H-DISTRIBUTIONS WITH UNBOUNDED MULTIPLIERS

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Abstract. We investigate H-distributions for sequences in the dual pairs of Bessel spaces, $(H^q_s, H^p_{-s}), s \in \mathbb{R}, q > 1$ and $q = p/(p-1)$, by the use of unbounded multipliers, with the finite regularity, as test functions. The results relating weak convergence, H-distributions and strong convergence are applied in the analysis of strong convergence for a sequence of approximated solutions to a class of differential equations $P(x,D)u_n = f_n$, where $P(x,D)$ is a differential operator of order k with coefficients in the Schwartz class and (f_n) is a strongly convergent sequence in an appropriate Bessel potential space. H-distributions, weak and strong convergence, Bessel potential spaces, multipliers
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1. Introduction, notation and definitions

The aim of this paper is to analyze H-distributions obtained by the use of a classes of unbounded multipliers which provide smaller test function spaces for testing the strong convergence of a weakly convergent sequence in Bessel potential spaces. General results are used for the proof of the strong convergence of a sequence of solutions extracted from a bounded set of solutions for a class of partial differential equations of order k, with coefficients in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. In our analysis we have considered finite order distribution spaces which enables us to use spaces of multipliers with the finite type regularities of symbols. Especially, classes of multipliers denoted by $s^m_{q,N}$ are obtained, similarly as in the case of Sobolev space, as completions of the respective normed spaces. Because of that in our investigation there were many estimates to be analyzed in order to determine the finite regularity of symbols.

H-measures and distributions are microlocal tools that can be used to investigate the strong convergence of weakly convergent sequences in the Lebesgue and Bessel potential spaces. H-measures, also known as microlocal defect measures, are associated to weakly convergent sequences in $L^2(\mathbb{R}^d)$. They are introduced independently by Tartar, [14], and Gerard, [7], as functionals on product of continuous functions compactly supported in $\mathbb{R}^d$ and continuous functions on the unit sphere $\mathbb{S}^{d-1}$, i.e. on $C_c(\mathbb{R}^d) \otimes C(\mathbb{S}^{d-1})$. More precisely, for a weakly convergent sequence $u_n \rightharpoonup 0$ in $L^2_{\rm loc}(\mathbb{R}^d)$ there exists a Radon measure μ such that for every $\varphi_1, \varphi_2 \in C_c(\mathbb{R}^d)$ and every $\psi \in C(\mathbb{S}^{d-1})$, up to a subsequence (which means that there exists a subsequence of (u_n) denoted again by the same symbol),

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \mathcal{F}(\varphi_1 u_n)\overline{\mathcal{F}(\varphi_2 u_n)}\psi\Big(\frac{\xi}{|\xi|}\Big)d\xi = \langle \mu, \varphi_1\bar{\varphi}_2 \otimes \psi\rangle,$$

where $\mathcal{F}$ denotes Fourier transform. The well-known example of oscillating sequence $u_k(x) = e^{ikx\xi_0}$ (cf.[7]) with the associated microlocal defect measure $\mu(x,\xi) =$

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$\nu(x)\otimes\delta(\xi-\xi_0)$, where ν is the Lebesgue measure on $\mathbb{R}^d$ and δ is the Dirac measure on $\mathbb{S}^{d-1}$, implies that the support of H-measure provides an information about the set of points $x\in\mathbb{R}^d$ where the strong convergence is lost and its dependence on the directions, frequencies, of oscillations (ξ_0, in this example). The original concept and its generalization for sequences of uncountable dimensions, given in [11], are mainly applied to hyperbolic PDEs, see e.g. [3], where $L^1_{\rm loc}(\mathbb{R}^d)$-precompactness of solutions to diffusion-dispersion approximation for a scalar conservation law was obtained. Many improvements have been done in adopting applications of H-measures to parabolic (e.g. [5]) and ultraparabolic (e.g. [12]) problems.

H-distributions from [6], resp. [4], generalize the concept of H-measures from $L^2(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, $p\neq 2$ and Bessel potential spaces, respectively. Hörmander-Mikhlin's theorem has to be fulfilled and test functions in ξ have to be more regular, namely $\psi\in C^\kappa(\mathbb{S}^{d-1})$, $\kappa=[d/2]+1$. For the later use, we recall:

Theorem 1 (Hörmander-Mikhlin theorem, cf. e.g. [10], Theorem 8.2)**.** *Let $1<p<\infty$, $\kappa>d/2$ and $\psi\in C^\kappa(\mathbb{R}^d)$. If $\psi:\mathbb{R}^d\backslash\{0\}\to\mathbb{C}$ satisfies $|\partial^\alpha\psi(\xi)|\leq B|\xi|^{-|\alpha|}$, for all $|\alpha|\leq\kappa$ and $\xi\neq 0$, then there is a constant $C=C(d,p)$ such that*

$$\|\mathcal{A}_\psi(f)\|_{L^p}\leq CB\|f\|_{L^p},\ \textit{for all}\ f\in\mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{F}[\mathcal{A}_\psi(f)]=\psi\mathcal{F}f$.

In order to associate an H-distribution to a pair of sequences in the dual Bessel potential spaces, the authors proposed the use of (non-local) test functions $\varphi\in\mathcal{S}(\mathbb{R}^d)$. In this way, H-distribution becomes a functional on $\mathcal{S}(\mathbb{R}^d)\otimes C^\kappa(\mathbb{S}^{d-1})$ belonging to the space denoted by $\mathcal{SE}'(\mathbb{R}^d\times\mathbb{S}^{d-1})$, whose topology is well described in [4]. The existence theorem for H-distribution associated to a sequence in a Bessel potential space reads as follows [4]: *If $u_n\rightharpoonup 0$ in $W^{-k,p}(\mathbb{R}^d)$ and $v_n\rightharpoonup 0$ in $W^{k,q}(\mathbb{R}^d)$, then there exist subsequences $(u_{n'}),(v_{n'})$ and a H-distribution $\mu\in\mathcal{SE}'(\mathbb{R}^d\times\mathbb{S}^{d-1})$ such that for every $\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R}^d)$ and every $\psi\in C^\kappa(\mathbb{S}^{d-1})$,*

$$\lim_{n'\to\infty}\langle\mathcal{A}_\psi(\varphi_1u_{n'})\,,\,\varphi_2v_{n'}\rangle=\lim_{n'\to\infty}\langle\varphi_1u_{n'}\,,\,\mathcal{A}_{\overline{\psi}}(\varphi_2v_{n'})\rangle=\langle\mu,\varphi_1\bar{\varphi}_2\otimes\psi\rangle.$$

Moreover, in [4] the strong convergence is tested on all $v_n\rightharpoonup 0$ in $W^{k,q}(\mathbb{R}^d)$.

In this paper, in testing of strong convergence, we are confronting smaller test spaces and larger spaces of multipliers, that is, ψ is not bounded and can not be the Fourier multiplier in the sense of Hörmander-Mikhlin theorem. Therefore, we use a class of pseudo-differential operators to ensure the boundedness of operator $\mathcal{A}_\psi:W^{k+m,q}(\mathbb{R}^d)\to W^{k,q}(\mathbb{R}^d)$, minimizing the assumptions on ψ. Moreover, in application to a class of differential equations the strong convergence of a sequence u_n of solutions converging weakly to zero in $W^{-k,p}(\mathbb{R}^d)$ is tested over all $v_n\rightharpoonup 0$ in $W^{k+m,q}(\mathbb{R}^d)\subset W^{k,q}(\mathbb{R}^d)$, if $m>0$.

The paper is organized as follows. In Section 2 we present results related to a class of pseudo-differential operators considering them as bilinear mappings and we prove the continuity with respect to a certain class of symbols with the finite regularity conditions. The analysis of such symbols is based on the proof of Theorem 10.7 in [16]. We give in the Appendix A a complete proof of Theorem 2, since we follow proof of quoted theorem in [16], with the explanations needed for the proof of continuous bilinearity and the finiteness of regularity assumptions. Section 3 is devoted to a class of symbols depending only on ξ; the composition of a corresponding multiplier and an operator of multiplication is given. In the first part of Section 4 we give a theorem for a commutator important in the construction of H-distributions and the analysis of their properties. Also, in Section 4 we consider a class of symbols with respect to the $L^q(\mathbb{R}^d)-$norm, $q \in [1,2]$, as well as the existence of the corresponding H-distributions. In Section 5 we apply the results to a weakly convergent sequence of solutions of a class of partial differential equation of order $k \in \mathbb{N}$ with coefficients in $\mathcal{S}(\mathbb{R}^d)$.

2. Notation and bilinear continuity

The Bessel potential space $H^p_s(\mathbb{R}^d)$, $1 \le p < \infty$, $s \in \mathbb{R}$, is defined as a space of all tempered distributions u such that $\mathcal{A}_{\langle\xi\rangle^s} u := \mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}u)$ is in $L^p(\mathbb{R}^d)$. It is a Banach space with the norm $\|u\|_{H^p_s} = \|\mathcal{A}_{\langle\xi\rangle^s} u\|_{L^p}$. Moreover, $(H^p_s(\mathbb{R}^d))' = H^q_{-s}(\mathbb{R}^d)$, cf. [2]. Also, recall that for $m \in \mathbb{N}_0$ and $1 \le p < \infty$, the Bessel potential spaces coincide with the Sobolev spaces, i.e. $H^p_m(\mathbb{R}^d) = W^{m,p}(\mathbb{R}^d)$, cf. [1].

