Szegedi Tudományegyetem
Informatikai Tanszékcsoport
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Identities of Two-Dimensional Languages

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$$\text{WORD} = \text{a finite sequence of letters}$$

Version B. (Algebraic)

$$\text{WORD} = \text{an element of a free monoid}$$

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A **two-dimensional word** is a matrix of letters – a **picture**:

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where $a_{ij} \in \Sigma$ for some alphabet Σ.

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A **picture language** is a set of pictures.

Operations on pictures and picture languages

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The **row product** $P \downarrow Q$ is defined only if $n = \ell$, and its result is

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$$L^{>} = \bigcup_{n \geqslant 0} L^{\overset{n}{\rightarrow}}, \qquad L^{\vee} = \bigcup_{n \geqslant 0} L^{\downarrow n},$$

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A **binoid language** (or **bi-language**) is a subset of a free binoid.

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$x \in \Sigma$ is identified with the singleton poset S_x, labelled by x.

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New biposets are obtained by two binary operations $\circ_1, \circ_2$, where $\mathcal{A} \circ_i \mathcal{B}$ $(i = 1, 2)$ is defined on $A \cup B$ by

$$<_j^{\mathcal{A}\circ_i\mathcal{B}} = \begin{cases} <_j^{\mathcal{A}} \cup <_j^{\mathcal{B}} & \text{if } j \neq i, \\ <_j^{\mathcal{A}} \cup <_j^{\mathcal{B}} \cup (A \times B) & \text{if } j = i. \end{cases}$$

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A biposet is series-parallel (**sp** for short) if it is generated from the singletons by the two product operations.

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The set of all bi-words over Σ: BW_{Σ}

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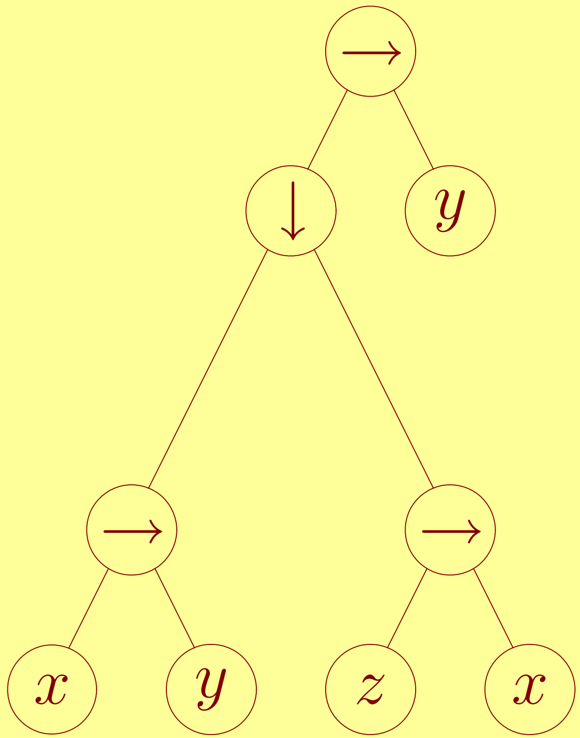
Example. $b(x, y, z) = ((x \to y) \downarrow (z \to x)) \to y$



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As an example, we show how the horizontal product works. We have three cases.

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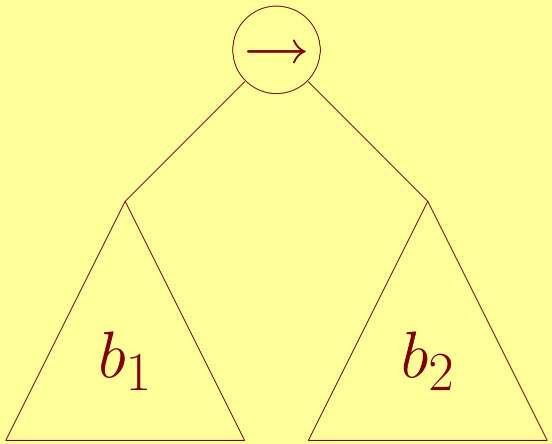
Case 1: b_1, b_2 are vertical/neutral



