The word problem for free idempotent-generated semigroups: an overview and elaboration for $\mathcal{T}_n$

Igor Dolinka

Department of Mathematics and Informatics, University of Novi Sad
☞ Serbian Academy of Sciences & Arts
dockie@dmi.uns.ac.rs

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Idempotents in a semigroup

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Induced quasi-orders:

$$e \leq_\ell f \text{ if and only if } ef = e, \qquad e \leq_r f \text{ if and only if } ef = f,$$

$$\leq = \leq_\ell \cap \leq_r \text{ – this is the usual Rees order.}$$

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This is the free idempotent-generated semigroup over $\mathcal{E}$:

$$\mathrm{IG}(\mathcal{E}) = \langle \overline{E} \mid \overline{e}\overline{f} = \overline{e \cdot f} \text{ whenever } \{e, f\} \text{ is a basic pair in } \mathcal{E} \rangle.$$

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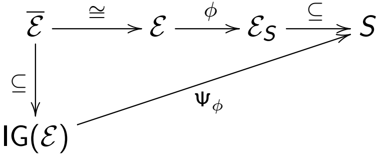

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Assume we have fixed a homomorphism $\Psi : \mathsf{IG}(\mathcal{E}) \to S$ extending the map $\overline{e} \mapsto e$, $e \in E(S)$.

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This third property was (partially) responsible for spawning

Conjecture (Folklore, 80s)

Maximal subgroups of free idempotent-generated semigroups must always be free.

(Spectacular) failure of the freeness conjecture

Brittenham, Margolis, Meakin (2009): A 73-element semigroup S generated by its 37 idempotents (arising from a combinatorial design) such that $\mathsf{IG}(\mathcal{E}_S)$ contains $\mathbb{Z} \times \mathbb{Z}$ as a subgroup

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IgD, Ruškuc (2013): For finitely presented G, (the biorder of) a finite band S will do

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2-cells: singular squares = 4-cycles $e \mathscr{R} e' \mathscr{L} f' \mathscr{R} f \mathscr{L} e$ such that $(\exists h \in E)$ with

- either $eh = e'$, $fh = f'$, $he = e$, $hf = f$ ("left-right"), or
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☞ turns out to be a specific instance of the above for a particular spanning tree of the GH-complex

Computing the maximal subgroups (2)

S	max. subgroups	who & when
$\mathbb{T}_n$	$\mathbb{S}_r$ $r \leq n-2$	Gray, Ruškuc (2012, PLMS)
$\mathbb{PT}_n$	$\mathbb{S}_r$ $r \leq n-2$	IgD (2013, Comm. Alg.)
$\mathcal{M}_n(\mathbb{F})$	$\mathrm{GL}_r(\mathbb{F})$ $r < n/3$	IgD, Gray (2014, TrAMS)
$\mathrm{End}(F_n(G))$	$G \wr \mathbb{S}_r$ $r \leq n-2$	Yang, IgD, Gould (2015, J. Algebra)

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- There is a finite (20-element) band S such that all max. subgroups of IG($\mathcal{E}_S$) are either trivial or products of two free groups (so they have decidable WP), and yet the WP is undecidable (by using the Mikhailova construction).

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So, what is the WP for IG($\mathcal{E}$) really all about?
☞ Yang, IgD, Gould (2019, Adv. Math.)
& IgD (2021, Israel J. Math.)

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- [YDG 19]: There is an algorithm for computing $\mathbf{w} \to (i, g, \lambda)$.

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- $\overline{\mathbf{u}_i} \mathscr{D} \overline{\mathbf{v}_i}$ *for all* $1 \leq i \leq k$, *and, furthermore*
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So, we have an invariant: $\overline{\mathbf{w}} \to \mathscr{D}$-fingerprint $(D_1, \dots, D_k)$ of $\overline{\mathbf{w}}$

The moral of the story

The WP for $\mathrm{IG}(\mathcal{E})$ (for finite $\mathcal{E}$) comes down to comparing elements of the form

$$(i_1, g_1, \lambda_1)(i_2, g_2, \lambda_2) \ldots (i_k, g_k, \lambda_k)$$

of a given $\mathscr{D}$-fingerprint $(D_1, \ldots, D_k)$.

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$$\overline{e} \cdot i = i' \quad \text{if and only if} \quad \overline{e}(i, g, \lambda) = (i', b_{\overline{e},i,i'} g, \lambda)$$

$$\lambda \cdot \overline{e} = \lambda' \quad \text{if and only if} \quad (i, g, \lambda)\overline{e} = (i, g a_{\overline{e},\lambda,\lambda'}, \lambda')$$

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(The coefficients a, b depend solely on the displayed indices, and are easily expressed in terms of the generators of G.)