We use standard notation $|\alpha| = \alpha_1 + \cdots + \alpha_d$ for multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d_0$, $\langle\xi\rangle^s = (1+|\xi|^2)^{s/2}, s \in \mathbb{R}$ and $đ\xi = (2\pi)^{-d}d\xi, \xi \in \mathbb{R}^d$. The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ is defined as $\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-ix\xi} f(x)dx$. For $x \in \mathbb{R}^d$ with $\langle D_x\rangle = \sqrt{1-\Delta}$ is denoted the pseudo-differential operator with symbol $\langle\xi\rangle$, hence $\langle D_x\rangle f = \int e^{ix\xi}\langle\xi\rangle \hat{f}(\xi)đ\xi$. Actually, in the sequel, we will use even powers of $\langle D_x\rangle$ and the corresponding Leibniz rule in the partial integration.

We define the space of symbols of pseudo-differential operators that we shall use. Let $m \in \mathbb{R}, N \in \mathbb{N}_0$ and $\sigma \in C^N(\mathbb{R}^d \times \mathbb{R}^d)$. Then the symbol σ is an element of S^m_N if for every $\alpha, \beta \in \mathbb{N}^d_0$ such that $|\alpha| \le N, |\beta| \le N$ the norm given by

$$|\sigma|_{S^m_N} := \max_{|\alpha|,|\beta| \le N} \sup_{x,\xi \in \mathbb{R}^d} |\partial^\alpha_\xi \partial^\beta_x \sigma(x,\xi)| \langle\xi\rangle^{-m+|\alpha|}$$

is finite. The space $(S^m_N, |\cdot|_{S^m_N})$ is a Banach space. Define $S^m = \text{proj} \lim_{N\to\infty} S^m_N$. Then S^m is a Fréchet space. Function $\sigma \in S^m \subset C^\infty$ is a symbol of a pseudo-differential operator of order m defined by

$$T_\sigma(u)(x) := \int_{\mathbb{R}^d} e^{ix\xi}\sigma(x,\xi)\hat{u}(\xi)đ\xi, \ u \in \mathcal{S}(\mathbb{R}^d) \tag{1}$$

and it is defined as an oscillatory integral for $u \in \mathcal{S}'(\mathbb{R}^d)$. Pseudo-differential operators of order m can also be defined by (1) for symbols with finite regularity $\sigma \in S^m_N \subset C^N$. This is the subject of Theorem 2.

If additionally there exists bounded function $c_0(x) \to 0$, as $|x| \to \infty$, such that

$$|\sigma(x,\xi)| \le c_0(x)\langle\xi\rangle^m, \tag{2}$$

then the symbol $\sigma \in S^m_N$ belongs also to the class of symbols denoted by $S^m_{0,N}$ (cf., [9] or [17]). It can be shown (cf. [9]) that if (2) holds for $\sigma \in S^m_N$, then a similar estimate can be achieved for all derivatives, i.e. for every $\alpha, \beta \in \mathbb{N}^d_0$ such that $|\alpha| \le N, |\beta| \le N$ there exists a bounded function $c_{\alpha\beta}(x) = o(1)$ as $|x| \to \infty$, and

$$|\partial^\alpha_\xi \partial^\beta_x \sigma(x,\xi)| \le c_{\alpha\beta}(x)\langle\xi\rangle^{m-|\alpha|}.$$

The operator T_σ, defined in (1), with smooth symbol $\sigma \in S^m$, is linear and continuous from $H^q_s(\mathbb{R}^d)$ to $H^q_{s-m}(\mathbb{R}^d)$, $s \in \mathbb{R}, 1 < q < \infty$, cf. Theorem 11.9. in [16]. This result is a generalization of Theorem 10.7. in [16], where $m = s = 0$.

We extend this assertion twofold, considering $T_\sigma u$ as a bilinear mapping and taking $\sigma \in S^0_N$, for any integer $N > 2d$.

Theorem 2. *Let N be integer such that $N > 2d$, $1 < p < \infty$ and let the operator T be defined by*

$$T(\sigma, u) = T_\sigma u, \ \sigma \in S^0_N, \ u \in L^p(\mathbb{R}^d).$$

Then T is a continuous bilinear operator from $S^0_N \times L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ and there exists $C_N > 0$ such that

$$\|T_\sigma u\|_{L^p} \leq C_N |\sigma|_{S^0_N} \|u\|_{L^p}.$$

The proof can be found in the Appendix A.

Corollary 3. *Let $1 < p < \infty$, $s, m \in \mathbb{R}$, $N > 2d$ and $\sigma \in S^m_N$. Then there exists $C_N > 0$ such that the following estimate holds*

$$\|T(\sigma, u)\|_{H^p_s} = \|T_\sigma u\|_{H^p_s} \leq C_N |\sigma|_{S^m_N} \|u\|_{H^p_{m+s}}, \ u \in H^p_{m+s}(\mathbb{R}^d). \tag{3}$$

Proof: From the definition of norm on $H^p_s(\mathbb{R}^d)$ spaces it follows that

$$\begin{aligned} \|T_\sigma u\|_{H^p_s} &= \|\mathcal{F}^{-1}(\langle\xi\rangle^s \mathcal{F}(T_{\langle\xi\rangle^{-s}}(T_{\langle\xi\rangle^{-m}\sigma(x,\xi)}(T_{\langle\xi\rangle^{m+s}} u)))\|_{L^p} \\ &= \|T_{\sigma_1(x,\xi)}(T_{\langle\xi\rangle^{m+s}} u)\|_{L^p}, \end{aligned} \tag{4}$$

where $\sigma_1(x,\xi) = \langle\xi\rangle^{-m}\sigma(x,\xi) \in S^0$. Applying Theorem 2 to (4) we obtain that for any integer $N > 2d$ there exists $c_N > 0$ such that

$$\|T_\sigma u\|_{H^p_s} \leq c_N |\sigma_1|_{S^0_N} \|T_{\langle\xi\rangle^{m+s}} u\|_{L^p} = c_N |\sigma_1|_{S^0_N} \|u\|_{H^p_{m+s}}.$$

Since $|\sigma_1|_{S^0_N} \leq |\langle\xi\rangle|_{S^{-m}_N} |\sigma|_{S^m_N} \leq c|\sigma|_{S^m_N}$, we obtain estimate (3). ❑

3. Multipliers

3.1. **Spaces of symbols with finite type regularities.** In the sequel we shall consider a class of symbols depending only on ξ with finite smoothness, defined as follows.

Let $m \in \mathbb{R}$, $q \in [1, \infty]$, $N \in \mathbb{N}_0$. Then, consider the space of all $\psi \in C^N(\mathbb{R}^d)$ for which the norm

$$|\psi|_{s^m_{q,N}} := \max_{|\alpha| \leq N} \|\partial^\alpha_\xi \psi(\xi)\langle\xi\rangle^{-m+|\alpha|}\|_{L^q} \tag{5}$$

is finite. The completion of this space, with respect to this norm, is denoted by $(s^m_{q,N}, |\cdot|_{s^m_{q,N}})$. In the case when $q = \infty$ we already have Banach space of symbols, i.e. the introduced space is the same as its completion. We consider operator T_ψ with symbol $\psi \in s^m_{q,N}$, defined as in (1). Since ψ depends only on ξ the operator T_ψ is called multiplier operator. Note that

$$|\partial^\alpha \psi(\xi)| \leq |\psi|_{s^0_{\infty,N}} |\xi|^{-|\alpha|}, |\alpha| \leq N, |\xi| > |\xi_0| > 0,$$

if $\psi \in s^0_{\infty,N}$ and $N > d/2$. Therefore, by Theorem 1, we have the following result.

Corollary 4. *Let $N > d/2$, $1 < p < \infty$. Then, $\mathcal{A}_\psi$ is a continuous bilinear operator on $s^0_{\infty,N} \times L^p(\mathbb{R}^d)$, and*

$$\|\mathcal{A}(\psi, u)\|_{L^p} = \|\mathcal{A}_\psi(u)\|_{L^p} \leq C|\psi|_{s^0_{\infty,N}} \|u\|_{L^p}. \tag{6}$$

With $(s^m_{\infty,N})_0 \subset s^m_{\infty,N}$ we denote the class of multipliers such that $\psi \in (s^m_{\infty,N})_0$ means that

$$\sup_{|\xi|\to\infty} \frac{|\partial^\alpha_\xi \psi(\xi)|}{\langle\xi\rangle^{m-|\alpha|}} = 0, \quad \text{for all } \ |\alpha| \le N. \tag{7}$$

Separability of symbol classes is important in the construction of H-distributions. The following results hold.

Theorem 5. *a) The space $((s^m_{\infty,N+1})_0, |\cdot|_{s^m_{\infty,N}})$ is separable.*

b) Let $1 \le q < \infty$. Then the space $(s^m_{q,N+1}, |\cdot|_{s^m_{q,N}})$ is separable.

Proof: a) We will prove that $\mathcal{S}(\mathbb{R}^d)$ is dense in $((s^m_{\infty,N+1})_0, |\cdot|_{s^m_{\infty,N}})$. Since $\mathcal{S}(\mathbb{R}^d)$ is separable, this implies separability of $((s^m_{\infty,N+1})_0, |\cdot|_{s^m_{\infty,N}})$. Let $\psi \in (s^m_{\infty,N+1})_0$. Then by the standard arguments, one can prove that $\psi_n(\xi) = (\psi * \phi_n)\chi(\xi/n)$ converges to ψ in the norm $|\cdot|_{s^m_{\infty,N}}$, where $\chi \in C^\infty_c$, $\chi(\xi) = 1$ for $|\xi| \le 1$ and $\chi(\xi) = 0$ for $|\xi| \ge 2$ and ϕ_n is standard sequence of mollifiers. In the proof the key point is that for suitable constants,

$$\xi \in \text{ supp } \psi^{(\alpha)}_n \Rightarrow \ \ C_1/\xi \le 1/n \le C_2/\xi, \ \ |\xi| > |\xi_0|, \ \ |\alpha| > 0.$$

b) The proof uses the same estimate as well as the well known properties of Lebesgues spaces. ❑

3.2. **Composition of multiplier and multiplication operators.** Next we analyze compactness properties of the operator $\mathcal{A}_\psi T_\varphi$. According to our previous notation we denote the operator of multiplication with $\varphi = \varphi(x) \in \mathcal{S}(\mathbb{R}^d)$ as T_φ.

