The prefix membership problem for one-relator groups, and its semigroup-theoretical cousins

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The eNBSAN Online Meeting 2020

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Starring

Robert D. Gray
(Uni of East Anglia, Norwich)

Lt. Col. Frank Slade
(US Army, retired)

Also starring

UEA campus bunnies

(providing the much-required positivity...)

Intro & Some History

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The beginning of the story: Back to the Great Depression

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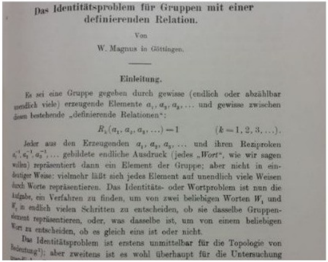

Das Identitätsproblem für Gruppen mit einer definierenden Relation.

Von

W. Magnus in Göttingen.

Einleitung.

Es sei eine Gruppe gegeben durch gewisse (endlich oder abzählbar unendlich viele) erzeugende Elemente $a_1, a_2, a_3, \ldots$ und gewisse zwischen diesen bestehende „definierende Relationen":

$$R_k(a_1, a_2, a_3, \ldots) = 1 \qquad (k = 1, 2, 3, \ldots).$$

Jeder aus den Erzeugenden $a_1, a_2, a_3, \ldots$ und ihren Reziproken $a_1^{-1}, a_2^{-1}, a_3^{-1}, \ldots$ gebildete endliche Ausdruck (jedes „*Wort*", wie wir sagen wollen) repräsentiert dann ein Element der Gruppe; aber nicht in eindeutiger Weise: vielmehr läßt sich jedes Element auf unendlich viele Weisen durch Worte repräsentieren. Das Identitäts- oder Wortproblem ist nun die Aufgabe, ein Verfahren zu finden, um von zwei beliebigen Worten W_1 und W_2 in endlich vielen Schritten zu entscheiden, ob sie dasselbe Gruppenelement repräsentieren, oder, was dasselbe ist, um von einem beliebigen Wort zu entscheiden, ob es gleich eins ist oder nicht.

Das Identitätsproblem ist erstens unmittelbar für die Topologie von Bedeutung[1]); aber zweitens ist es wohl überhaupt für die Untersuchung

Gimme some old time rock'n'roll

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Theorem (Shirshov, 1962)

Every one-relator Lie algebra has decidable word problem.

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NB. RIP S. I. Adyan (1 January 1931 – 5 May 2020).

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Adyan & Oganessyan (1987): The word problem for one-relator monoids can be reduced to the special case of

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where $a, b, c \in X$, $b \neq c$ and $s, t \in X^*$ (and their duals).

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This holds basically because $M = \mathrm{Mon}\langle X \mid asb = atc \rangle$ embeds into $I = \mathrm{Inv}\langle X \mid asbc^{-1}t^{-1}a^{-1} = 1 \rangle$.

The plot thickens

	$\mathrm{Gp}\langle X \mid w = 1\rangle$	$\mathrm{Mon}\langle X \mid w = 1\rangle$	$\mathrm{Inv}\langle X \mid w = 1\rangle$
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Theorem (RD Gray, 2019; Invent. Math., March 2020)

There exists a one-relator inverse monoid $\mathrm{Inv}\langle X \mid w = 1\rangle$ *with undecidable word problem.*

Inverse monoid basics (1): Definitions & FIM

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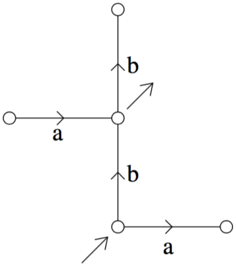

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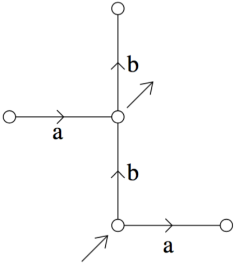

Elements of $FIM(X)$ are represented as Munn trees = birooted finite subtrees of the Cayley graph of $FG(X)$. The Munn tree on the left illustrates the equality

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If w is cyclically reduced, then $M = \mathrm{Inv}\langle X \mid w = 1\rangle$ is E-unitary.

