Representing semigroups and groups by endomorphisms of Fraïssé limits

Part II. Groups: overt & covert

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Sherlock Holmes: A Game of Shadows (2011)

It's so overt, it's covert – a more brutal version

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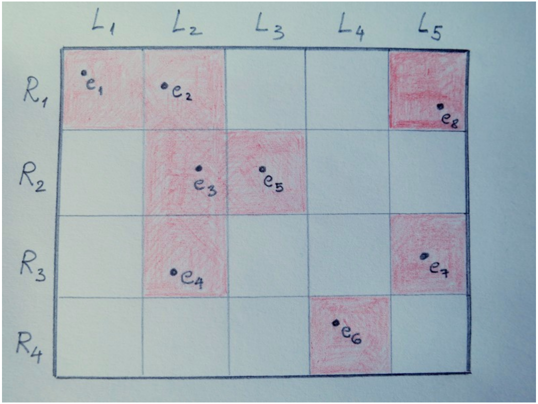
$$a \mathscr{J} b \Leftrightarrow S^1 a S^1 = S^1 b S^1 \Leftrightarrow (\exists x, y, u, v \in S^1)\, uax = b \,\&\, vby = a$$

The eggbox picture of a $\mathscr{D}$-class



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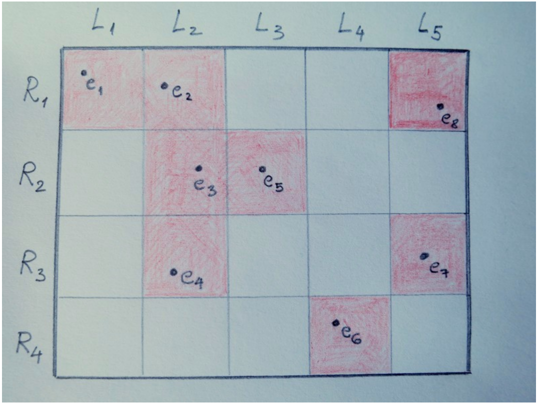

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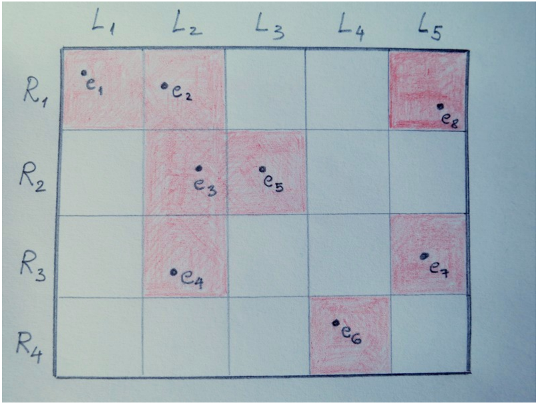

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maximal subgroups of a semigroup = $\mathscr{H}$-classes containing idempotents

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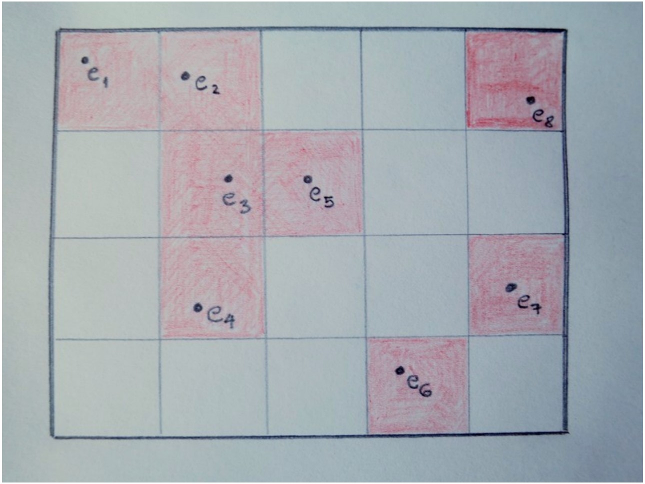
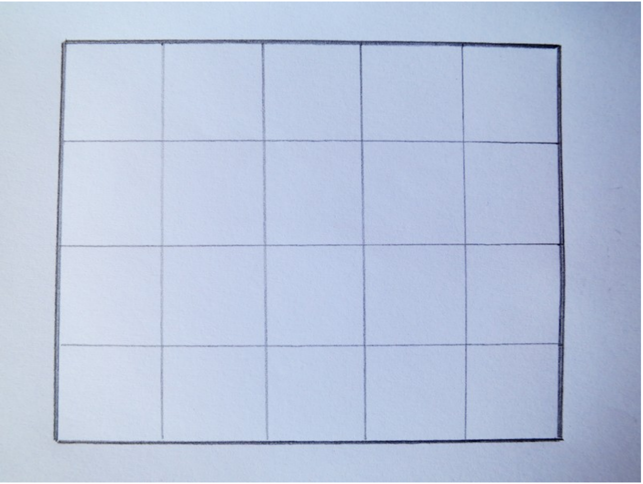
Hence, a is regular $\iff$ $a \ \mathscr{D} \ e$ for and idempotent e.