Theorem 6. *Let $m \in \mathbb{R}$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then, $\mathcal{A}_\psi T_\varphi$ is a compact operator from $H^q_m(\mathbb{R}^d)$ into $H^q_{-\varepsilon}(\mathbb{R}^d)$, for any $\varepsilon > 0$ if*

(1) *$\psi \in s^m_{\infty,N}$, $N \ge 3d+3$;*

(2) *$\psi \in s^m_{q,N}$, $N \ge d+3$, $1 \le q \le 2$.*

Remark 7. In our proof of Theorem 6, in the first part, we use the operator $\langle D_\eta\rangle^{2k} = (1-\Delta)^k$ and partial integration. Because of that we need the assumption $2k = d+1$, for d odd and $2k = d+2$, for d even and in the first case we need that $N > 3d+1$ and in the second that $N > 3d+2$, since then we can use Theorem 2. Because of that we assume that N is an integer such that $N \ge 3d+3$. Considering the second part of the proof we have to apply again $(1-\Delta)^k$ so that k should be integer equal to $(d+1)/2$ or $(d+2)/2$. This implies, in both cases $N \ge d+3$.

Proof: (1) We shall show that the symbol of the composition $\mathcal{A}_\psi T_\varphi$, denoted by σ, is in $S^m_{0,N-d-1}$, if d is odd or in $S^m_{0,N-d-2}$, if d is even. Recall, if $\sigma_1 \in S^{m_1}$ and $\sigma_2 \in S^{m_2}$, there exists $\sigma \in S^{m_1+m_2}$ such that $T_{\sigma_1}T_{\sigma_2} = T_\sigma$ and

$$\sigma(x,\xi) = \iint e^{-iy\eta}\sigma_1(x,\xi+\eta)\sigma_2(x+y,\xi)dy đ\eta, \ x,\xi \in \mathbb{R}^d$$

exists as an oscillatory integral. Therefore, we need to prove that for $\psi \in s^m_{\infty,N}$, $N \ge 3d+3$ and for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the symbol of the composition σ, given by $\sigma(x,\xi) = \iint e^{-iy\eta}\psi(\xi+\eta)\varphi(x+y)dy đ\eta, \ x,\xi \in \mathbb{R}^d$ belongs to the class $S^m_{0,N-d-1}$, if d is odd or to the class $S^m_{0,N-d-2}$, if d is even. We assume that d is odd. The proof is analogous in the case when d is even. Using Peetre's inequality we can estimate:

$$|\sigma(x,\xi)| = \Big| \iint e^{-iy\eta} \langle y\rangle^{-2k} \langle D_\eta\rangle^{2k} \Big(\langle \eta\rangle^{-2l} \psi(\xi+\eta) \langle D_y\rangle^{2l} \varphi(x+y)\Big) dy đ\eta \Big|$$

$$\leq \iint \langle y\rangle^{-2k} \langle \eta\rangle^{-2l} \langle \xi+\eta\rangle^{m} |\langle D_y\rangle^{2l} \varphi(x+y)| dy đ\eta$$

$$\leq c \iint \langle y\rangle^{-2k} \langle \eta\rangle^{-2l} \langle \xi\rangle^{m} \langle \eta\rangle^{|m|} |\langle D_y\rangle^{2l} \varphi(x+y)| dy đ\eta \leq C \langle \xi\rangle^{m},$$

for $2k > d$ and $2l - |m| > d$. Since d is odd we can choose $2k = d+1$. Moreover, since $\varphi \in \mathcal{S}(\mathbb{R}^d)$ it follows that for any $M > 0$ there exists $c_M > 0$ such that

$$\langle D_y\rangle^{2l} \varphi(x+y) \leq c_M \langle x+y\rangle^{-M} \leq C_M \langle x\rangle^{-M} \langle y\rangle^{M}.$$

Then,

$$|\sigma(x,\xi)| \leq c\langle\xi\rangle^{m} \langle x\rangle^{-M}, \tag{8}$$

if we choose $0 < M < 1$, since in that case $2k - M > d$. Next we estimate the derivatives of $\sigma(x,\xi)$. We have

$$\Big| \iint e^{-iy\eta} \partial_\xi^\alpha \psi(\xi+\eta) \partial_x^\beta \varphi(x+y) dy đ\eta \Big| =$$

$$\Big| \iint e^{-iy\eta} \langle y\rangle^{-2k} \langle D_\eta\rangle^{2k} \Big(\langle\eta\rangle^{-2l} \partial_\xi^\alpha \psi(\xi+\eta)\Big) \langle D_y\rangle^{2l} \partial_x^\beta \varphi(x+y) dy đ\eta \Big| \leq$$

$$c \iint \langle y\rangle^{-2k} \langle\eta\rangle^{-2l} \langle\xi\rangle^{m-|\alpha|} \langle\eta\rangle^{|m-|\alpha||} |\langle D_y\rangle^{2l} \partial_x^\beta \varphi(x+y)| dy đ\eta \leq c\langle\xi\rangle^{m-|\alpha|}.$$

Therefore, $|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)| \leq c\langle\xi\rangle^{m-|\alpha|}$ when $2k = d+1$, $2l > d + |m - |\alpha||$. Since $\psi \in s^m_{\infty,N}$, we have that $|\alpha| + 2k \leq N$. Then, the assumption $N \geq 3d+3$, that is, $N-d-1 > 2d$, allow us to use Theorem 2 in the sequel. We have proved that $\sigma \in S^m_{N-d-1}$ for odd d and (8) implies that $\sigma \in S^m_{0,N-d-1}$.

The rest of the proof is similar to the proof of Theorem 3.2 in [17] which claims that if $\sigma \in S_0^m$, then $T_\sigma : H^q_m(\mathbb{R}^d) \to H^q_{-\varepsilon}(\mathbb{R}^d)$ is a compact operator, for $m \in \mathbb{R}, 1 < q < \infty$. We apply similar techique to the one used in the proof of Theorem 3.2 [17]. Take $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. For $\nu \in \mathbb{N}$ let $\sigma_\nu(x,\xi) = \phi\Big(\dfrac{x}{\nu}\Big)\sigma(x,\xi)$. Then, $T_{\sigma_\nu} = \phi_\nu T_\sigma$, for $\phi_\nu(x) = \phi\Big(\dfrac{x}{\nu}\Big)$. The operator T_{σ_ν} is compact because T_σ is bounded from $H^q_m(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$ (Theorem 2) and the operator of multiplication by ϕ_ν is compact from $L^q(\mathbb{R}^d)$ into $H^q_{-\varepsilon}(\mathbb{R}^d)$, for any $\varepsilon > 0$. If $v \in H^q_m(\mathbb{R}^d)$, $1 < q < \infty$, then Theorem 2 implies that there exists $c > 0$ such that

$$\|(T_{\sigma_\nu} - T_\sigma)v\|_{H^q_{-\varepsilon}} \leq \|(T_{\sigma_\nu} - T_\sigma)v\|_{L^q} \leq c|\sigma_\nu - \sigma|_{S^m_{N-d-1}} \|v\|_{H^q_m}.$$

We estimate:

$$|\sigma_\nu - \sigma|_{S^m_{N-d-1}} = \max_{|\alpha|,|\beta| \leq N-d-1} \sup_{x,\xi\in\mathbb{R}^d} \frac{|\partial_\xi^\alpha \partial_x^\beta ((\phi(\frac{x}{\nu}) - 1)\sigma(x,\xi))|}{\langle\xi\rangle^{m-|\alpha|}}$$

$$\leq \max_{|\alpha|,|\beta| \leq N-d-1} \sup_{|x|\geq\nu,\xi\in\mathbb{R}^d} \frac{|\sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \partial_x^{\beta-\gamma} (\phi(\frac{x}{\nu}) - 1) \partial_\xi^\alpha \partial_x^\gamma \sigma(x,\xi)|}{\langle\xi\rangle^{m-|\alpha|}} \leq C_1 c_{\alpha,\gamma}(\nu).$$

Since $\sigma \in S^m_{0,N-d-1}$, it follows that $c_{\alpha,\gamma}(\nu) = o(1)$ as $\nu \to \infty$. We conclude that $\|T_{\sigma_\nu} - T_\sigma\|_{\mathcal{L}(H^q_m, H^q_{-\varepsilon})} \to 0$ as $\nu \to \infty$, which implies that T_σ is also a compact operator.

