$

Example 2: 2×2 rectangular band

$S = \langle e_{ij} \mid e_{ij} e_{kl} = e_{il} \ (i, j, k, l \in \{1, 2\}) \rangle$:

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$\mathrm{RIG}(S) = \mathrm{IG}(S)$.

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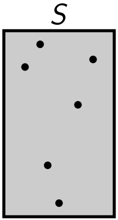
Question

Which groups arise as maximal subgroups of $\mathrm{IG}(E)$ (and thus of $\mathrm{RIG}(E)$)?

The big picture



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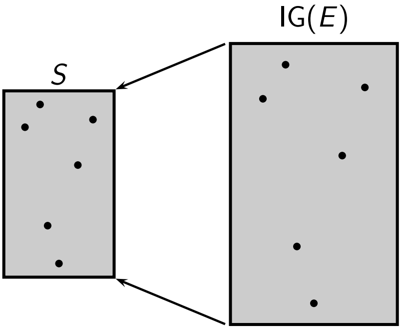

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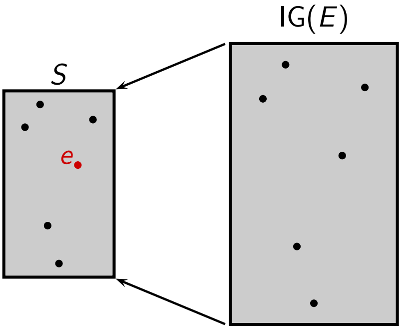

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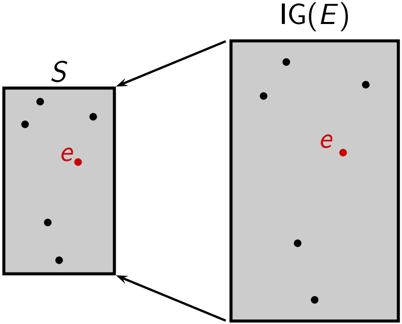

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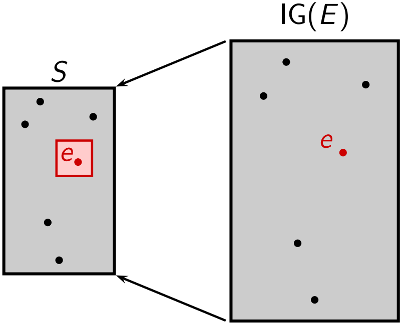

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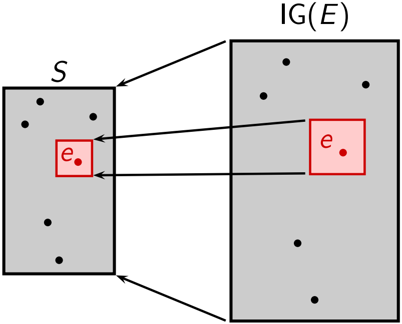

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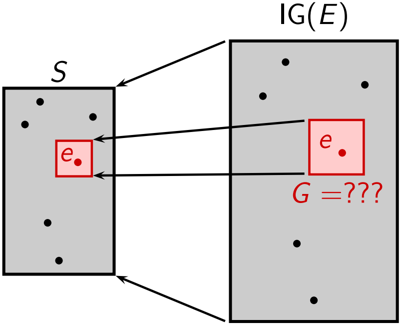
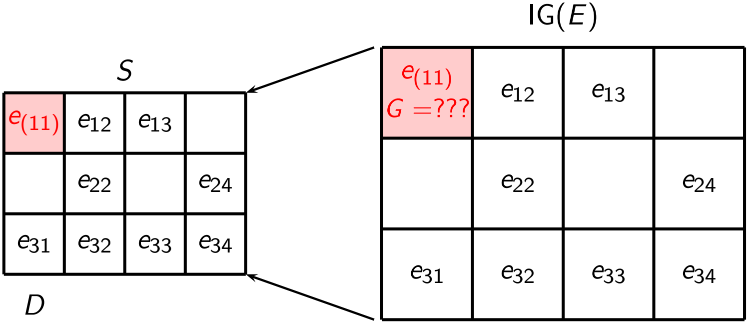


