Finite groups are big as semigroups

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Big algebraic structures

Let $\mathcal{K}$ be a class of algebraic structures of a given similarity type (usually a variety or some other well-behaved class).

A finite algebra $B \in \mathcal{K}$ is $\mathcal{K}$-big if there exists a countably infinite algebra $A \in \mathcal{K}$ such that B is isomorphic to a maximal proper subalgebra of A.

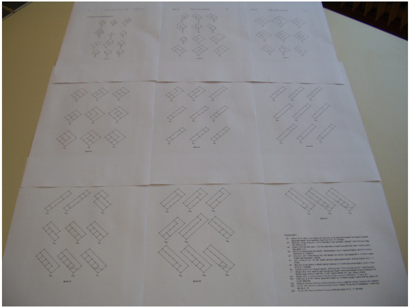
In other words, $A = \langle B, a \rangle$ for any $a \in A \setminus B$.

Motivation: Lattices

In 2001, Freese, Ježek and Nation published a paper where they fully described big lattices.

Theorem

There exist a list of 145 lattices (in fact, only 81 of them, up to dual isomorphism) such that a finite lattice is big if and only if it contains one of the lattices from the list as a sublattice.



Groups: serious issues

Open Problem

Characterise big groups.

This is closely related to difficult Burnside-type problems.

Ol'shanskii (1982): Constructed the first *Tarski monster group* — for each prime $p > 10^{75}$ there exists a 2-generated infinite group all of whose nontrivial proper subgroups have order p.

$\Longrightarrow \mathbb{Z}_p$ is a big group for any prime $p > 10^{75}$.

Adyan & Lysionok (1991): For any odd $n \geq 1003$ there exists a 2-generated infinite group G such that any proper subgroup of G is contained in a cyclic subgroup of order n.

$\Longrightarrow \mathbb{Z}_{2k+1}$ is a big group for any $k \geq 501$.

Relaxing the problem

Problem (R. Gray, P. Marković, 2011)

Which finite groups are big with respect to the class of all semigroups?

Theorem (ID & N. Ruškuc)

A finite group G is big with respect to the class of all semigroups if and only if $|G| \geq 3$.

Theorem (ID & NR)

Each finite semigroup S such that the kernel (the unique minimal ideal) of S contains a subgroup G such that $|G| \geq 3$ is a big semigroup.

Also: we should take care of $\mathbb{Z}_2$ and the trivial group...

A simple, yet important fact

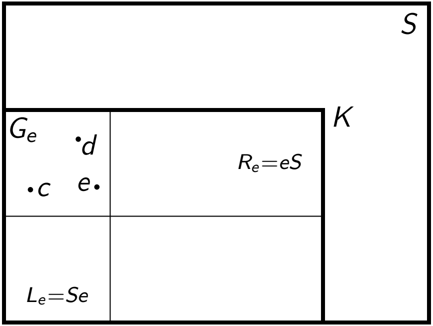
If S is a big (finite) semigroup such that it is $\cong$ a maximal proper subsemigroup of an infinite semigroup T, then T is called a witness for S.

Lemma

If $T \supset S$ is a witness for a big semigroup S, then $T \setminus S$ is contained in a single $\mathscr{J}$-class of T. In particular, if S is a group, then T can have at most two $\mathscr{J}$-classes.

Idea: Construct a witness Σ_S for S as an ideal extension of an <u>infinite</u> Rees matrix semigroup M by S^0, so that $\Sigma = S \cup M$, where S acts on M (from left and right) sufficiently 'transitively' to move around an arbitrary $a \in M$ along a generating set of M.

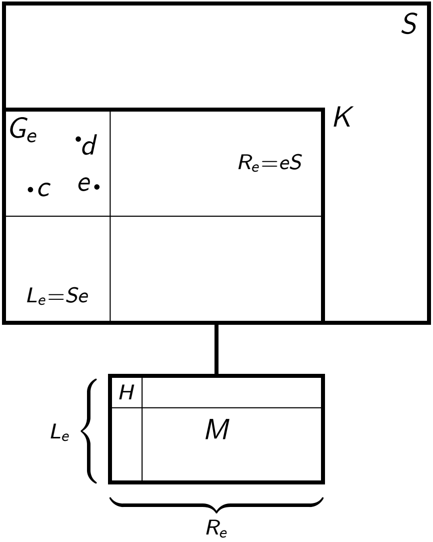
The kernel K of S



By assumption, we must have $|G_e| \geq 3$.