(2) We denote $\psi_\nu(\xi) = \phi\Big(\dfrac{x}{\nu}\Big)\psi(\xi)$, where $\phi \in C_c^\infty(\mathbb{R}^d)$ is introduced in the same way as in part (1). We will show that $\mathcal{A}_\psi T_\varphi : H^q_m(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$. Then, as in the previous part of the proof, $T_{\psi_\nu} T_\varphi$ is a compact operator from $H^q_m(\mathbb{R}^d)$ into $H^q_{-\varepsilon}(\mathbb{R}^d)$. Therefore, we need to prove that $T_{\psi_\nu}T_\varphi \to \mathcal{A}_\psi T_\varphi$ in norm as $\nu \to \infty$. We have that

$$\|T_{\psi_\nu - \psi}(\varphi v)\|_{L^q} = \Big\|\Big(\phi\Big(\frac{x}{\nu}\Big) - 1\Big)T_\psi(\varphi v)\Big\|_{L^q} =$$

$$\Big(\int_{\mathbb{R}^d}\Big|\Big(\phi\Big(\frac{x}{\nu}\Big) - 1\Big)\int_{\mathbb{R}^d} e^{ix\xi}\psi(\xi)\mathcal{F}(\varphi v)(\xi)d\!\!{}^-\xi\Big|^q dx\Big)^{\frac{1}{q}} =$$

$$\Big(\int_{\mathbb{R}^d}\Big|\Big(\phi\Big(\frac{x}{\nu}\Big) - 1\Big)\int_{\mathbb{R}^d} (L^k_\xi e^{ix\xi})\psi(\xi)\mathcal{F}(\varphi v)(\xi)d\!\!{}^-\xi\Big|^q dx\Big)^{\frac{1}{q}},$$

where $L_\xi = (1+|x|^2)^{-1}(1-\Delta_\xi)$ and $L_\xi e^{ix\xi} = e^{ix\xi}$. After integration by parts for $k = \lfloor d/2 \rfloor + 1$, that is $2k = d+1$ for d odd or $2k = d+2$ for d even it holds that

$$\|T_{\psi_\nu - \psi}(\varphi v)\|_{L^q} = \Big(\int_{\mathbb{R}^d}\frac{|(\phi(\frac{x}{\nu}) - 1)|^q}{(1+|x|^2)^{kq}}\Big|\int_{\mathbb{R}^d} e^{ix\xi}\sum_{|r|=0}^{2k} a_r \partial^r(\psi(\xi)\mathcal{F}(\varphi v)(\xi))d\xi\Big|^q dx\Big)^{\frac{1}{q}}$$

$$= \Big(\int_{\mathbb{R}^d}\frac{|(\phi(\frac{x}{\nu}) - 1)|^q}{\langle x\rangle^{2kq}}\Big|\int_{\mathbb{R}^d} e^{ix\xi}\sum_{|r|=0}^{2k} a_r \sum_{\alpha+\beta=r}\binom{r}{\alpha}\frac{\partial^\alpha\psi(\xi)\langle\xi\rangle^{m-|\alpha|}}{\langle\xi\rangle^{m-|\alpha|}}\partial^\beta\mathcal{F}(\varphi v)(\xi)d\xi\Big|^q dx\Big)^{\frac{1}{q}}$$

Since $2k > d$, we can write $2k = d + \varepsilon_1 + \varepsilon_2$, $\varepsilon_i > 0, i = 1, 2$. This implies

$$\|T_{\psi_\nu - \psi}(\varphi v)\|_{L^q} \le c \sup_{x\in\mathbb{R}^d}\Big|\frac{\phi(\frac{x}{\nu}) - 1}{\langle x\rangle^{\varepsilon_1}}\Big||\psi|_{s^m_{q,N}}\Big\|\mathcal{F}((-ix)^\beta\varphi v)(\xi)\langle\xi\rangle^{m-|\alpha|}\Big\|_{L^p}.$$

Putting $h_\beta = x^\beta\varphi$ it follows that

$$\|T_{\psi_\nu - \psi}(\varphi v)\|_{L^q} \le c \sup_{x\in\mathbb{R}^d}\Big|\frac{\phi(\frac{x}{\nu}) - 1}{\langle x\rangle^{\varepsilon_1}}\Big||\psi|_{s^m_{q,N}}\Big(\int_{\mathbb{R}^d}\langle\xi\rangle^{p(m-|\alpha|)}|\mathcal{F}(h_\beta v)(\xi)|^p d\xi\Big)^{1/p}$$

$$= c \sup_{x\in\mathbb{R}^d}\Big|\frac{\phi(\frac{x}{\nu}) - 1}{\langle x\rangle^{\varepsilon_1}}\Big||\psi|_{s^m_{q,N}}\Big(\int_{\mathbb{R}^d}\langle\xi\rangle^{p(m-|\alpha|)}|\hat{h}_\beta * \hat{v}|^p d\xi\Big)^{1/p}$$

Next, we use Peetre's and Young's inequality to conclude that $\|T_{\psi_\nu - \psi}(\varphi v)\|_{L^q}$

$$\le c \sup_{x\in\mathbb{R}^d}\Big|\frac{\phi(\frac{x}{\nu}) - 1}{\langle x\rangle^{\varepsilon_1}}\Big||\psi|_{s^m_{q,N}}\Big\|\mathcal{F}(h^\beta)(\cdot)(1+|\cdot|^2)^{\frac{|m-|\alpha||}{2}} * \mathcal{F}v(\xi)(1+|\cdot|^2)^{\frac{m-|\alpha|}{2}}\Big\|_{L^p}$$

$$\le c \sup_{x\in\mathbb{R}^d}\Big|\frac{\phi(\frac{x}{\nu}) - 1}{\langle x\rangle^{\varepsilon_1}}\Big||\psi|_{s^m_{q,N}}\Big\|\mathcal{F}v(\xi)|(1+|\xi|^2)^{\frac{m-|\alpha|}{2}}\Big\|_{L^p}\Big\|\mathcal{F}(h^\beta)(1+|\xi|^2)^{\frac{|m-|\alpha||}{2}}\Big\|_{L^1}$$

$$\le c \sup_{x\in\mathbb{R}^d}\Big|\frac{\phi(\frac{x}{\nu}) - 1}{\langle x\rangle^{\varepsilon_1}}\Big||\psi|_{s^m_{q,N}}\Big\|\mathcal{F}(h^\beta)(\xi)(1+|\xi|^2)^{\frac{|m-|\alpha||}{2}}\Big\|_{L^1}\|v\|_{H^q_m}$$

$$\le c_1 \sup_{|x|\ge\nu}\frac{1}{\langle x\rangle^{\varepsilon_1}}|\psi|_{s^m_{q,N}}\|v\|_{H^q_m} \to 0,\ \nu\to\infty.$$

Hence, $\mathcal{A}_\psi T_\varphi$ is a limit of compact operators $T_{\psi_\nu} T_\varphi$ and the proof also implies that $\mathcal{A}_\psi T_\varphi : H^q_m(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$. Since $\|T_{\psi_\nu-\psi}(\varphi v)\|_{H^q_{-\varepsilon}} \le \|T_{\psi_\nu-\psi}(\varphi v)\|_{L^q}$ it follows that $\mathcal{A}_\psi T_\varphi$ is a compact operator from $H^q_m(\mathbb{R}^d)$ to $H^q_{-\varepsilon}(\mathbb{R}^d)$. ❑

4. H-distributions

From now on we assume that N is an integer such that $N \ge 3d+5$. Actually, because of the use of Theorem 2 we explained that $N \ge 3d+3$ (cf. Remark 7). We enlarge N because we need to assume that $N-d-3 > 2d$, if d is odd and $N-d-4 > 2d$, if d is even.

4.1. Compactness of the commutator $C = [\mathcal{A}_\psi, T_\varphi]$.

Theorem 8. *Let $\psi \in s^m_{\infty,N}$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $m \in \mathbb{R}$. Then the commutator $C = [\mathcal{A}_\psi, T_\varphi] = \mathcal{A}_\psi T_\varphi - T_\varphi \mathcal{A}_\psi$ is a compact operator from $H^q_m(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$. If p denotes the symbol of C, then $p \in S^{m-1}_{0,N-d-3}$, if d is odd or $p \in S^{m-1}_{0,N-d-4}$, if d is even.*

Proof: Let $\psi \in s^m_{\infty,N}$, $N \ge 3d+5$, d odd and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The symbol of the composition $\mathcal{A}_\psi T_\varphi$ is given by $\sigma(x,\xi) = \iint e^{-iy\eta}\psi(\xi+\eta)\varphi(x+y)dy đ\eta$, $x,\xi \in \mathbb{R}^d$. Using Taylor expansion, we obtain that $\sigma(x,\xi) = I_1(x,\xi) + I_2(x,\xi)$, where

$$I_1(x,\xi) = \sum_{|\alpha|\le 1} \frac{1}{\alpha!} \iint e^{-iy\eta}\eta^\alpha \partial^\alpha_\xi \psi(\xi)\varphi(x+y)dy đ\eta$$

and $I_2(x,\xi) = 2\sum_{|\alpha|=2} \frac{1}{\alpha!} \iint e^{-iy\eta}\eta^\alpha \Big(\int_0^1 (1-\theta)^2 \partial^\alpha_\xi \psi(\xi+\theta\eta)d\theta\Big)\varphi(x+y)dy đ\eta$. Then, $I_1(x,\xi) = \sum_{|\alpha|\le 1} \frac{1}{\alpha!}\partial^\alpha_\xi\psi(\xi)D^\alpha_y\varphi(y)|_{y=x}$ and similarly,

$$I_2(x,\xi) = 2\sum_{|\alpha|=2} \frac{1}{\alpha!} \iint e^{-iy\eta} \Big(\int_0^1 (1-\theta)^2 \partial^\alpha_\xi \psi(\xi+\theta\eta)d\theta\Big) D^\alpha_y\varphi(x+y)dy đ\eta.$$