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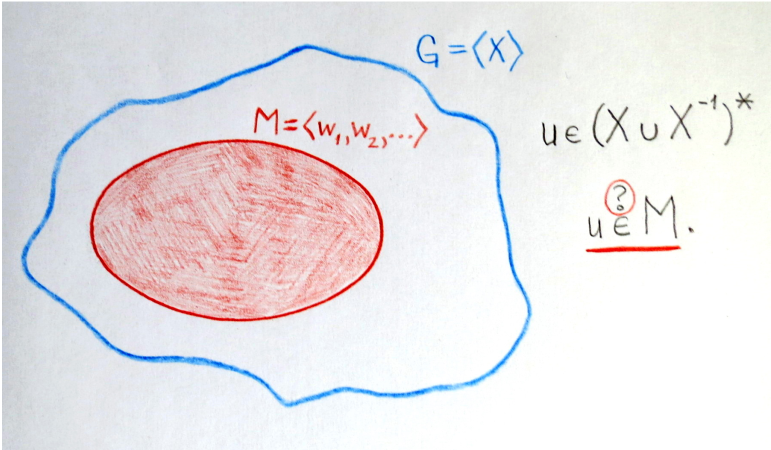
Remark

$G = \mathrm{Gp}\langle X \mid w = 1\rangle$ is the maximum group image of $M = \mathrm{Inv}\langle X \mid w = 1\rangle$.

A Glimpse into the Toolbox

Membership problem (for a submonoid M of a group G)



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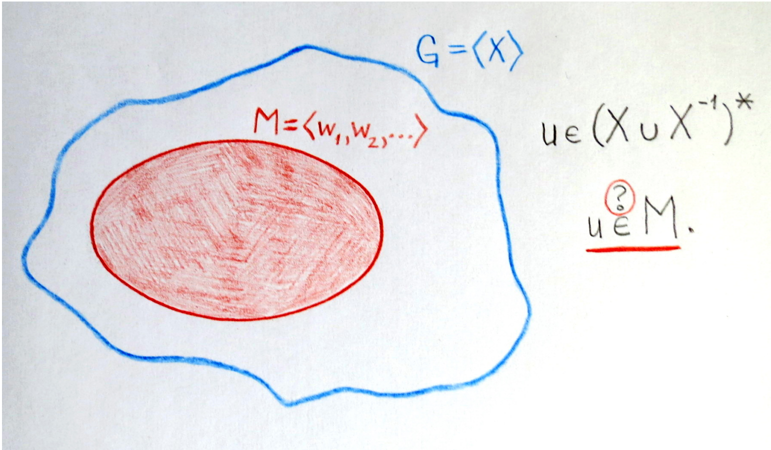
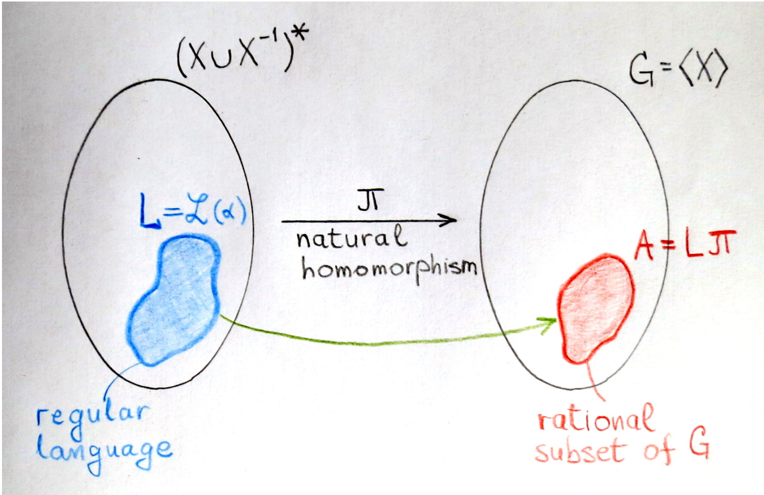


Submonoid membership problem for G: Is there an algorithm which, given $u, w_1, w_2, \dots \in \overline{X}^*$, decides if $u \in \mathrm{Mon}\langle w_1, w_2, \dots \rangle$?