Let's zoom in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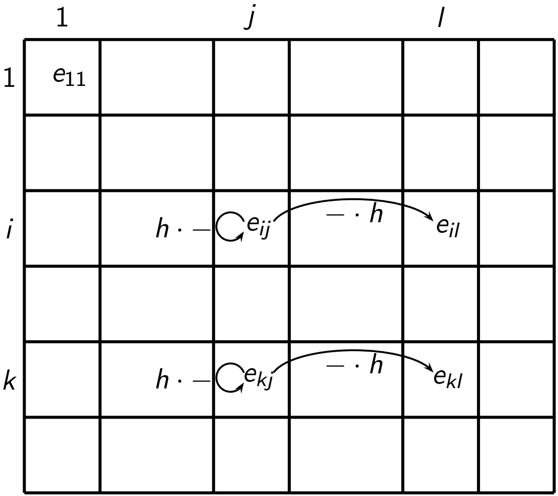
$$G = \langle f_{ij} \ (e_{ij} \in D \cap E) \mid ??? \rangle$$

Typical relations: $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$

- $h = h^2$



Singular square $\begin{bmatrix} e_{ij} & e_{il} \\ e_{kj} & e_{kl} \end{bmatrix} \Rightarrow$ relation $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$.

Presentation – Approach #1

Theorem (Nambooripad 1979; Gray, Ruškuc 2012)

The maximal subgroup G of $e \in E$ in $IG(E)$ or $RIG(E)$ is defined by the presentation:

$$\begin{aligned}\langle f_{ij} \mid f_{i,\pi(i)} &= 1 && (i \in I),\\ f_{ij} &= f_{il} && (\text{if } r_j e_{il} = r_l \text{ is a Schreier rep.}),\\ f_{ij}^{-1} f_{il} &= f_{kj}^{-1} f_{kl} && \left(\begin{bmatrix} e_{ij} & e_{il} \\ e_{kj} & e_{kl} \end{bmatrix} \text{ sing. sq.}\right)\rangle.\end{aligned}$$

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Proof: Reidemeister–Schreier rewriting process followed by Tietze transformations.

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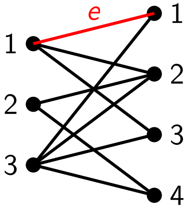

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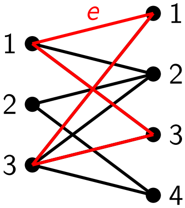

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Theorem (Brittenham, Margolis, Meakin, 2009)

The fundamental group of $GH(S)$ at any point of its connected component C_e containing the edge $e \cong$ the maximal subgroup of $RIG(E(S))$ (and thus of $IG(E(S))$) containing e.

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Obviously, a clever choice of $\mathcal{T}$ may speed up the computation.

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Question

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- Finally, Gray and Ruškuc (2012) proved that every group arises as a maximal subgroup of some free idempotent generated semigroup (!!!).

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Remark
Maximal subgroups of free idempotent generated semigroups arising from finite semigroups have to be finitely presented by Reidemeister–Schreier.

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- Endomorphism monoid of a free G-act: IgD, Gould, Yang (wreath products of G by symmetric groups).
  (J. Algebra, 2015)

Bands

Theorem (IgD, 2012)

For every left- or right seminormal band B, all maximal subgroups of $IG(B)$ are free. For every variety $\mathbf{V}$ *not contained in* $\mathbf{LSNB} \cup \mathbf{RSNB}$ *there exists* $B \in \mathbf{V}$ *such that* $IG(B)$ *contains a non-free maximal subgroup.*

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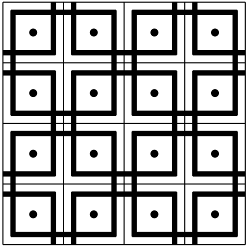
An example of the GH-complex in a 20-element regular band (the top 2×2 $\mathcal{D}$-class not shown):



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Answer (IgD, Ruškuc – IJAC, 2013): All of them! (Resp. all finitely presented ones.)