Since the symbol of $T_\varphi \mathcal{A}_\psi$ is $\varphi(x)\psi(\xi)$, the symbol of the commutator C is

$$p(x,\xi) = \sum_{|\alpha|=1} \frac{1}{\alpha!}\partial^\alpha_\xi\psi(\xi)D^\alpha_y\varphi(y)|_{y=x} + I_2(x,\xi),$$

where $\tilde{I}_1(x,\xi) := \sum_{|\alpha|=1} \frac{1}{\alpha!}\partial^\alpha_\xi\psi(\xi)D^\alpha_y\varphi(y)|_{y=x}$. Clearly, $\tilde{I}_1(x,\xi) \in S^{m-1}_{0,N-1}$. Next, we need to estimate $I_2(x,\xi)$. Note that $I_2(x,\xi) = 2\sum_{|\alpha|=2}\frac{1}{\alpha!}\int_0^1 (1-\theta)^2 I_3(x,\xi)d\theta$, where $I_3(x,\xi) = \iint e^{-iy\eta}\partial^\alpha_\xi\psi(\xi+\theta\eta)D^\alpha_y\varphi(x+y)dy đ\eta$. As in the proof of Theorem 6, we have:

$$|I_3(x,\xi)| \le \iint \langle y\rangle^{-2k}\langle D_\eta\rangle^{2k}\Big(\langle\eta\rangle^{-2l}\partial^\alpha_\xi\psi(\xi+\theta\eta)\Big)\langle D_y\rangle^l\Big[D^\alpha_y\varphi(x+y)\Big]dy đ\eta$$
$$\le C\langle\xi\rangle^{m-2}\langle x\rangle^{-M},$$

for $2k = d+1$, $0 < M < 1$, $2l > d + |m-2|$. Also, from the proof of Theorem 6 it follows that $I_2 \in S^{m-2}_{0,N-d-3}$. Since $\tilde{I}_1(x,\xi) \in S^{m-1}_{0,N-1} \subset S^{m-1}_{0,N-d-3}$ and $I_2 \in$

$S^{m-2}_{0,N-d-3} \subset S^{m-1}_{0,N-d-3}$ it follows that $p \in S^{m-1}_{0,N-d-3}$. Now we apply the proof of Theorem 6, part 1, to conclude that $C = T_p$ is a compact operator from $H^q_m(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$. The proof is analogous in the case when d is even. ❑

Corollary 9. *Let $\psi \in s^m_{\infty,N}$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $m, s \in \mathbb{R}$. Then the commutator $C = [T_\psi, T_\varphi] = T_\psi T_\varphi - T_\varphi T_\psi$ is a compact operator from $H^q_{m+s}(\mathbb{R}^d)$ into $H^q_s(\mathbb{R}^d)$. If p denotes the symbol of C, then $p \in S^{m-1}_{0,N-d-3}$, if d is odd or $p \in S^{m-1}_{0,N-d-4}$, if d is even.*

Proof: Note that $\mathcal{A}_\psi T_\varphi$ is a compact operator from $H^q_{m+s}(\mathbb{R}^d)$ to $H^q_{s-\varepsilon}(\mathbb{R}^d)$ for $\psi \in s^m_{\infty,N}$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\varepsilon > 0$. This easily follows from the proof of Theorem 6 and Corollary 3. Thus, the proof of Theorem 8 implies the claim. ❑

4.2. **H-distributions with $\boldsymbol{\psi \in (s^m_{\infty,N+1})_0}$.** Note that the completion of the tensor product $\mathcal{S}(\mathbb{R}^d) \otimes (s^m_{\infty,N+1})_0$ is the same for both the ε and the π topology, since $\mathcal{S}(\mathbb{R}^d)$ is nuclear ([15], Theorem 50.1). We use the notation $\mathcal{S}(\mathbb{R}^d)\hat{\otimes}(s^m_{\infty,N+1})_0$ for the completion.

Theorem 10. *Let $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$ and $v_n \rightharpoonup 0$ in $H^q_m(\mathbb{R}^d)$, $m \in \mathbb{R}$. Then, up to subsequences, there exists a distribution $\mu \in (\mathcal{S}(\mathbb{R}^d)\hat{\otimes}(s^m_{\infty,N+1})_0)'$ such that for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and all $\psi \in (s^m_{\infty,N+1})_0$,*

$$\lim_{n\to\infty} \langle \varphi_1 u_n, \overline{\mathcal{A}_{\bar{\psi}}(\varphi_2 v_n)} \rangle = \langle \mu, \varphi_1 \bar{\varphi}_2 \otimes \psi \rangle.$$

Proof: Since $\psi \in (s^m_{\infty,N+1})_0 \subset s^m_{\infty,N}$ we have that $\mathcal{A}_{\bar{\psi}}(\varphi_2 v_n) \in L^q(\mathbb{R}^d)$. We write $\psi(\xi) = \psi_1(\xi)\psi_2(\xi)$, $\psi_1(\xi) = \langle\xi\rangle^m \in s^m_{\infty,N+1}$, $\psi_2(\xi) = \langle\xi\rangle^{-m}\psi(\xi) \in (s^0_{\infty,N+1})_0$. Hence, using (6), it follows

$$\|\mathcal{A}_{\overline{\psi}}(\varphi v_n)\|_{L^q} \leq c|\psi_2|_{s^0_{\infty,N}} \|\mathcal{A}_{\overline{\psi_1}}(\varphi v_n)\|_{L^q} \leq c_1 |\psi|_{s^m_{\infty,N}} \|\varphi v_n\|_{H^q_m}.$$

In the last inequality we have used the estimate

$$|\psi_2|_{s^0_{\infty,N}} = |\langle\xi\rangle^{-m}\psi(\xi)|_{s^0_{\infty,N}} \leq C|\langle\xi\rangle^{-m}|_{s^{-m}_{\infty,N}} |\psi|_{s^m_{\infty,N}} \leq C_1 |\psi|_{s^m_{\infty,N}}.$$

Using Peetre's inequality and the exchange formula for the inverse Fourier transform of convolution, we have

$$\|\varphi v_n\|_{H^q_m} = \Big(\int_{\mathbb{R}^d} \Big|\mathcal{F}^{-1}((1+|\xi|^2)^{\frac{m}{2}} \hat{\varphi} \star \hat{v}_n)\Big|^q dx\Big)^{\frac{1}{q}}$$

$$\leq \Big(\int_{\mathbb{R}^d} \Big|\mathcal{F}^{-1}(\hat{v}_n(1+|\cdot|^2)^{\frac{m}{2}})\Big|^q \Big|(\mathcal{F}^{-1}(\hat{\varphi}(1+|\cdot|^2)^{\frac{m}{2}})\Big|^q dx\Big)^{\frac{1}{q}}$$

$$\leq C \sup_{x\in\mathbb{R}^d} \Big|\mathcal{F}^{-1}((1+|\xi|^2)^{\frac{m}{2}}\hat{\varphi})\Big| \|v_n\|_{H^q_m}$$

$$\leq C \int_{\mathbb{R}^d} \frac{1}{\langle\xi\rangle^{d+1}} \|\langle\xi\rangle^{d+1+m}\hat{\varphi}\|_\infty d\xi \leq C\|\langle\xi\rangle^{d+1+m}\hat{\varphi}\|_\infty.$$

Recall that for $\mathcal{S}(\mathbb{R}^d)$ we have a plenty of equivalent sequences of norms among which we can use

$$|\varphi|_k = \sup_{|\alpha|\leq k} \|\langle\xi\rangle^k \hat{\varphi}^{(\alpha)}(\xi)\|_\infty,\ k \in \mathbb{N}_0.$$

Therefore,

$$\Big|\int_{\mathbb{R}^d} u_n \overline{\mathcal{A}_{\overline{\psi}}(\varphi v_n)} dx\Big| \leq C|\psi|_{s^m_{\infty,N}} |\varphi|_{d+1+\lceil |m| \rceil}. \tag{9}$$

For fixed $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the mapping $\psi \mapsto \mu_n(\varphi, \psi) := \int_{\mathbb{R}^d} u_n \overline{\mathcal{A}_{\overline{\psi}}(\varphi v_n)} dx$ is linear and continuous, and for fixed $\psi \in (s^m_{\infty,N+1})_0$, the mapping $\varphi \mapsto \mu_n(\varphi, \psi)$ is anti-linear and continuous. The rest of the proof follows the standard steps for proving the existence of H-distributions, as was done in the proof of Theorem 2.1. in [4].

Fix $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and consider a sequence of mappings

$$\Phi^{\varphi}_n : \psi \mapsto \mu_n(\varphi, \psi).$$

Thus $\Phi^{\varphi}_n \in ((s^m_{\infty,N+1})_0)'$ and we can apply Sequential Banach Alaoglu theorem to extract weakly star convergent subsequence $\Phi^{\varphi}_{\nu} \overset{*}{\rightharpoonup} \Phi^{\varphi}$, since $((s^m_{\infty,N+1})_0, |\cdot|_{s^m_{\infty,N}})$ is separable. More precisely, for every fixed $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we construct a linear mapping Φ^{φ} such that $\langle \Phi^{\varphi}_{\nu}, \psi \rangle \to \langle \Phi^{\varphi}, \psi \rangle$, $\nu \to \infty$ and $\psi \in (s^m_{\infty,N+1})_0$. Actually, by diagonalization we find a sequence (Φ^{φ}_{ν}) converging on a dense countable subset of $(s^m_{\infty,N+1})_0$ and by the Banach - Steinhaus theorem we extend it to $(s^m_{\infty,N+1})_0$. Then, for fixed $\psi \in (s^m_{\infty,N+1})_0$, the mapping $\varphi \mapsto \langle \Phi^{\varphi}_{\nu}, \psi \rangle$ is a pointwise bounded sequence in $\mathcal{S}'(\mathbb{R}^d)$ which converges on a dense set $M \subset \mathcal{S}(\mathbb{R}^d)$; this is again obtained by diagonalization procedure. By the Banach-Steinhaus theorem, see [8, p. 169], $\langle \Phi^{\varphi}_{\nu}, \psi \rangle$ converges to $\langle \Phi^{\varphi}, \psi \rangle$ on $\mathcal{S}(\mathbb{R}^d)$. In this way we show that for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and every $\psi \in (s^m_{\infty,N+1})_0$,

$$\lim_{\nu \to \infty} \langle \Phi^{\varphi}_{\nu}, \psi \rangle = \langle \Phi^{\varphi}, \psi \rangle.$$