Rational subsets in groups



RSMP + Benois

Rational subset membership problem for a group $G = \langle X \rangle$:

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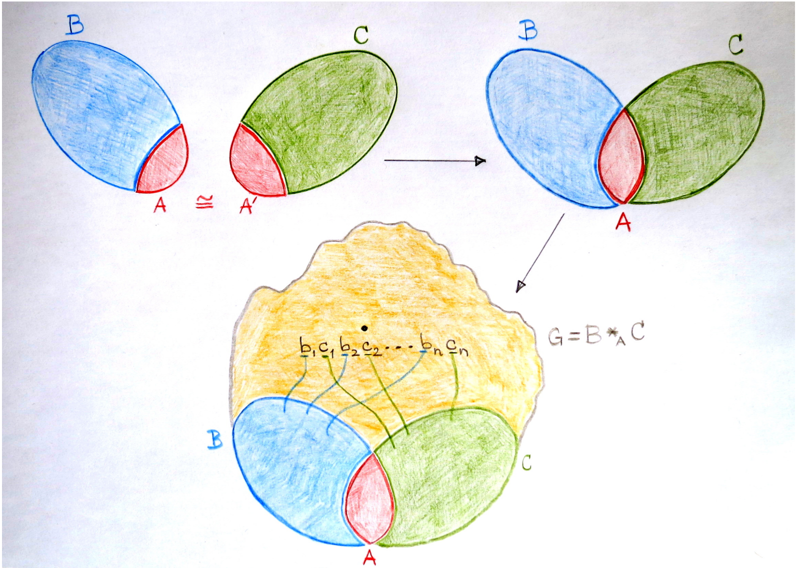
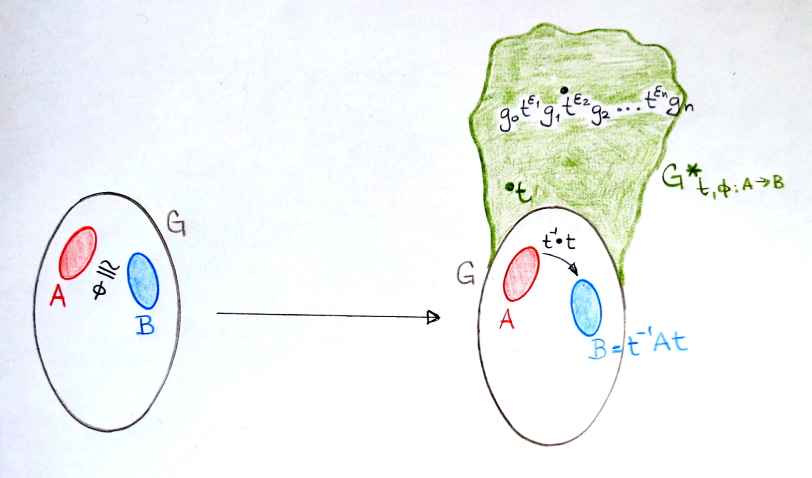
(ii) *If $M = \mathrm{Inv}\langle X \mid w = 1\rangle$ is E-unitary then every conservative factorisation if unital.*

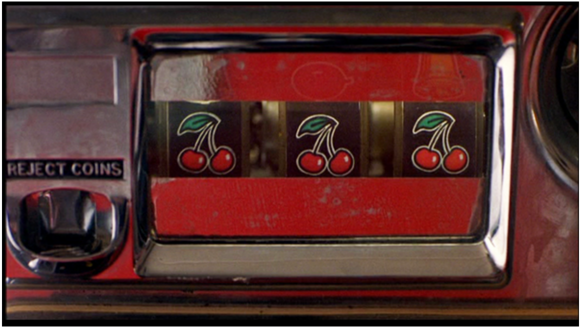
Amalgamated free product of groups $B *_A C$



HNN extension of a group $G*_{t,\phi:A\to B}$



The Results



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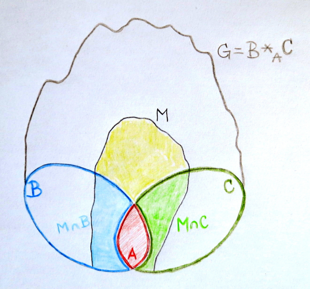

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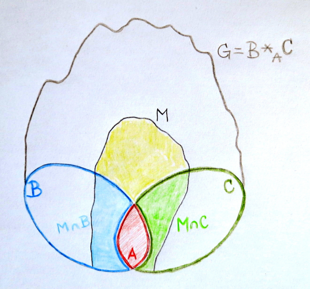

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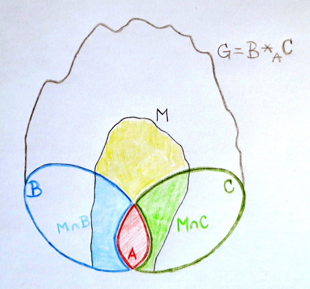

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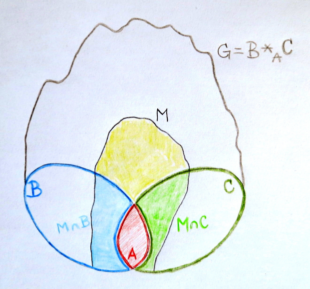