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- $\mathcal{T} = \mathcal{T}_I^* \times \mathcal{T}_J$;
- the minimal ideal: $K = \{(\sigma, \tau) : \sigma, \tau \text{ constants}\}$;
- K is an $I \times J$ rectangular band.

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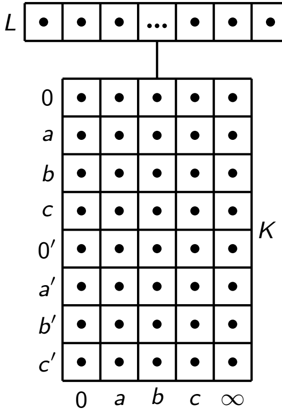
- $B = K \cup L$, where L is a left zero semigroup.



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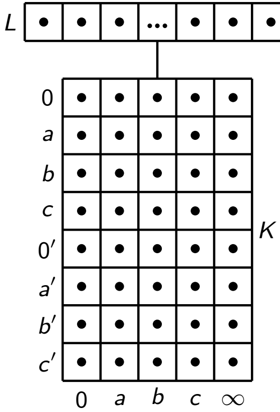

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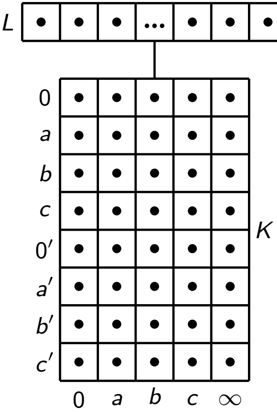

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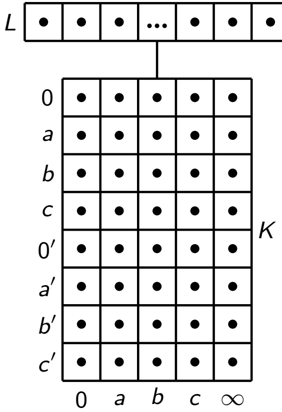

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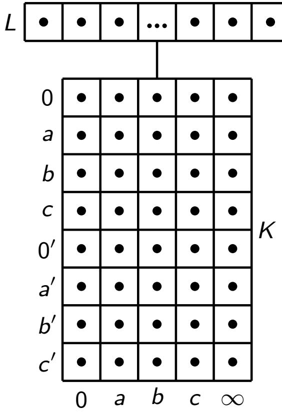

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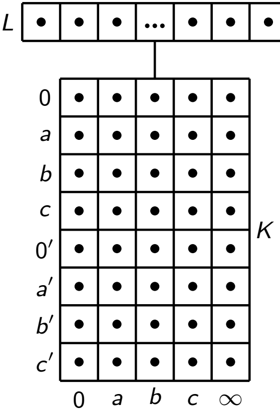

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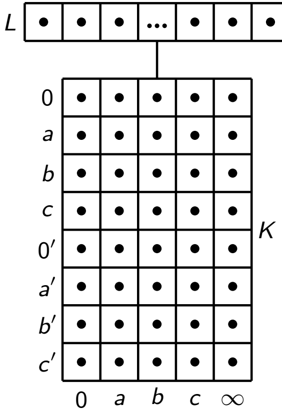
  - thus τ is specified by $(\infty)\tau$.