Moreover, by (9),

$$|\langle \Phi^{\varphi}, \psi \rangle| \leq c |\varphi|_{\lceil m \rceil + d + 1} |\psi|_{s^m_{\infty,N}}.$$

By the kernel theorem ([15][Part III, Chap. 50, Proposition 50.7, p. 524]) we have that there exists $\mu \in (\mathcal{S}(\mathbb{R}^d) \hat{\otimes} (s^m_{\infty,N+1})_0)'$ defined as

$$\langle \mu(x,\xi), \varphi(x)\psi(\xi) \rangle = \lim_{\nu \to \infty} \langle \Phi^{\varphi}_{\nu}, \psi \rangle = \lim_{\nu \to \infty} \int u_{\nu} \overline{\mathcal{A}_{\overline{\psi}}(\varphi v_{\nu})} dx,$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\psi \in (s^m_{\infty,N+1})_0$, where u_{ν} is a subsequence of u_n and v_{ν} is a subsequence of v_n. Since every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ can be written as $\varphi = \overline{\varphi}_1 \varphi_2$ for some $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ ([13]), we have that $\langle \mu, \varphi\psi \rangle = \lim_{\nu \to \infty} \int u_{\nu} \overline{\mathcal{A}_{\overline{\psi}}(\overline{\varphi}_1 \varphi_2 v_{\nu})} dx$. Using Theorem 8, we obtain that for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and $\psi \in (s^m_{\infty,N+1})_0$,

$$\langle \mu, \overline{\varphi}_1 \varphi_2 \psi \rangle = \lim_{\nu \to \infty} \int_{\mathbb{R}^d} \varphi_1 u_{\nu} \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{\nu})} dx.$$

This completes the proof.

❑

Remark 11. Let $\psi_0 \in s^m_{\infty,N}$. The above proof implies that $\mu_n(\cdot, \psi_0), n \in \mathbb{N}$ is a bounded sequence of linear mappings on $\mathcal{S}(\mathbb{R}^d)$. Thus, it has a convergent subsequence $\mu_{n_k}(\cdot, \psi_0)$ converging to $\mu(\cdot, \psi_0)$ in $\mathcal{S}'(\mathbb{R}^d)$. If we choose another $\psi_1 \in s^m_{\infty,N}$ we can find subsequence of $\mu_{n_k}(\cdot, \psi_1)$ denoted by $\mu_l(\cdot, \psi_1)$ converging in $\mathcal{S}'(\mathbb{R}^d)$. We do not have the same sequence for all $\psi \in s^m_{\infty,N}$. Because of that we need to introduce separable class of symbols $(s^m_{\infty,N+1})_0$.

Corollary 12. *Let $u_n \rightharpoonup 0$ in $H^p_{-s}(\mathbb{R}^d)$ and $v_n \rightharpoonup 0$ in $H^q_{s+m}(\mathbb{R}^d)$, $s, m \in \mathbb{R}$. Then, up to subsequences, there exists $\mu \in (\mathcal{S}(\mathbb{R}^d)\hat{\otimes}(s^m_{\infty,N+1})_0)'$ such that for all $\varphi_1, \varphi_2 \in \mathcal{S}$ and all $\psi \in (s^m_{\infty,N+1})_0$,*

$$\lim_{n\to\infty} \langle \varphi_1 u_n, \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_n)}\rangle = \langle \mu, \varphi_1\overline{\varphi_2} \otimes \psi\rangle.$$

Proof: We consider the sequence of functionals $\mu_n(\varphi,\psi) = \langle \mathcal{A}_{\langle\xi\rangle^{-s}}(u_n), \overline{\mathcal{A}_{\overline{\langle\xi\rangle^s\psi}}(\varphi v_n)}\rangle$. Let $\psi_1(\xi) = \langle\xi\rangle^{-s}$ and $\psi_2(\xi) = \langle\xi\rangle^s\psi(\xi)$. Then,

$$|\mu_n(\varphi,\psi)| \le \|\mathcal{A}_{\psi_1}(u_n)\|_{L^p}\|\mathcal{A}_{\bar{\psi}_2}(\varphi v_n)\|_{L^q} \le c|\psi|_{s^m_{\infty,N}}|\varphi|_{2d+2+\lceil|m+s|\rceil}.$$

Notice that $\lim_{n\to\infty}\mu_n(\varphi,\psi) = \lim_{n\to\infty}\langle u_n, \overline{\mathcal{A}_{\overline{\psi}}(\varphi v_n)}\rangle$. Applying the proof of Theorem 10 and using Corollary 9, we obtain the assertion. ❑

Theorem 13. *Let $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$. If for every sequence $v_n \rightharpoonup 0$ in $H^q_m(\mathbb{R}^d)$, $m \in \mathbb{R}$ it holds that*

$$\lim_{n\to\infty}\langle u_n, \mathcal{A}_{\langle\xi\rangle^m}(\varphi v_n)\rangle = 0, \tag{10}$$

then for every $\theta \in \mathcal{S}(\mathbb{R}^d)$, $\theta u_n \to 0$ strongly in $L^p(\mathbb{R}^d)$, $n \to \infty$.

Proof: We will prove that for all $\theta \in \mathcal{S}(\mathbb{R}^d)$ and every bounded $B \subseteq L^q(\mathbb{R}^d)$,

$$\sup\{\langle\theta u_n, \phi\rangle \,:\, \phi \in B\} \to 0,\ n \to \infty.$$

Assume the opposite, i.e. that there exist $\theta \in \mathcal{S}(\mathbb{R}^d)$, a bounded set B_0 in $L^q(\mathbb{R}^d)$, an $\varepsilon_0 > 0$ and a subsequence θu_ν of θu_n such that

$$\sup\{|\langle\theta u_\nu, \phi\rangle| \,:\, \phi \in B_0\} \ge \varepsilon_0, \text{ for every } \nu \in \mathbb{N}.$$

Choose $\phi_\nu \in B_0$ such that $|\langle\theta u_\nu, \phi_\nu\rangle| > \varepsilon_0/2$. Since $\phi_\nu \in B_0$ and B_0 is bounded in $L^q(\mathbb{R}^d)$, then (ϕ_ν) is weakly precompact in $L^q(\mathbb{R}^d)$, i.e. up to a subsequence, $\phi_\nu \rightharpoonup \phi_0$ in $L^q(\mathbb{R}^d)$. Moreover, since ϕ_0 is fixed, we have $\langle u_\nu, \phi_0\rangle \to 0$ and

$$|\langle\theta u_\nu, \phi_\nu - \phi_0\rangle| > \frac{\varepsilon_0}{4}, \quad \nu > \nu_0. \tag{11}$$

Applying (10) on $u_\nu \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$ and $\mathcal{A}_{\langle\xi\rangle^{-m}}(\phi_\nu - \phi_0) \rightharpoonup 0$ in $H^q_m(\mathbb{R}^d)$, we obtain that for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$\lim_{\nu\to\infty}\langle u_\nu, \mathcal{A}_{\langle\xi\rangle^m}(\varphi\mathcal{A}_{\langle\xi\rangle^{-m}}((\phi_\nu - \phi_0))\rangle = 0. \tag{12}$$

Choosing $\varphi = \theta$ and using Theorem 8, we get $\lim_{\nu\to\infty}\langle\theta u_\nu, \phi_\nu - \phi_0\rangle = 0$, which contradicts (11). ❑

Following the proof of Theorem 13 and using Corollary 12, it is easy to prove the next corollary.

Corollary 14. *Let $u_n \rightharpoonup 0$ in $H^p_{-s}(\mathbb{R}^d)$, $m, s \in \mathbb{R}$. If for every sequence $v_n \rightharpoonup 0$ in $H^q_{s+m}(\mathbb{R}^d)$ it holds that $\lim_{n\to\infty}\langle u_n, \mathcal{A}_{\langle\xi\rangle^m}(\varphi v_n)\rangle = 0$, then $\theta u_n \to 0$ strongly in $H^p_{-s}(\mathbb{R}^d)$, $n \to \infty$, for every $\theta \in \mathcal{S}(\mathbb{R}^d)$.*

4.3. **H-distributions with $\psi \in s^m_{q,N}, 1 < q \le 2$.** As in the case of symbols in the class $s^m_{\infty,N}$, we need to prove that $C(\varphi_2 v_n) = T_{\varphi_1}\mathcal{A}_\psi(\varphi_2 v_n) - \mathcal{A}_\psi T_{\varphi_1}(\varphi_2 v_n) \to 0$ in $L^q(\mathbb{R}^d)$, if $v_n \rightharpoonup 0$ in $H^q_m(\mathbb{R}^d)$. The following theorem holds.