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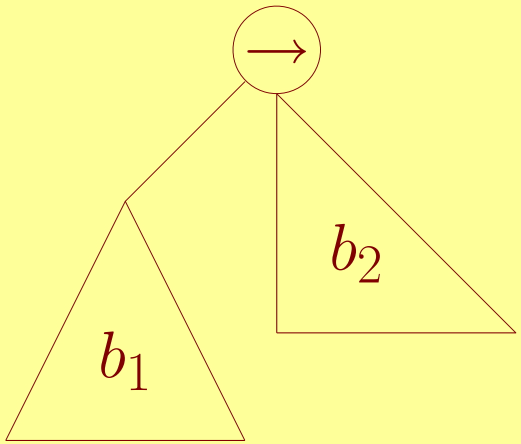
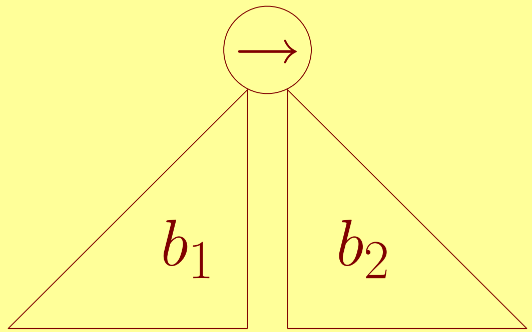
Case 2: b_1 is vertical/neutral, b_2 is horizontal



How to represent elements of a free binoid? Take 2.

Case 3: b_1, b_2 are horizontal



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Algebras

Algebra of bi-languages over Σ:

$$\mathbf{BiLang}_\Sigma = (\mathcal{P}(\mathsf{BW}_\Sigma), +, \rightarrow, \downarrow, {}^{>}, {}^{\vee}, \varnothing, \{\epsilon\})$$

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A word of caution: Recognizable picture languages (REC) require, besides the above operations, the intersection and the so-called alphabetic projection.

A result ($\sim$, 2005)

Theorem. Identities satisfied by all algebras $\mathbf{BiLang}_\Sigma$ = identities satisfied by all algebras $\mathbf{Pict}_\Sigma$.

I.Dolinka, A note on identities of two-dimensional languages, *Discrete Applied Mathematics* **146** (2005), 43–50.

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In the sequel, we denote the above equational theory by Θ.

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such that for any bi-word $b' = b'(x_1, \ldots, x_n)$ we have

$$P_b \in b'(L_1, \ldots, L_n) \iff b' \equiv b.$$

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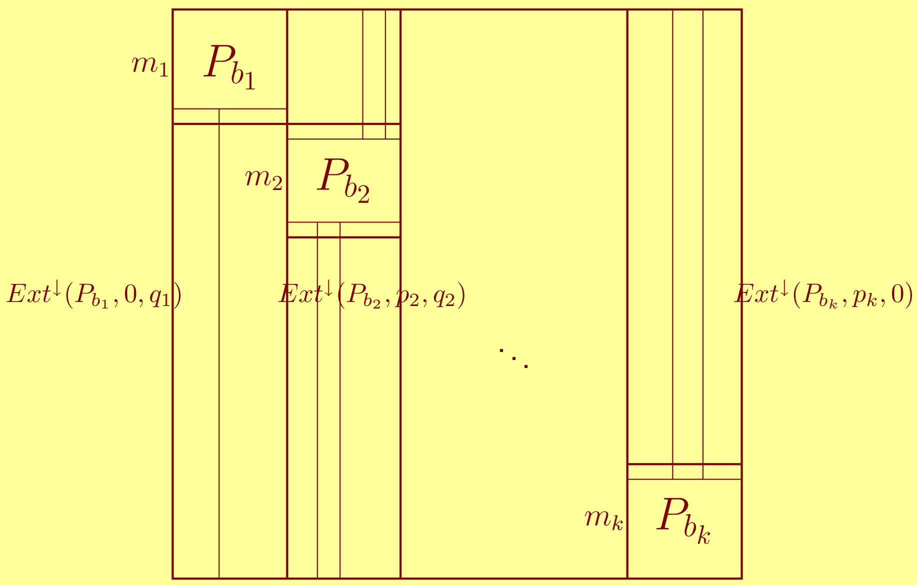

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The witness picture is:

$$P_b = \begin{bmatrix} 1 & 2 & 2 & 2 & 5 \\ 1 & 2 & 2 & 2 & 5 \\ 3 & 3 & 3 & 4 & 5 \\ 3 & 3 & 3 & 4 & 5 \\ 3 & 3 & 3 & 4 & 5 \end{bmatrix}.$$

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Conjecture. The identities of ordinary string languages in the 'horizontal' signature $\{+, \rightarrow, ^{>}, \varnothing, \epsilon\}$ & the same identities in the 'vertical' signature $\{+, \downarrow, ^{\vee}, \varnothing, \epsilon\}$ will do.

