One-relator groups, monoids, inverse monoids: An update on the word problem

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The word problem (in groups, monoids,...)

Assume we have given a (finitely generated) group $G = \langle X \rangle$ (e.g. by a presentation, etc.). So, elements of G are represented by words over $\overline{X} = X \cup X^{-1}$.

The word problem for G is the following decision (algorithmic) problem:

INPUT: A word $w \in \overline{X}^*$.
QUESTION: Does w represent the identity element 1 in G?

Similarly, one can ask about the word problem for monoids / inverse monoids / ..., with the difference being that the input requires two words u, v (over X^* or $\overline{X}^*$, respectively), and then we want to decide if $u = v$ holds in the corresponding monoid.

One-relator groups

A one-relator group is a group defined by the presentation of the form

$$G = \mathrm{Gp}\langle X \mid w = 1 \rangle = \mathrm{FG}(X)/\langle\!\langle w \rangle\!\rangle$$

for a word $w \in \overline{X}$.

W. Magnus (1932): Every one-relator group has a decidable word problem.

Examples:

- $\mathbb{Z} \times \mathbb{Z} = \mathrm{Gp}\langle x, y \mid [x, y] = 1 \rangle$
- Baumslag-Solitar groups
  $B(m, n) = \mathrm{Gp}\langle a, b \mid b^{-1} a^m b a^{-n} = 1 \rangle$
- (orientable) surface groups
  $\mathrm{Gp}\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle$

So, what about one-relator monoids?

Open Problem (as of 2021 (!))

Is the word problem decidable for all one-relator monoids $\mathrm{Mon}\langle X \mid u = v \rangle$?

Theorem (Adyan, 1966)

The word problem for $\mathrm{Mon}\langle X \mid u = v \rangle$ *is decidable if either:*

- *one of* u, v *is empty (e.g.* $u = 1$ *– special monoids), or*
- *both* u, v *are non-empty, and have different initial letters and different terminal letters.*

Lallement (1977) and L. Zhang (1992) provided alternative proofs for the result about special monoids. (The proof of Zhang is particularly compact and elegant.)

The connection to the inverse realm

Adyan & Oganessyan (1987): The word problem for one-relator monoids can be reduced to the special case of

$$\mathrm{Mon}\langle X \mid asb = atc \rangle$$

where $a, b, c \in X$, $b \neq c$ and $s, t \in X^*$ (and their duals).

So, where do (one-relator) inverse monoids come into play?

Theorem (Ivanov, Margolis & Meakin, 2001)

If the word problem is decidable for all special inverse monoids $\mathrm{Inv}\langle X \mid w = 1 \rangle$ *— where w is a reduced word over $\overline{X}$ — then the word problem is decidable for every one-relator monoid.*

This holds basically because $M = \mathrm{Mon}\langle X \mid asb = atc \rangle$ embeds into $I = \mathrm{Inv}\langle X \mid asbc^{-1}t^{-1}a^{-1} = 1 \rangle$.

The plot thickens

	$\mathrm{Gp}\langle X \mid w = 1\rangle$	$\mathrm{Mon}\langle X \mid w = 1\rangle$	$\mathrm{Inv}\langle X \mid w = 1\rangle$
decidable WP	✓ (Magnus, 1932)	✓ (Adyan, 1966)	? ✗ (Gray, 2019)

Conjecture (Margolis, Meakin, Stephen, 1987)

Every inverse monoid of the form $\mathrm{Inv}\langle X \mid w = 1\rangle$ has decidable word problem.

Theorem (RD Gray, Invent. Math., 2020)

There exists a one-relator inverse monoid $\mathrm{Inv}\langle X \mid w = 1\rangle$ *with undecidable word problem.*

This result follows from the existence of a particular one-relator group G and its finitely generated submonoid $N \leq G$ such that the membership problem for N in G is undecidable.

The (prefix) membership problem (1)

Let G be a finitely generated group, and let $M \leq G$ be a finitely generated submonoid. The membership problem for M in G is the following decision problem:

INPUT: A word $w \in \overline{X}^*$.
QUESTION: Does the element of G represented by w belong to M?

Now let $G = \mathrm{Gp}\langle X \mid w = 1 \rangle$. The prefix monoid of G is the submonoid of G generated by all the elements represented by the prefixes of w:

$$P_w = \mathrm{Mon}\langle \mathrm{pref}(w) \rangle \leq G.$$

The prefix membership problem for the (one-relator) group G asks if the membership problem for P_w in G is decidable.
(Caution: depends on the presentation!)

The (prefix) membership problem (2)

Example

Let $G = \mathrm{Gp}\langle a, b \mid aba^{-1}b^{-1} = 1\rangle$.
$P_w = \mathrm{Mon}\langle a, ab, aba^{-1}\rangle = \mathrm{Mon}\langle a, b\rangle$ (because $aba^{-1} = b$ in G).
So P_w consists of elements of G represented by all positive words.
The prefix membership problem is decidable in this case, as it suffices to bring a given word into a normal form $a^i b^j$ and then check whether $i, j \geq 0$.
For example:

$$a^5 b^{-7} a^{-10} b^8 a^9 = a^4 b \in P_w, \quad a^3 b^5 a^{-5} b^2 = a^{-2} b^7 \notin P_w.$$

Problem

Does every one-relator group $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ has decidable prefix membership problem? If the answer is "no", characterise words w which do have this property.