IgD & Ruškuc construction: the action of L on K

Notation	**Indexing**	$\mathrm{im}(\sigma)$	$(\infty)\tau$
(σ_0, τ_0)	–	$\{0, 0'\}$	0
(σ_a, τ_a)	$a \in A$	$\{0, a'\}$	a
$(\overline{\sigma}_a, \overline{\tau}_a)$	$a \in A$	$\{a, a'\}$	0
$(\sigma_{\mathbf{r}}, \tau_{\mathbf{r}})$	$\mathbf{r} = (ab, c) \in R$	$\{b, c'\}$	a

IgD & Ruškuc construction: the endgame

	0	a	b	c	∞
0	f_{00}	f_{0a}	f_{0b}	f_{0c}	$f_{0\infty}$
a	f_{a0}	f_{aa}	f_{ab}	f_{ac}	$f_{a\infty}$
b	f_{b0}	f_{ba}	f_{bb}	f_{bc}	$f_{b\infty}$
c	f_{c0}	f_{ca}	f_{cb}	f_{cc}	$f_{c\infty}$
$0'$	$f_{0'0}$	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	$f_{a'0}$	$f_{a'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	$f_{b'0}$	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
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$$\xrightarrow[]{(\sigma_0,\tau_0)\ (\sigma_a,\tau_a)\ (\overline{\sigma}_a,\overline{\tau}_a)}$$

	0	a	b	c	∞
0	1	1	1	1	1
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Theorem

There exists a finite semigroup S such that $IG(E)$ has an undecidable word problem.

The word problem for $\mathrm{IG}(E)$ – reloaded

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Question (Updated)

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Does $\mathrm{IG}(E)$ have a decidable word problem if E is finite and every maximal subgroup of $\mathrm{IG}(E)$ has a decidable word problem?

Theorem

There exists an algorithm deciding whether $w \in E^+$ represents a regular element of $IG(E)$.

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If every maximal subgroup of $IG(E)$ has a solvable word problem, then there is an algorithm which, given $u, v \in E^+$ such that u represents a regular element, decides whether $u = v$ holds in $IG(E)$.

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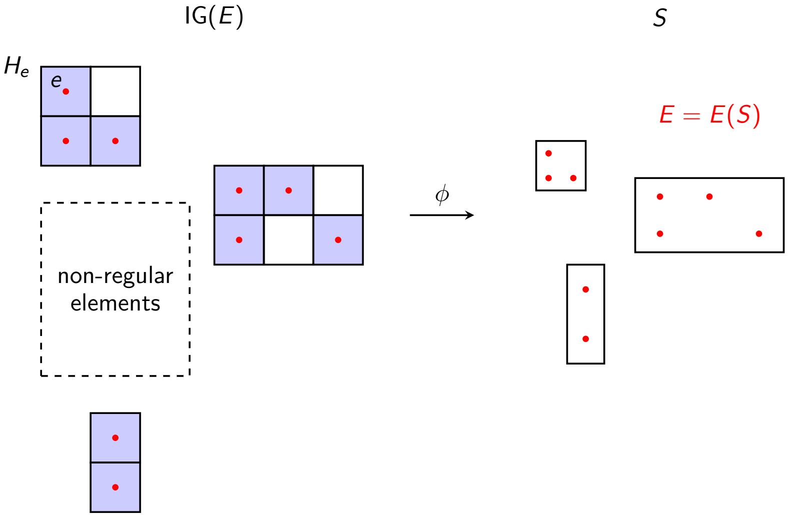
Corollary

The word problem for $RIG(E)$ is solvable iff the word problem for each of its maximal subgroups is solvable.

The general picture



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Theorem

There exists a finite band $B_{G,H}$ (constructed from a f.p. group G and its f.g. subgroup H) such that:

(i) *All maximal subgroups of $IG(B_{G,H})$ have decidable word problems.*

(ii) *The word problem for $IG(B_{G,H})$ is undecidable.*

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For this result we make use of another decision problem...