Thus we may fix two non-identity elements $c, d \in G_e = eSe$.

Σ_S



M: Choosing the structure group and the sandwich matrix

Let H be a 2-generated infinite periodic group, $H = \langle \gamma_1, \gamma_2 \rangle$ (i.e. a counterexample to the Burnside conjecture).

Remark

By this, we have ensured that M is infinite, periodic, finitely generated, and has finitely many left and right ideals.

Consider a function $\lambda : L_e \cup R_e \to H$ with the following properties:

1. $\lambda(e) = 1_H$,
2. $\lambda(c) = \gamma_1$,
3. $\lambda(d) = \gamma_2$,
4. $\lambda(se) = \lambda(ese)$ for all $s \in S$. (That is, the value of λ on $L_e = Se$ is completely determined by its values on $G_e = eSe$.)

M: Choosing the sandwich matrix (2)

Recall that

$$M = \mathcal{M}(L_e, H, R_e, P)$$

where $P = [p_{a,b}]$ is a $R_e \times L_e$ matrix.

For $a \mathscr{R} e \mathscr{L} b$ we define

$$p_{a,b} = \lambda(a)^{-1}\lambda(ab)\lambda(b)^{-1}$$

Remark

Since $be = b$ we have $p_{e,b} = 1_H$.

The multiplication between S and M is defined by:

$$\begin{aligned} s \cdot (a, h, b) &= (sa, \lambda(sa)\lambda(a)^{-1}h, b) \\ (a, h, b) \cdot s &= (a, h\lambda(b)^{-1}\lambda(bs), bs) \end{aligned}$$

The definition is OK

Lemma

Σ_S is a semigroup.

Remark

In the particular case when S is a group, the associativity of Σ_S boils down to an elementary fact in geometric group theory: there is a balanced labelling of the Cayley graph of S by elements of H such that two given non-loop edges are labelled by γ_1 and γ_2 respectively. (A spanning tree argument...)

The definitions of λ, P and $\cdot$ between S and M are motivated by (and are one implementation of) this.

Proof of the Main Theorem (1)

Let $h_0 \in H$ and $a \mathscr{R} e \mathscr{L} b$ be arbitrary.

Goal: Prove $T \equiv \langle S, (a, h_0, b) \rangle = \Sigma_S$.

There is no loss of generality in assuming that $a = b = e$, for otherwise $\exists s, t \in S$ such that $sa = bt = e$, and so

$$s(a, h_0, b)t = (e, \lambda(sa)\lambda(a)^{-1} h_0 \lambda(b)^{-1} \lambda(bt), e) \in T,$$

and we may continue working with

$$h_0' = \lambda(sa)\lambda(a)^{-1} h_0 \lambda(b)^{-1} \lambda(bt)$$

instead of h_0.

Proof of the Main Theorem (2)

Revised goal: Prove $T \equiv \langle S, (e, h_0, e)\rangle = \Sigma_S$.

Recall that we have picked $c, d \in G_e \setminus \{e\}$ carrying λ-labels γ_1 and γ_2, respectively. Since H is periodic, $h_0^m = 1_H$ for some $m \in \mathbb{N}$. So, the following are elements of T:

$$\begin{aligned}
(e, h_0, e)^m c (e, h_0, e)^m &= (e, 1_H, e)(c, \lambda(c), e) \\
&= (e, p_{e,c}\lambda(c), e) \\
&= (e, \gamma_1, e),
\end{aligned}$$

$$\begin{aligned}
(e, h_0, e)^m d (e, h_0, e)^m &= (e, 1_H, e)(d, \lambda(d), e) \\
&= (e, p_{e,d}\lambda(d), e) \\
&= (e, \gamma_2, e).
\end{aligned}$$

Proof of the Main Theorem (3)

Therefore, $H_e = \{e\} \times H \times \{e\} \subseteq T$.

However, then for any $a \mathscr{R} e \mathscr{L} b$ we have

$$aH_e b = \{a\} \times H \times \{b\} \subseteq T,$$

because

$$a(e, h, e)b = (a, \lambda(a)h\lambda(b), b),$$

and $x \mapsto \lambda(a)x\lambda(b)$ is a permutation of H.