The (prefix) membership problem (3)

Proved true in several special cases, when w...

- ...represents an idempotent of the free inverse monoid (Birget, Margolis, Meakin, 1993/4);
- ...is "strictly positive" (Ivanov, Margolis, Meakin, 2001);
- ...defines certain Adyan-type or Baumslag-Solitar-type groups (Margolis, Meakin, Šunić, 2005);
- ...satisfies certain small-cancellation-type conditions (Juhász, 2012, 2014).

Tying everything together

Theorem (Ivanov, Margolis & Meakin, 2001)

If $M = \mathrm{Inv}\langle X \mid w = 1 \rangle$ is E-unitary, then

word problem for M = prefix membership problem for $G = \mathrm{Gp}\langle X \mid w = 1 \rangle$.

E-unitary inverse semigroups = the "good guys" of inverse semigroup theory:

- For any $e \in E(S)$ and $x \in S$,
  $e \leq x$ (in the natural inverse semigroup order) $\Rightarrow x \in E(S)$.
- The minimum group congruence σ on S is idempotent-pure, meaning that $E(S)$ constitutes a single σ-class (the identity element of the group S/σ).
- ...

Theorem (Ivanov, Margolis & Meakin, 2001)

If w is cyclically reduced, then $M = \mathrm{Inv}\langle X \mid w = 1 \rangle$ is E-unitary.

The paper

In our recently published paper,

I.Dolinka, R.D.Gray, New results on the prefix membership problem for one-relator groups, *Trans. Amer. Math. Soc.* **374** (2021), 4309–4358.

our strategy was first to prove several general sufficient conditions for a finitely generated submonoid in

- free amalgamated products of f.g. groups,
- HNN-extensions of a f.g. group,

and then to apply these results to deduce decidability of the prefix membership problem in certain classes of one-relator groups.

Here is a sample of such a type of result.

Theorem A

Let $G = H *_L K$, where H, K, L are finitely generated groups such that both H, K have decidable word problems, and the membership problem for L in both H and K is decidable.
Let M be a submonoid of G such that:

(i) $L \subseteq M$;

(ii) both $M \cap H$ and $M \cap K$ are finitely generated, and

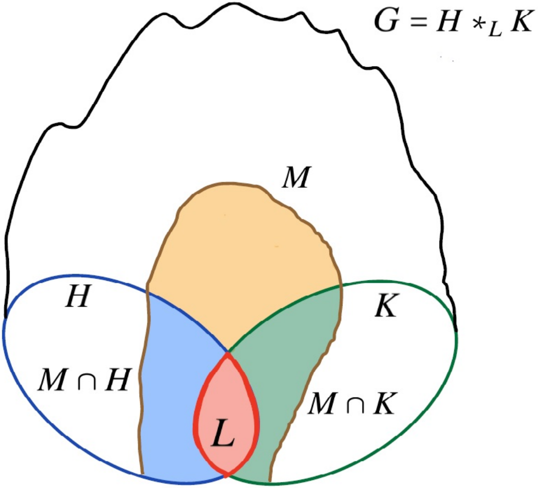
$$M = \mathrm{Mon}\langle (M \cap H) \cup (M \cap K) \rangle;$$

(iii) the membership problem for $M \cap H$ in H is decidable;

(iv) the membership problem for $M \cap K$ in K is decidable.

Then the membership problem for M in G is decidable.

Picture for Theorem A



Conservative factorisations (1)

Let $G = \mathrm{Gp}\langle X \mid w = 1 \rangle$ and consider a factorisation

$$w \equiv w_1 \dots w_k.$$

Define $P(w_1, \dots, w_k)$ to be the submonoid of G generated by $\bigcup_{1 \le i \le k} \mathrm{pref}(w_i)$.

Since every prefix of w is a product of prefixes of w_i's, clearly $P_w \subseteq P(w_1, \dots, w_k)$. When equality takes place, we say that the considered factorisation is conservative.

The factors w_i in a conservative factorisation are called pieces.

Conservative factorisations (2)

Methods for finding conservative factorisations:

- Adyan overlap algorithm: based on the fact that if $\alpha\beta$ and $\beta\gamma$ are pieces, so are α, β, γ
  - Example: w.r.t. $G = \mathrm{Gp}\langle a, b, c, d \mid abcdabcdcd = 1\rangle$, the factorisation $(ab)(cd)(ab)(cd)(cd)$ is conservative
- Benois method [Gray & Ruškuc, 2021]: based on Benois' Theorem (stating that rational subsets in f.g. free groups have uniformly decidable membership problem)
  - Example:
    $G = \mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$ is a conservative factorisation

Application 1: Unique marker letter theorem

Let $G = \mathrm{Gp}\langle X \mid w = 1\rangle$, where $w \equiv w_1 \dots w_k$ is a conservative factorisation. Let $U = \{u_1, \dots, u_m\}$ be the set of the pieces appearing in this factorisation. Suppose that

- for all $1 \le i \le m$ there is a letter $a_i \in X$ appearing exactly once in u_i and not appearing in any u_j, $j \neq i$.