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Vertex group $W_{(\lambda,i)}$: the subgroup of $G_1 \times G_2$ consisting of the labels of all closed walks based at (λ, i)

It's all about new relations

Assume we have the following data:

- groups $G_1, \ldots, G_m$ $(m \geq 2)$,
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$$\begin{aligned}(a_1^{-1} g b_1, x_2) &\in \rho_1,\\ (a_2^{-1} x_2 b_2, x_3) &\in \rho_2,\\ &\vdots\\ (a_{m-1}^{-1} x_{m-1} b_{m-1}, x_m) &\in \rho_{m-1},\\ a_m^{-1} x_m b_m &= h.\end{aligned}$$

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Clearly, ρ induces a map $\varphi_\rho : \mathcal{P}(G_1) \to \mathcal{P}(G_m)$.

The map θ

Now let

$$\begin{aligned} \mathbf{x} &= (i_1, a_1, \lambda_1) \ldots (i_m, a_m, \lambda_m) \\ \mathbf{y} &= (j_1, b_1, \mu_1) \ldots (j_m, b_m, \mu_m) \end{aligned}$$

be two elements of $\mathrm{IG}(\mathcal{E})$ of $\mathscr{D}$-fingerprint $(D_1, \ldots, D_m)$.

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where $W_{(\lambda_k, i_{k+1})}$ is the vertex group of $\mathcal{A}(D_k, D_{k+1})$ at (λ_k, i_{k+1}), and (g_k, h_k) is the label of any walk $(\lambda_k, i_{k+1}) \rightsquigarrow (\mu_k, j_{k+1})$.

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It can be calculated in terms of standard computational tasks within group theory.

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Theorem (IgD, 2021)

$\mathbf{x} = \mathbf{y}$ *holds in* IG($\mathcal{E}$) *if and only if* $i_1 = j_1$, $\lambda_m = \mu_m$, *and*

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Also, $\mathscr{D} = \mathscr{J}$ + Sch-group of $\mathbf{x} \cong (G_1, \mathbf{x}, \mathbf{x})\theta / (\{1\}, \mathbf{x}, \mathbf{x})\theta$.

$\mathsf{IG}(\mathcal{E}_{\mathcal{T}_n})$ facts

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- a typical element of $\overline{D}_m$ is of the form

$$(P, g, A)$$

P – a partition of $[1, n]$ into m classes; A – a subset of $[1, n]$ of size m; g – an element of the max. subgroup (see above)

Contact graph $\mathcal{A}(\overline{D}_m, \overline{D}_r)$ in $\mathrm{IG}(\mathcal{E}_{\mathcal{T}_n})$

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Lemma

For $(P, g, A) \in \overline{D}_m$ and $(P', g', A') \in \overline{D}_r$ the product $(P, g, A)(P', g', A')$ is regular if and only if either (1) $m \geq r$ and A saturates P', or (2) $m \leq r$ and P' separates A.

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So, such pairs (A, P') are regular (= uninteresting).

Connected components in $\mathcal{A}(\overline{D}_m, \overline{D}_r)$

The type of (A, P) ($|A| = m$, $|P| = r$): the sequence

$$|A \cap P_1|, \ldots, |A \cap P_r|$$

sorted in a non-increasing order.

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Example

$n = 9$, $A = \{1, 3, 5, 7\}$, $P = \{\{1, 2, 6\}, \{3, 5, 7, 9\}, \{4, 8\}\}$.
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Homeomorphism $(\phi, \psi) : (A, P) \sim (B, Q)$ – a pair of bijections $\phi : A \to B$, $\psi : P \to Q$ such that

$$a_i \in P_j \quad \text{if and only if} \quad a_i\phi \in P_j\psi.$$

Connected components in $\mathcal{A}(\overline{D}_m, \overline{D}_r)$ (cont'd)

(A, P) is stationary if all P-classes containing elements from $[1, n] \setminus A$ are singletons.

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(A, P) and (B, Q) are connected in $\mathcal{A}(\overline{D}_m, \overline{D}_r)$ iff they are homeomorphic and not stationary.

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Remark

Stationary pairs are always isolated vertices.

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If $m = n - 1$ or $r = n - 1$ then (A, P) is non-regular in $\mathcal{A}(\overline{D}_m, \overline{D}_r)$ iff it is stationary.

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So, in the rest of the talk assume that $m, r \leq n - 2$.

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Assume there is an edge $(A, P) \longrightarrow (B, Q)$ *in* $\mathcal{A}(\overline{D}_m, \overline{D}_r)$ *induced by* $e \in E$.

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Theorem (IgD, 2022)

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Namely, for the "coset representatives" (g_k, h_k) in the WP it suffices to take any homeomorphism $(A_k, P_{k+1}) \sim (B_k, Q_{k+1})$.

Thank you!

Questions and comments to:

dockie@dmi.uns.ac.rs

Further information may be found at:

http://people.dmi.uns.ac.rs/∼dockie