A regular $\mathscr{D}$-class



A regular eggbox

A non-regular $\mathscr{D}$-class



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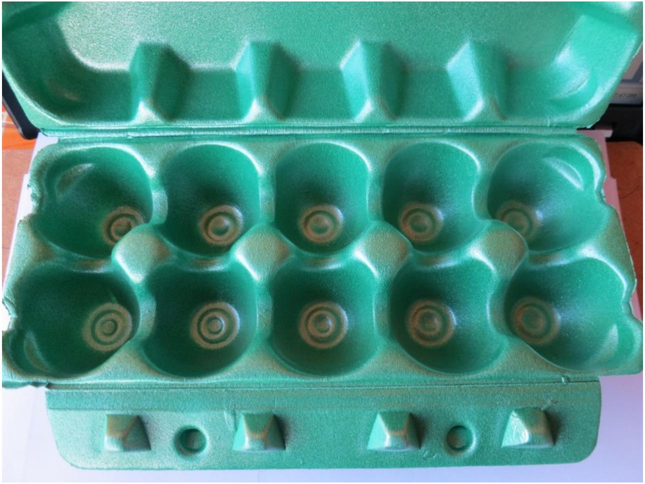

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If H is a group (so that D is regular), then $S_H \cong H$.

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(6) $\mathcal{T}_X$ is regular.

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Lemma

$f \mathrel{\mathscr{D}} g \implies \langle Af \rangle \cong \langle Ag \rangle$.

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(iv) *if e is idempotent, then* $H_e \cong \mathrm{Aut}(\mathrm{im}(e)) \cong \mathrm{Aut}(\mathrm{im}(f))$ *for any $f \in D_e$.*

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(i) *If $t \in T_H$, then $t|_{Af}$ is an automorphism of both $\langle Af \rangle$ and $\mathrm{im}(f)$;*

(ii) *the mapping $\phi : \rho_t \mapsto t|_{Af}$ is an embedding of S_H into $\mathrm{Aut}(\langle Af \rangle) \cap \mathrm{Aut}(\mathrm{im}(f))$.*

So, what the heck are the images of (idempotent) endomorphisms of Fraïssé limits?

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Theorem (ID, 2012)

Let **C** *be a neat Fraïssé class enjoying the strict AP and the 1PHEP. Then there exists and (idempotent) endomorphism f of F, the Fraïssé limit of* **C***, such that $A \cong \mathrm{im}(f)$ if and only if A is algebraically closed in* $\overline{\mathbf{C}}$.

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Let $\mathbf{K}$ be a class of L-structures. An L-structure A is existentially (algebraically) closed (in $\mathbf{K}$) if for any primitive (positive) formula $\Phi(\mathbf{x})$ and any tuple $\mathbf{a}$ from A we have already $A \models \Phi(\mathbf{a})$ whenever there is an extension $A' \in \mathbf{K}$ of A such that $A' \models \Phi(\mathbf{a})$.

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In the rest of this talk we will be concerned with simple graphs and study $\mathrm{End}(R)$. However, all these results can be adapted for

- the random digraph,
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Proposition

A countable graph (V, E) is a.c. if and only if there exists $E' \subseteq E$ such that $(V, E') \cong R$ (that is, it is e.c.). Consequently, for any a.c. graph Γ there is a bijective homomorphism $R \to \Gamma$.

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Name of the game: Strengthen this for countable a.c. graphs.

The team

Point Guard:
Martyn Quick

Forward:
"Baby" James Mitchell

Center:
Jillian "Jay" McPhee

Shooting Guard:
Robert "Bob" Gray

Power Forward:
Dr. D

Happy 30th birthday, Jay !!! (July 28)

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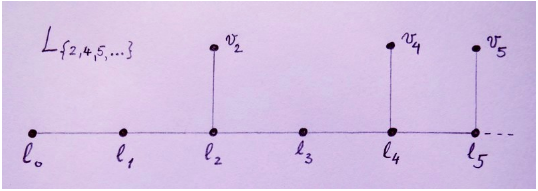
- $\mathrm{Aut}(\Delta^\dagger) = \mathrm{Aut}(\Delta)$.
- Δ any graph, Λ infinite locally finite graph $\Rightarrow (\Delta \uplus \Lambda)^\dagger$ is a.c.
- The central idea – consider l.f. graphs L_S for $S \subseteq \mathbb{N} \setminus \{0, 1\}$:



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Proof (cont'd).