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Theorem (Mihailova, 1958)

Their exists a finitely presented group G with a finitely generated subgroup H such that

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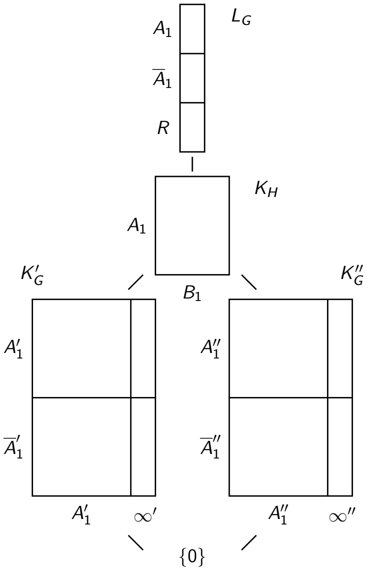
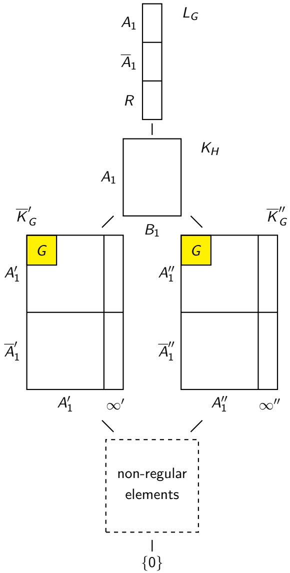
Example

$G = F \times F$ (F a f.g. free group), $H = \ker(\nu)$ ($\nu : F \to W$ a natural homomorphism onto a group with an undecidable WP).

The $B_{G,H}$ construction



Encoding the membership problem



Structure of IG$(B_{G,H})$

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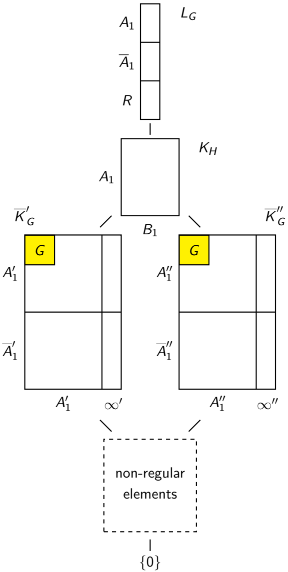

Structure of IG$(B_{G,H})$

Each of $\overline{K}'_G$ and $\overline{K}''_G$ is a Rees matrix semigroup over G

$$\overline{K}'_G \cong I' \times G \times J', \quad \overline{K}''_G \cong I'' \times G \times J''.$$

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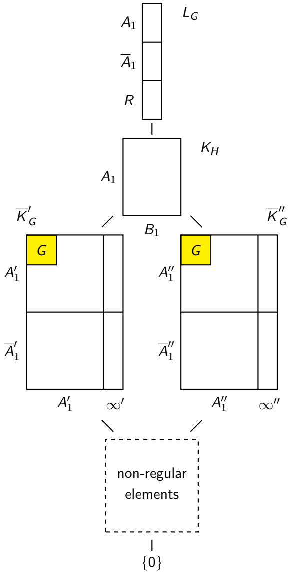

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For any word w over A the equality $(1', 1, 1')(1'', 1, 1'') = (1', w^{-1}, 1')(1'', w, 1'')$ holds in IG$(B_{G,H}) \Leftrightarrow w \in H$.

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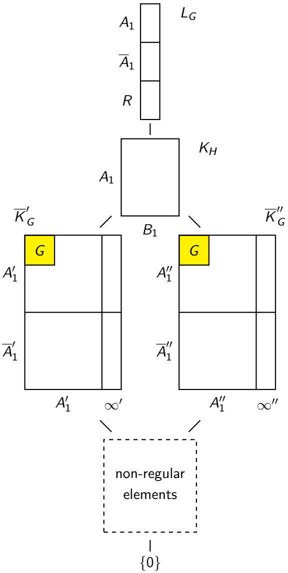

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Conclusion: If $\mathrm{IG}(B_{G,H})$ had a decidable word problem this would imply the membership problem for H in G is decidable, which is a contradiction. ↯

THANK YOU!

Questions and comments to:

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Further information may be found at:

http://people.dmi.uns.ac.rs/∼dockie