Then $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ has decidable prefix membership problem.

Example

In $G = \mathrm{Gp}\langle a, b, x, y \mid axbaybaybaxbaybaxb = 1\rangle$, the Adyan overlap method produces a conservative factorisation

$$(axb)(ayb)(ayb)(axb)(ayb)(axb),$$

where the pieces axb and ayb have the unique marker letter property, so G has decidable prefix membership problem.

Application 1: Unique marker letter theorem

$G = \mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$ is a conservative factorisation. Here, the pieces don't have the unique marker letter property. However, upon transforming

$$\begin{aligned} G &= \mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle \\ &= \mathrm{Gp}\langle a, b, c, d \mid (aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad)(ad) \\ &\quad (aba^{-1})(aba^{-1})(aca^{-1})(ad)(aca^{-1})(ad) = 1\rangle \end{aligned}$$

we obtain a conservative factorisation, where the pieces aba^{-1}, aca^{-1} and ad do have the unique marker letter property.

Since the prefix monoid turns out to be unchanged in this modified presentation of G, we conclude that G (w.r.t. the initial presentation) has decidable prefix membership problem.
(This answers a question harking back to the 1987 paper of Margolis, Meakin and Stephen.)

Application 2: Disjoint alphabets theorem

Let $G = \mathrm{Gp}\langle X \mid w = 1\rangle$, where w is cyclically reduced and $w \equiv w_1 \dots w_k$ is a conservative factorisation. Let $U = \{u_1, \dots, u_m\}$ be the set of the pieces appearing in this factorisation. Suppose that $m \geq 2$ and

- for all $1 \leq i \neq j \leq m$, u_i and u_j have no letters in common.

Then $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ has decidable prefix membership problem.

Example

In $G = \mathrm{Gp}\langle a, b, c, d \mid ababcdcdababcdcdcdcdabab = 1\rangle$, the Adyan overlap method produces a conservative factorisation

$$(abab)(cdcd)(abab)(cdcd)(cdcd)(abab),$$

where the pieces *abab* and *cdcd* have no letters in common.
So, G has decidable prefix membership problem.

Application 3: Exponent sum zero theorem

Let $G = \mathrm{Gp}\langle X \mid w = 1 \rangle$ where $t \in X$ and w (containing t) has t-exponent sum 0. (E.g. $w \equiv atat^2a^2t^{-3}$.)
Then (by the observation of Moldavanskiĭ (1967)), the following exists:

- a one-relator group $G' = \mathrm{Gp}\langle X' \mid w' = 1 \rangle$ with $|w'| < |w|$;
- subsets $A, B \subset X'$ forming bases of free subgroups F_1, F_2 of G';
- an isomorphism $\phi : F_1 \to F_2$;

such that G is a HNN extension of G' w.r.t. ϕ.

Theorem

With the above notation, if G' is a free group and w is prefix t-positive, then G has decidable prefix membership problem.

Further applications

- Cyclically pinched groups: $\mathrm{Gp}\langle X, Y \mid uv^{-1} = 1\rangle$ where $u \in \overline{X}^*$ and $v \in \overline{Y}^*$
  - These include the orientable surface groups

$$\mathrm{Gp}\langle a_1, \dots, a_n, b_1, \dots, b_n \mid [a_1, b_1] \dots [a_n, b_n] = 1\rangle$$

  - and the non-orientable surface groups

$$\mathrm{Gp}\langle a_1, \dots, a_n \mid a_1^2 \dots a_n^2 = 1\rangle$$

- Conjugacy pinched groups: $\mathrm{Gp}\langle X, t \mid t^{-1}utv^{-1} = 1\rangle$ where $u, v \in \overline{X}^*$ are non-empty and reduced (these include the Baumslag-Solitar groups)
- Some Adyan-type groups: $\mathrm{Gp}\langle X \mid uv^{-1} = 1\rangle$, $u, v \in X^*$ are positive words such that the first letters of u, v are different and also the last letters of u, v are different (some new cases are covered)

A negative result and a problem

Theorem

There exists a finite alphabet X and a reduced word $w \in \overline{X}^*$ such that $G = \mathrm{Gp}\langle X \mid w = 1 \rangle$ has undecidable prefix membership problem.

Open Problem

Characterise the words w such that $G = \mathrm{Gp}\langle X \mid w = 1 \rangle$ has decidable prefix membership problem. Are all cyclically reduced words among them?

Thank you!

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