Theorem 15. *Let $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, $\psi \in s^m_{q,N}$, $1 \le q \le 2$, $N \ge 2d+4$, $m \in \mathbb{R}$. Then, $(T_{\varphi_1}\mathcal{A}_\psi - \mathcal{A}_\psi T_{\varphi_1})T_{\varphi_2}$ is a compact operator from $H^q_m(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$.*

Proof: Denote by $p(x,\xi)$ the symbol of $T_{\varphi_1}\mathcal{A}_\psi - \mathcal{A}_\psi T_{\varphi_1}$. Then,

$$p(x,\xi) = \sum_{|\alpha|=1} \frac{1}{\alpha!}\partial^\alpha_\xi \psi(\xi) D^\alpha_y \varphi_1(y)|_{y=x} + p^2(x,\xi),$$

where

$$p^2(x,\xi) = 2\sum_{|\alpha|=2} \frac{1}{\alpha!}\int_0^1 (1-\theta)^2 \iint e^{-iy\eta}\partial^\alpha_\xi \psi(\xi+\theta\eta) D^\alpha_y \varphi_1(x+y) dy đ\eta d\theta.$$

As in the proof of Theorem 8, we will approximate T_p by the sequence of compact operators $T_{p_\nu}(x) = \phi(x/\nu)T_p(x)$, where ϕ is constructed as in the proof of Theorem 6, part (1). We will first show that $T_pT_{\varphi_2} : H^q_m(\mathbb{R}^d) \to H^q_1(\mathbb{R}^d)$. Then multiplying with ϕ implies that $T_{p_\nu}T_{\varphi_2} : H^q_m(\mathbb{R}^d) \to H^q_{1-\varepsilon}(\mathbb{R}^d)$ is a compact operator for every $\varepsilon > 0$, so $T_{p_\nu}T_{\varphi_2} : H^q_m(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is a compact operator.
Let $p^1(x,\xi) = \sum_{|\alpha|=1} \frac{1}{\alpha!}\partial^\alpha_\xi \psi(\xi) D^\alpha_y \varphi_1(y)|_{y=x}$. Since $\psi \in s^m_{q,N}$, it follows that $\psi_\alpha(\xi) := \partial^\alpha_\xi \psi(\xi) \in s^{m-1}_{q,N-1}$, because $|\alpha| = 1$, i.e. (5) holds for $m-1$. If $v \in H^q_m(\mathbb{R}^d)$ and $\varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, it follows that $\mathcal{A}_{\psi_\alpha}(\varphi_2 v) \in H^q_1(\mathbb{R}^d)$. Indeed,

$$\|\mathcal{A}_{\psi_\alpha}(\varphi_2 v)\|_{H^q_1} = \|\mathcal{A}_{\langle\xi\rangle}(\mathcal{A}_{\psi_\alpha}(\varphi_2 v))\|_{L^q} = \|\mathcal{A}_{\psi_\alpha(\xi)\langle\xi\rangle}(\varphi_2 v)\|_{L^q}.$$

Because $\psi_\alpha(\xi) := \partial^\alpha_\xi \psi(\xi) \in s^{m-1}_{q,N-1}$, it follows that $\psi_\alpha(\xi)\langle\xi\rangle \in s^m_{q,N-1}$, which is easy to derive from the definition (5). Since $\varphi_2 v \in H^q_m(\mathbb{R}^d)$, and $N-1 \ge d+3$, we have that $\mathcal{A}_{\psi_\alpha(\xi)\langle\xi\rangle}(\varphi_2 v) \in L^q(\mathbb{R}^d)$ (this property is shown in the proof of Theorem 6). Therefore, $\|\mathcal{A}_{\psi_\alpha}(\varphi_2 v)\|_{H^q_1} < \infty$ and $\|T_p(\varphi_2 v)\|_{H^q_1} \le \|T_{p^1}(\varphi_2 v)\|_{H^q_1} + \|T_{p^2}(\varphi_2 v)\|_{H^q_1}$. Hence,

$$\|T_{p^1}(\varphi_2 v)\|_{H^q_1} \le \sum_{|\alpha|=1} \frac{1}{\alpha!}\|T_{\partial^\alpha_\xi \psi(\xi) D^\alpha_y \varphi_1(y)|_{y=x}}(\varphi_2 v)\|_{H^q_1}$$

$$= \sum_{|\alpha|=1} \frac{1}{\alpha!}\|((D^\alpha_y \varphi_1(y)|_{y=x})T_{\partial^\alpha_\xi \psi(\xi)}(\varphi_2 v))\|_{H^q_1}.$$

Let $D^\alpha_y \varphi_1(y)|_{y=x} =: \varphi^\alpha_1(x) \in \mathcal{S}(\mathbb{R}^d)$. Then, $\|\varphi^\alpha_1(x)\mathcal{A}_{\partial^\alpha_\xi \psi(\xi)}(\varphi_2 v)\|_{H^q_1} = \|\varphi^\alpha_1(x)v_\alpha(x)\|_{H^q_1}$, where $v_\alpha(x) = \mathcal{A}_{\partial^\alpha_\xi \psi(\xi)}(\varphi_2 v) \in H^q_1(\mathbb{R}^d)$. Therefore,

$$\|\varphi^\alpha_1(x)\mathcal{A}_{\partial^\alpha_\xi \psi(\xi)}(\varphi_2 v)\|_{H^q_1} \le |\varphi^\alpha_1(x)|_p \|v_\alpha(x)\|_{H^q_1},$$

where $|\varphi^\alpha_1(x)|_p$ is a semi-norm of function $\varphi^\alpha_1 \in \mathcal{S}(\mathbb{R}^d)$ for some $p \in \mathbb{N}$. So we have proved that

$$\|T_{p^1}(\varphi_2 v)\|_{H^q_1} \le \sum_{|\alpha|=1} \frac{1}{\alpha!}|\varphi^\alpha_1(x)|_p \|v_\alpha(x)\|_{H^q_1} < \infty.$$

Next, we want to prove that $T_{p^2}(\varphi_2 v) \in H_1^q(\mathbb{R}^d)$. By the definition, $\|T_{p^2}(\varphi_2 v)\|_{H_1^q} = \|\mathcal{A}_{\langle\xi\rangle} T_{p^2}(\varphi_2 v)\|_q$. Therefore,

$$\mathcal{A}_{\langle\xi\rangle} T_{p^2}(\varphi_2 v)(x) = \int_{\mathbb{R}^d} e^{ix\xi}\Big[\iint e^{-ix'\eta}\langle\xi+\eta\rangle p^2(x+x',\xi)dx'đ\eta\Big]\mathcal{F}(\varphi_2 v)(\xi)đ\xi.$$

Let $\sigma(x,\xi) = \iint e^{-ix'\eta}\langle\xi+\eta\rangle p^2(x+x',\xi)dx'đ\eta$. It follows that

$$\|\mathcal{A}_{\langle\xi\rangle} T_{p^2}(\varphi_2 v)\|_q = \Big(\int_{\mathbb{R}^d_x}\Big|\int_{\mathbb{R}^d_\xi} e^{ix\xi}\sigma(x,\xi)\mathcal{F}(\varphi_2 v)(\xi)đ\xi\Big|^q dx\Big)^{1/q}.$$

Moreover, we have

$$\int_{\mathbb{R}^d_\eta}\int_{\mathbb{R}^d_{x'}} e^{-ix'\eta}\Big[\langle\xi+\eta\rangle p^2(x+x',\xi)\Big]dx'đ\eta$$

$$= 2\sum_{|\alpha|=2}\frac{1}{\alpha!}\int_0^1(1-\theta)^2\int_{\mathbb{R}^d_\eta}\int_{\mathbb{R}^d_{x'}} e^{-ix'\eta}\langle\xi+\eta\rangle I_\theta(x+x',\xi)dx'đ\eta d\theta,$$

where $I_\theta(x+x',\xi) = \int_{\mathbb{R}^d_{\tilde\eta}}\int_{\mathbb{R}^d_{\tilde y}} e^{-i\tilde y\tilde\eta}\partial_\xi^\alpha\psi(\xi+\theta\tilde\eta)D_{\tilde y}^\alpha\varphi_1(x+x'+\tilde y)d\tilde yđ\tilde\eta$. It follows that for large enough $l,l'\in\mathbb{N}_0$,

$$\int_{\mathbb{R}^d_\eta}\int_{\mathbb{R}^d_{x'}} e^{-ix'\eta}\langle\xi+\eta\rangle I_\theta(x+x',\xi)dx'd\eta$$

$$= \int_{\mathbb{R}^d_\eta}\int_{\mathbb{R}^d_{x'}} e^{-ix'\eta}\langle x'\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\Big[\langle\eta\rangle^{-2l}\langle\xi+\eta\rangle\Big]\langle D_{x'}\rangle^{2l} I_\theta(x+x',\xi)dx'd\eta.$$

Repeating the same procedure, we have

$$\int_{\mathbb{R}^d_{\tilde\eta}}\int_{\mathbb{R}^d_{\tilde y}} e^{-i\tilde y\tilde\eta}\partial_\xi^\alpha\psi(\xi+\theta\tilde\eta)\langle D_{x'}\rangle^{2l}D_{\tilde y}^\alpha\varphi_1(x+x'+\tilde y)d\tilde y d\tilde\eta$$

$$= \int_{\mathbb{R}^d_{\tilde\eta}}\int_{\mathbb{R}^d_{\tilde y}} e^{-i\tilde y\tilde\eta}\frac{1}{\langle\tilde y\rangle^{2k'}}\langle D_{\tilde\eta}\rangle^{2k'}\Big[\partial_\xi^\alpha\psi(\xi+\theta\tilde\eta)\langle\tilde\eta\rangle^{-2k}\Big]\langle D_{\tilde y}\rangle^{2k}\langle D_{x'}\rangle^{2l}D_{\tilde y}^\alpha\varphi_1(x+x'+\tilde y)d\tilde y d\tilde\eta.$$

Since $\varphi_1\in\mathcal{S}$, it follows that for every $M\in\mathbb{N}_0$ there exists $C>0$ so that

$$|\langle D_{\tilde y}\rangle^{2k}\langle D_{x'}\rangle^{2l}D_{\tilde y}^\alpha\varphi_1(x+x'+\tilde y)| \le C\langle x+x'+\tilde y\rangle^{-M}.$$