- Properties of L_S ($S, T \subseteq \mathbb{N} \setminus \{0, 1\}$):
  - Each L_S is rigid ($\mathrm{Aut}(L_S) = \mathbf{1}$).
  - $L_S \cong L_T \iff S = T$.

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  - Each L_S is rigid ($\mathrm{Aut}(L_S) = \mathbf{1}$).
  - $L_S \cong L_T \iff S = T$.
- If L_S is isomorphic to no connected component of Γ (and this excludes only countably many choices of S), then

$$\mathrm{Aut}(\Gamma \uplus L_S)^\dagger = \mathrm{Aut}(\Gamma \uplus L_S) \cong \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(L_S) \cong \mathrm{Aut}(\Gamma).$$

Automorphism groups of countable a.c. graphs

Proof (cont'd).

- Properties of L_S ($S, T \subseteq \mathbb{N} \setminus \{0, 1\}$):
  - Each L_S is rigid ($\mathrm{Aut}(L_S) = \mathbf{1}$).
  - $L_S \cong L_T \iff S = T$.
- If L_S is isomorphic to no connected component of Γ (and this excludes only countably many choices of S), then

$$\mathrm{Aut}(\Gamma \uplus L_S)^\dagger = \mathrm{Aut}(\Gamma \uplus L_S) \cong \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(L_S) \cong \mathrm{Aut}(\Gamma).$$

- $S_1 \neq S_2$ yield non-isomorphic a.c. graphs.

Images of idempotent endomorphisms

Theorem (Bonato, Delić, 2000; ID, 2012)

Let Γ *be a countable graph. There exists an idempotent* $f \in \mathrm{End}(R)$ *such that* $\mathrm{im}(f) \cong \Gamma$ *if and only if* Γ *is a.c.*

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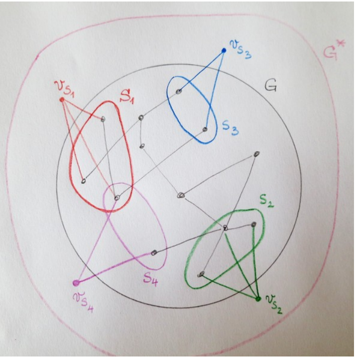
Theorem

If Γ is a countable a.c. graph, then there exists an (induced) subgraph $\Gamma' \cong \Gamma$ of R such that there are $2^{\aleph_0}$ idempotent endomorphisms f of R such that $\mathrm{im}(f) = \Gamma'$.

Images of idempotent endomorphisms

Proof.



At each stage of extending a homomorphism $\phi : \Gamma \to R_\Gamma$ to an endomorphism $\hat{\phi}$ of $R = R_\Gamma$, instead of mapping $v_S \mapsto v_{S\phi}$, if $\mathrm{im}(\phi)$ is a.c. one can find a common neighbour w for $S\phi$ within $\mathrm{im}(\phi)$.

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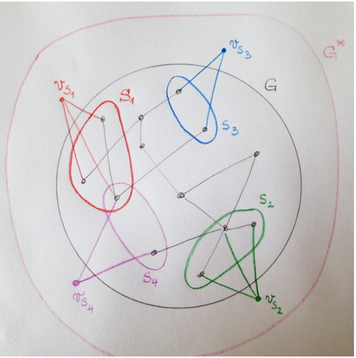

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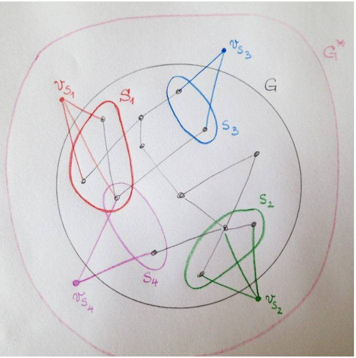

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In fact, at each stage there are infinitely many choices for w, which results in $\aleph_0^{\aleph_0} = 2^{\aleph_0}$ extensions.

The number of regular $\mathscr{D}$-classes with a given group $\mathscr{H}$-class

Theorem

(i) *Let* Γ *be a countable graph. Then there exist* $2^{\aleph_0}$ *distinct regular* $\mathscr{D}$*-classes of* $\mathrm{End}(R)$ *whose group* $\mathscr{H}$*-classes are* $\cong \mathrm{Aut}(\Gamma)$.

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$\mathrm{End}(R)$ *has* $2^{\aleph_0}$ *regular* $\mathscr{D}$*-classes. (You know, the ones with eggs...)*

The size of a regular eggbox

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Every regular $\mathscr{D}$-class of $\mathrm{End}(R)$ *contains* $2^{\aleph_0}$ *many* $\mathscr{R}$*- and* $\mathscr{L}$*-classes.*

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All such f are idempotents, and $f \;\mathscr{D}\; e$, moreover, $f \;\mathscr{L}\; e$.