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Recently, I succeeded in proving that this conjecture is true.

A short summary of the proof follows.

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Birational bi-language $=$ bi-language of the form $\mathcal{B}(\alpha)$

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Γ_1 (Γ_2) $=$ all identities of string languages in the horizontal (vertical) signature.

Decomposition Lemma

For any birational expression α, there are birational expressions α^h and α^v such that

- $\alpha = \alpha^h + \alpha^v$ follows from $\Gamma_1 \cup \Gamma_2$,
- $\mathcal{B}(\alpha_h)$ ($\mathcal{B}(\alpha_v)$) consists precisely of all horizontal (vertical) and neutral bi-words from $\mathcal{B}(\alpha)$.

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Lemma. Let α_1, α_2 be birational expressions, and let α_i^h, α_i^v ($i = 1, 2$) have the same meaning as above. Then $\alpha_1 = \alpha_2$ belongs to Θ if and only if both $\alpha_1^h = \alpha_2^h$ and $\alpha_1^v = \alpha_2^v$ belong to Θ.

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Lemma. For each α there is a trimmed expression α_0 such that $\Gamma_1 \cup \Gamma_2 \vdash \alpha = \alpha_0$.

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(i) There exist a linear (= each variable occurs exactly once) $\rightarrow$-rational expression $\alpha'(x_1, \ldots, x_n)$ and vertical expressions $\beta_1, \ldots, \beta_n$ such that

$$\alpha \equiv \alpha'(\beta_1, \ldots, \beta_n).$$

In such a case, if $\delta(\alpha) \geqslant 1$, we have $\delta(\alpha) = \max(\delta(\beta_1), \ldots, \delta(\beta_n)) + 1$.

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(ii) There exist a horizontal birational expression $\hat{\alpha}$, a linear $\longrightarrow$-rational expression $\alpha''(x_1, \ldots, x_k)$ and vertical expressions $\beta'_1, \ldots, \beta'_k$ such that

(a) the identity $\alpha = \hat{\alpha}$ follows from $\Gamma_1 \cup \Gamma_2$,

(b) $\hat{\alpha} \equiv \alpha''(\beta'_1, \ldots, \beta'_k)$, and

(c) $\epsilon \notin \mathcal{B}(\beta'_i)$ and $\mathcal{B}(\beta'_i) \neq \varnothing$ for all $1 \leqslant i \leqslant k$.

Definition: Doppelgänger (as in "Twin Peaks")

Let α_1, α_2 be two horizontal birational expressions (of depth $d \geqslant 1$). **Linearization Lemma** $\Rightarrow$

$$\begin{aligned}\alpha_1 &= \alpha_1''(\beta_1, \ldots, \beta_n),\\ \alpha_2 &= \alpha_2''(\beta_{n+1}, \ldots, \beta_m),\end{aligned}$$

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All there languages are (nonempty) subsets of E – the set of all neutral and vertical bi-words of depth $\leqslant d - 1$.

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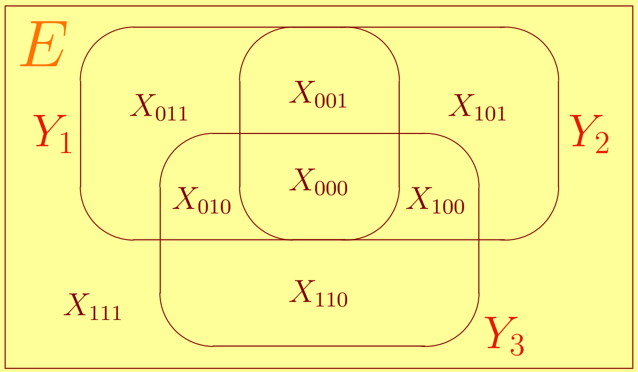
For convenience, let $Y_i^0 = Y_i$ and $Y_i^1 = E \setminus Y_i$.

For a binary sequence $\sigma \in \{0,1\}^m$, let

$$X_\sigma = \bigcap_{i=1}^{m} Y_i^{\sigma(i)}.$$

What the heck is this?



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For $1 \leqslant i \leqslant m$, define the sets $\Lambda_i \subseteq \{0,1\}^m$ by

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The 'horizontal' identity

$$\alpha_1''\left(\sum_{\sigma\in\Lambda_1} x_\sigma, \ldots, \sum_{\sigma\in\Lambda_n} x_\sigma\right) = \alpha_2''\left(\sum_{\sigma\in\Lambda_{n+1}} x_\sigma, \ldots, \sum_{\sigma\in\Lambda_m} x_\sigma\right)$$

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The idea behind this identity is that the above sums of letters (from $\Xi_m = \{x_\sigma : \ \sigma \in \{0,1\}^m\}$) indexed by Λ_i's record the **set-theoretical configuration** of the bi-languages Y_i.