Hence, $L_e \times H \times R_e \subseteq T$, so $T = \Sigma_S$, Q.E.D.

The trivial (semi)group is not big

Suppose, to the contrary, that S is a witness for $\{e\}$, $e \in E(S)$.

Both Se and eS are subsemigroups of S containing e, so $Se, eS \in \{\{e\}, S\}$.

If $Se = eS = S$, then e is an identity element of S, and if $Se = eS = \{e\}$, then e is the zero of S.

In either case, for any $s \in S \setminus \{e\}$ we have $S = \langle e, s\rangle = \{e, s, s^2, \dots\}$, where s is not periodic (because S is infinite), so $\{e, s^2, s^4, \dots\}$ is a proper subsemigroup of S containing e.

If $Se = S$ and $eS = \{e\}$ ($Se = \{e\}$ and $eS = S$) then S is a left (resp. right) zero semigroup $\Longrightarrow$ every subset of S is a subsemigroup. Contradiction!

Yet another useful...

Lemma

Let S be a big semigroup, and let T be any witness for S. Let J be the unique $\mathscr{J}$-class of T containing $T \setminus S$. Then J contains a J-primitive idempotent, that is, a minimal element in the restriction of the Rees order of idempotents of T to $J \cap E(T)$.

Steps:

(i) There exist $a, b \in J$ such that $ab \in J$.

(ii) There exists $t \in J$ such that $t^n \in J$ for all $n \in \mathbb{N}$.

(iii) J contains an idempotent.

(iv) J contains a J-primitive idempotent.

$\mathbb{Z}_2$ is not a big semigroup (1)

Assume to the contrary, that T is a witness for $\mathbb{Z}_2 = \{e, a\}$.

Since $T \setminus \mathbb{Z}_2$ is contained in a single $\mathscr{J}$-class J of T, there are two possibilities:

1. $T = J$ is simple, or
2. T has precisely two $\mathscr{J}$-classes: $\mathbb{Z}_2$ and J.

In either case, J is the kernel of T and, since it contains a J-primitive idempotent that must also be T-primitive, it follows that J is completely simple.

$\mathbb{Z}_2$ is not a big semigroup (2)

Case 1: $T \cong \mathcal{M}(I, G, \Lambda, P)$, and G has a subgroup of order 2.

Now $\mathbb{Z}_2$ is not big as a group (F+J+N — easy), so if G is infinite, there is a proper subgroup G_1 of G properly containing $\mathbb{Z}_2$, destroying T as a witness.

Thus G must be finite, so at least one of the index sets I, Λ are infinite.

At the same time, notice that we must have

$$T = \langle G_{i\mu}, (j, h, \nu) \rangle$$

for some $i \in I$, $\mu \in \Lambda$, and any $(j, h, \nu) \in T \setminus G_{i\mu}$.

However, $\langle G_{i\mu}, (j, h, \nu) \rangle \subseteq G_{i\mu} \cup G_{j\mu} \cup G_{i\nu} \cup G_{j\nu} \subsetneq T$.
Contradiction!

$\mathbb{Z}_2$ is not a big semigroup (3)

Case 2: T is an ideal extension of a completely simple semigroup J by $\mathbb{Z}_2^0$.

So, T has an idempotent $f \neq e$, whence $T = \langle a, f \rangle$. Furthermore, e can be assumed to be the identity of T, for otherwise $\{e, a\} \subsetneq eTe \subsetneq T$.

Hence, each element of T is an alternating product of a and f.

We have $faf \mathscr{J} f \implies f = t_1(faf)t_2 = ft_1faft_2f$ for some $t_1, t_2 \in J^1$.

Therefore, for some $k \geq 1$ we have

$$(faf)^k = faf \cdots faf = f$$

$\implies |J| \leq 4k + 1$ (i.e. J is finite). Contradiction!

OK, girls & boys, the last slide of this talk is SOOOOO predictable...

Open Problem

Characterise big semigroups.

Igor, now remember to make a sketch on the black-/white-board...
(For what is a lecture without a nice drawing...?)

Also, don't forget some handwaving to finish it off nicely. ♡

THANK YOU!

Questions and comments to:

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Further information may be found at:

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