However, all these idempotents are not $\mathscr{R}$-related.

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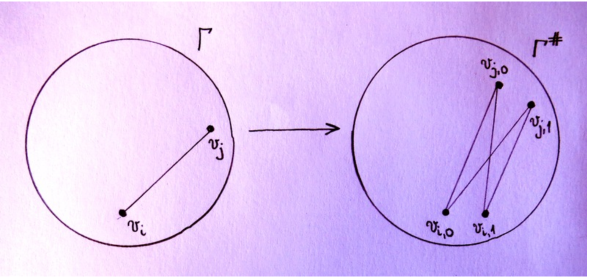

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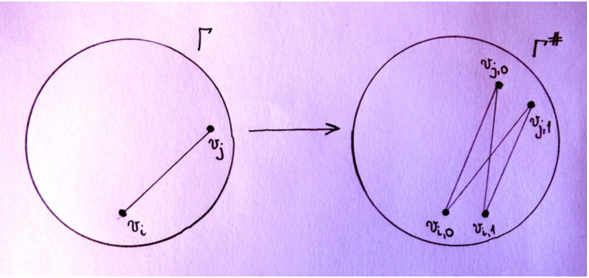

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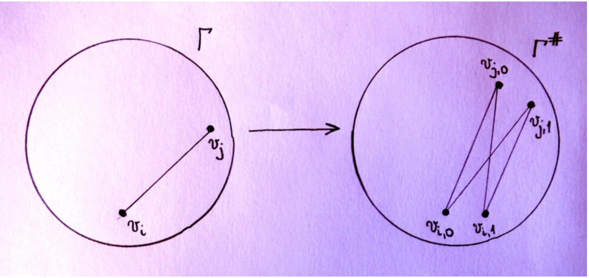

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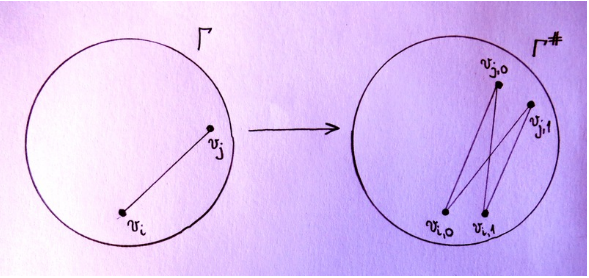

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Γ a.c. $\Longrightarrow$ $\Gamma^\sharp$ a.c. Hence, the identity map on $\Gamma^\sharp$ can be extended to an endomorphism $g : R \to \Gamma^\sharp$.

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For each binary sequence $\mathbf{b} = (b_i)_{i \in \mathbb{N}}$ define a map $\psi_{\mathbf{b}}$ on $\Gamma^\sharp$ by

$$v_{i,r}\psi_{\mathbf{b}} = v_{i,b_i}$$

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Different images $\Rightarrow$ they are not $\mathscr{L}$-related.

Non-regular eggboxes

Theorem

Let $\Gamma \not\cong R$ be a countable a.c. graph. Then there exists a non-regular endomorphism of R such that $\mathrm{im}(f) \cong \Gamma$ and D_f contains $2^{\aleph_0}$ many $\mathscr{R}$- and $\mathscr{L}$-classes.

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Open Problem

Are there any non-regular eggboxes of some other size?

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Proposition

Let f be an injective endomorphism of $R = (V, E)$ as described above, with $Vf = V_0$. Then

$$S_{H_f} \cong \mathrm{Aut}(\langle V_0 \rangle) \cap \mathrm{Aut}(\mathrm{im}(f))$$

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So, to show a universality result for Schützenberger groups in End(R), one needs to extend the Frucht-de Groot-Sabidussi Theorem to countable a.c. graphs with 2-coloured edges (blue and red, say) where the 'red graph' is $\cong R$.

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Let Γ be any countable graph. There are $2^{\aleph_0}$ non-regular $\mathscr{D}$-classes of End(R) *such that the Schützenberger groups of the $\mathscr{H}$-classes within them are* $\cong$ Aut(Γ).

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See arXiv:1408.4107 for details.

A few words on posets

A poset $(P, \leq)$ is a.c. if for any finite $A, B \subseteq P$ such that $A \leq B$ there exists $x \in P$ such that

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It follows all countable/finite groups arise as automorphism groups of countable/finite a.c. posets.

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Related work: G. Behrendt (PEMS, 1992)

THANK YOU!

Questions and comments to:

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