Example

$$x^{>} + (x^{\vee})^{>} = (x^{\vee})^{>}$$

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To get rid of ϵ from $\mathcal{B}(\beta_2) = \mathcal{B}(\beta_3)$, we make use of

$$x^{\vee} = \epsilon + x \downarrow x^{\vee}$$

and proceed with $x \downarrow x^{\vee}$ instead of $x^{\vee}$.

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For simplicity, write x for x_{000} and y for x_{100}. So, our doppelgänger is just

$$x^{>} + (x + y)^{>} = (x + y)^{>},$$

a familiar law telling that the Kleene star is monotone.

Doppelgänger Lemma

Assume $\alpha_1 = \alpha_2$ belongs to Θ (i.e. it is a valid bi-langauge identity). Then its doppelgänger is a valid string identity.

The main proof (outlined)

Goal: to prove that a valid identity $\alpha_1 = \alpha_2$ is a consequence of $\Gamma_1 \cup \Gamma_2$.

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So, we may assume that both α_1 and α_2 are e.g. horizontal.

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Linearization Lemma $\Rightarrow$ there are horizontal birational expressions $\hat{\alpha}_1, \hat{\alpha}_2$ such that

$$\Gamma_1 \cup \Gamma_2 \vdash \quad \alpha_1 = \hat{\alpha}_1, \quad \alpha_2 = \hat{\alpha}_2,$$

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while the identity $\hat{\alpha}_1 = \hat{\alpha}_2$ has the form

$$\alpha_1''(\beta_1', \ldots, \beta_k') = \alpha_2''(\beta_{k+1}', \ldots, \beta_m'),$$

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$$\alpha_1''(\beta_1', \ldots, \beta_k') = \alpha_2''(\beta_{k+1}', \ldots, \beta_m'),$$

where α_1'', α_2'' are linear $\longrightarrow$-rational expressions (involved later in the course of forming a doppelgänger identity), and $\beta_1', \ldots, \beta_m'$ are vertical expressions, all of them having depth at most $d-1$, whose values $Y_1, \ldots, Y_m$ satisfy $\epsilon \notin Y_i \neq \varnothing$, $1 \leqslant i \leqslant m$.

The main proof (outlined)

Let $\Lambda_1, \ldots, \Lambda_m$ and X_σ, $\sigma \in I$, be as in the definition of a doppelgänger. We already know that

$$Y_i = \bigcup_{\sigma \in \Lambda_i} X_\sigma$$

holds for all $1 \leqslant i \leqslant m$.

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Ésik–Németh (2004) $\Rightarrow$ birational bi-languages closed for intersections and set differences, so all X_σ's are birational,

$$X_\sigma = \mathcal{B}(\xi_\sigma).$$

The main proof (outlined)

Therefore, the following identities are valid:

$$\beta_i' = \sum_{\sigma \in \Lambda_i} \xi_\sigma, \quad (*)$$

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This is an identity of depth $\leqslant d - 1$, so it follows from $\Gamma_1 \cup \Gamma_2$ by induction hypothesis.

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Doppelgänger Lemma $\Rightarrow$ the adjoined string identity

$$\alpha_1''\left(\sum_{\sigma\in\Lambda_1} x_\sigma, \ldots, \sum_{\sigma\in\Lambda_n} x_\sigma\right) = \alpha_2''\left(\sum_{\sigma\in\Lambda_{n+1}} x_\sigma, \ldots, \sum_{\sigma\in\Lambda_m} x_\sigma\right)$$

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By combining $(*)$ and the above doppelgänger, we obtain the required formal proof for $\alpha_1 = \alpha_2$.

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So, the nonempty X_σ's are $X_{000} = \{x\}$ and $X_{100} = \{x \downarrow x, x \downarrow x \downarrow x, \ldots\}$.

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Thus, we have $\xi_{000} \equiv x$ and $\xi_{100} = x \downarrow x \downarrow x^{\vee}$.

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Thus, we have $\xi_{000} \equiv x$ and $\xi_{100} = x \downarrow x \downarrow x^{\vee}$.

Now, our identity follows from the above doppelgänger and

$$x + x \downarrow x \downarrow x^{\vee} = x \downarrow x^{\vee}.$$

THANK YOU!

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