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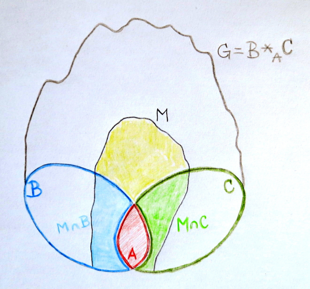

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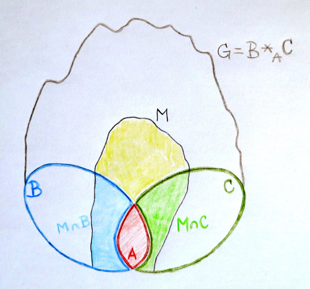

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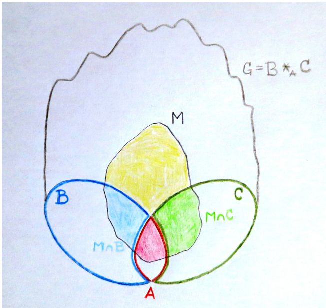

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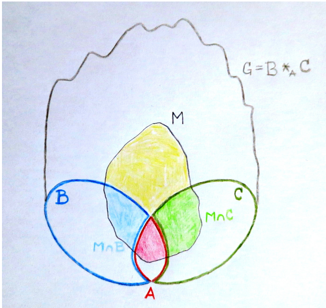

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Chicago O'Hare International Airport (IATA code: ORD)

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While waiting for a connecting flight at ORD sometime in the 1980s, Stuart Margolis and John Meakin came up with the following example, the (in)famous O'Hare (inverse) monoid:

$$\mathrm{Inv}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1 \rangle$$

Application #2: O'Hare-type examples

Proposition

Let $M = \mathrm{Inv}\langle Y, a, d \mid (au_{i_1}d)\dots(au_{i_m}d) = 1\rangle$*, where* a, d *do not appear in* u_{i_j}*'s.*

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Application #4: Cyclically pinched presentations

Theorem

The prefix membership problem is decidable for one-relator groups defined by cyclically pinched presentations:

$$G = \mathrm{Gp}\langle X \cup Y \mid uv^{-1} = 1 \rangle$$

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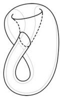

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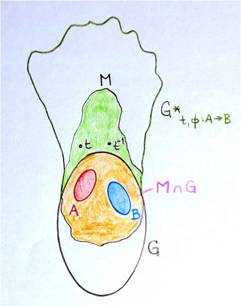

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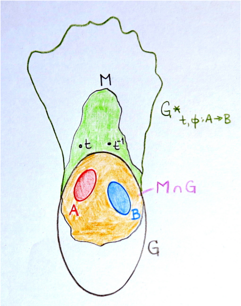

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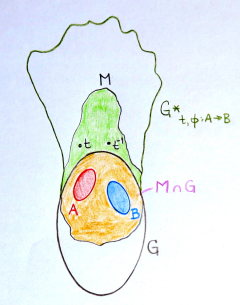

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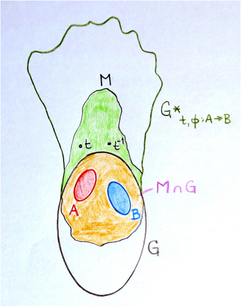

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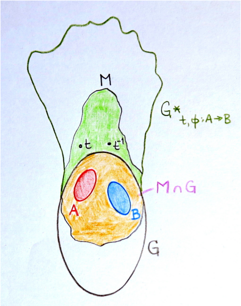

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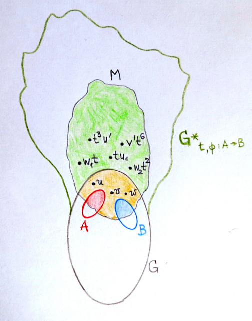

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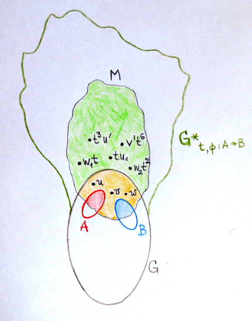

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The grand finale & an open problem

By modifying slightly the ideas from Bob's *Inventiones* paper, we obtain

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In particular, what about cyclically reduced words?

Thank you!

Questions and comments to:

dockie@dmi.uns.ac.rs

Further information may be found at:

http://people.dmi.uns.ac.rs/∼dockie