Then, applying Peetre's inequality we have that $C\langle x+x'+\tilde y\rangle^{-M} \le C_1\langle x'\rangle^M\langle x\rangle^{-M}\langle\tilde y\rangle^M$. Hence, by choosing large enough $M\in\mathbb{N}_0$, i.e. it is enough to assume that $M=d+1$ and $k'=d+1$, we have $\|\mathcal{A}_{\langle\xi\rangle}T_{p^2}(\varphi_2 v_n)\|_{L^q} \le$

$$\le C_1\Big(\int_{\mathbb{R}^d_x}\frac{1}{\langle x\rangle^{Mq}}\Big|\int_0^1(1-\theta)^2\int_{\mathbb{R}^d_{\xi'}}\int_{\mathbb{R}^d_{x'}}\frac{\langle x'\rangle^M}{\langle x'\rangle^{2l'}}dx'\int_{\mathbb{R}^d_{\tilde y}}\frac{\langle\tilde y\rangle^M}{\langle\tilde y\rangle^{2k'}}d\tilde y\int_{\mathbb{R}^d_\eta}\langle D_\eta\rangle^{2l'}\Big[\frac{\langle\xi+\eta\rangle}{\langle\eta\rangle^{2l}}\Big]d\eta\times$$

$$\times\int_{\mathbb{R}^d_{\tilde\eta}}\langle D_{\tilde\eta}\rangle^{2k'}\Big[\langle\tilde\eta\rangle^{-2k}\partial_\xi^\alpha\psi(\xi+\theta\tilde\eta)\Big]d\tilde\eta\mathcal{F}(\varphi_2 v_n)(\xi)d\xi d\theta\Big|^q dx\Big)^{1/q} \le$$

$$c\int_{\mathbb{R}^d_\xi}\Big|\int_{\mathbb{R}^d_\eta}\langle D_\eta\rangle^{2l'}\Big[\frac{\langle\xi+\eta\rangle}{\langle\eta\rangle^{2l}}\Big]d\eta\int_0^1(1-\theta)^2\int_{\mathbb{R}^d_{\tilde\eta}}\langle D_{\tilde\eta}\rangle^{2k'}\Big[\frac{\partial_\xi^\alpha\psi(\xi+\theta\tilde\eta)}{\langle\tilde\eta\rangle^{2k}}\Big]d\tilde\eta d\theta\mathcal{F}(\varphi_2 v_n)(\xi)\Big|d\xi.$$

So, we have

$$\Big|\langle D_\eta\rangle^{2l'}\Big[\langle\eta\rangle^{-2l}\langle\xi+\eta\rangle\Big]\Big| \le c_2\langle\xi\rangle\langle\eta\rangle^{-2l+1},$$

and

$$\Big|\langle D_{\tilde\eta}\rangle^{2k'}\Big[\langle\tilde\eta\rangle^{-2k}\partial_\xi^\alpha\psi(\xi+\theta\tilde\eta)\Big]\Big| \le c_3\langle\tilde\eta\rangle^{-2k}\sum_{|r|=0}^{2k'}\frac{\partial_\xi^{\alpha+r}\psi(\xi+\theta\tilde\eta)}{\langle\xi+\theta\tilde\eta\rangle^{m-2-|r|}}\langle\xi+\theta\tilde\eta\rangle^{m-2-|r|}$$

$$\le c_4\langle\tilde\eta\rangle^{-2k}\sum_{|r|=0}^{2k'}\frac{\partial_\xi^{\alpha+r}\psi(\xi+\theta\tilde\eta)}{\langle\xi+\theta\tilde\eta\rangle^{m-2-|r|}}\langle\xi\rangle^{m-2-|r|}\langle\tilde\eta\rangle^{|m-2-|r||}.$$

Hence,

$$c\int_{\mathbb{R}^d_\xi}\Big|\int_{\mathbb{R}^d_\eta}\langle D_\eta\rangle^{2l'}\Big(\frac{\langle\xi+\eta\rangle}{\langle\eta\rangle^{2l}}\Big)d\eta\int_0^1(1-\theta)^2\int_{\mathbb{R}^d_{\tilde\eta}}\langle D_{\tilde\eta}\rangle^{2k'}\Big[\frac{\partial_\xi^\alpha\psi(\xi+\theta\tilde\eta)}{\langle\tilde\eta\rangle^{2k}}\Big]d\tilde\eta d\theta\mathcal{F}(\varphi_2 v)(\xi)\Big|đ\xi$$

$$\le c_5\int_{\mathbb{R}^d_{\tilde\eta}}\langle\tilde\eta\rangle^{-2k}\int_{\mathbb{R}^d_\xi}\sum_{|r|=0}^{2k'}\frac{|\partial_\xi^{\alpha+r}\psi(\xi+\theta\tilde\eta)|}{\langle\xi+\theta\tilde\eta\rangle^{m-2-|r|}}\langle\xi\rangle^{m-2-|r|+1}\langle\tilde\eta\rangle^{|m-2-|r||}|\mathcal{F}(\varphi_2 v)(\xi)|đ\xi d\tilde\eta$$

$$\le c_6|\psi|_{s^m_{q,2d+4}}\|\langle\xi\rangle^{m-1}|\mathcal{F}(\varphi_2 v)(\xi)|\|_{L^p}\le c_6|\psi|_{s^m_{q,N}}\|\langle\xi\rangle^{m-1}\mathcal{F}(\varphi_2 v)(\xi)\|_{L^p}<\infty.$$

We have proved that $T_pT_{\varphi_2}: H^q_m(\mathbb{R}^d)\to H^q_1(\mathbb{R}^d)$. It remains to prove that $T_{p_\nu}T_{\varphi_2}\to T_pT_{\varphi_2}$ in norm, as $\nu\to\infty$. We have

$$\|T_{p_\nu-p}(\varphi_2 v)\|_{L^q}=\Big\|\Big(\phi\Big(\frac{x}{\nu}\Big)-1\Big)T_p(\varphi_2 v)\Big\|_{L^q}$$

$$\le\Big\|\Big(\phi\Big(\frac{x}{\nu}\Big)-1\Big)T_{p_1}(\varphi_2 v)\Big\|_{L^q}+\Big\|\Big(\phi\Big(\frac{x}{\nu}\Big)-1\Big)T_{p_2}(\varphi_2 v)\Big\|_{L^q}.$$

Then, for $\Big\|\Big(\phi\Big(\frac{x}{\nu}\Big)-1\Big)T_{p_1}(\varphi_2 v)\Big\|_{L^q}$ we apply the same procedure as in the proof of Theorem 6, part 2 to conclude that $\Big\|\Big(\phi\Big(\frac{x}{\nu}\Big)-1\Big)T_{p_1}(\varphi_2 v)\Big\|_{L^q}\to 0,\nu\to\infty$. For $\Big\|\Big(\phi\Big(\frac{x}{\nu}\Big)-1\Big)T_{p_2}(\varphi_2 v)\Big\|_{L^q}$ we can apply previous proof and obtain that $\Big\|\Big(\phi\Big(\frac{x}{\nu}\Big)-1\Big)T_{p_2}(\varphi_2 v)\Big\|_{L^q}\to 0,\nu\to\infty$, as well. ❑

Corollary 16. *Let $\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R}^d)$, $\psi\in s^m_{q,N}$, $1\le q\le 2$, $N\ge 2d+4$, $m,s\in\mathbb{R}$. Then, $(T_{\varphi_1}\mathcal{A}_\psi-\mathcal{A}_\psi T_{\varphi_1})T_{\varphi_2}$ is a compact operator from $H^q_{m+s}(\mathbb{R}^d)$ to $H^q_s(\mathbb{R}^d)$.*

Proof: Proof of Theorem 6, part (2) implies that $\mathcal{A}_\psi T_\varphi$ is a compact operator from $H^q_{m+s}(\mathbb{R}^d)$ to $H^q_{s-\varepsilon}(\mathbb{R}^d)$, for any $\varepsilon>0$ and $\varphi\in\mathcal{S}(\mathbb{R}^d)$. Indeed, if $\psi\in s^m_{q,N}$, then $\langle\xi\rangle^s\psi\in s^{m+s}_{q,N}$. Therefore, if $v\in H^q_{m+s}(\mathbb{R}^d)$, then $\|\mathcal{A}_\psi(\varphi v)\|_{H^q_s}=\|\mathcal{A}_{\psi\langle\xi\rangle^s}(\varphi v)\|_{L^q}<\infty$. Hence, for weakly convergent sequence $v_n\rightharpoonup 0$ in $H^q_{m+s}(\mathbb{R}^d)$ we have that

$$\|\mathcal{A}_\psi(\varphi v_n)\|_{H^q_{s-\varepsilon}}\le\|\mathcal{A}_\psi(\varphi v_n)\|_{H^q_s}=\|\mathcal{A}_{\psi\langle\xi\rangle^s}(\varphi v_n)\|_{L^q}\to 0,n\to\infty,$$

which follows from Theorem 6, part (2), i.e. $T_{\psi\langle\xi\rangle^s}T_\varphi$ is a compact operator from $H^q_{m+s}(\mathbb{R}^d)$ into $H^q_{-\varepsilon}$, if $\langle\xi\rangle^s\psi\in s^{m+s}_{q,N}$. If we apply the proof of Theorem 15, we can conclude the compactness of $(T_{\varphi_1}\mathcal{A}_\psi-\mathcal{A}_\psi T_{\varphi_1})T_{\varphi_2}$ from $H^q_{m+s}(\mathbb{R}^d)$ to $H^q_s(\mathbb{R}^d)$. ❑

Theorem 17. *Let $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$, $v_n \rightharpoonup 0$ in $H^q_m(\mathbb{R}^d)$, $1 < q \le 2$, $m \in \mathbb{R}$. Then, up to subsequences, there exists $\mu \in (\mathcal{S}(\mathbb{R}^d)\hat{\otimes} s^m_{q,N})'$ such that for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and all $\psi \in s^m_{q,N}$,*

$$\lim_{n\to\infty} \langle \varphi_1 u_n, \mathcal{A}_{\overline{\psi}}(\varphi_2 v_n)\rangle = \langle \mu, \varphi_1\overline{\varphi_2} \otimes \psi\rangle.$$

Proof: As in the proof of Theorem 6, for $\psi \in s^m_{q,N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\|\mathcal{A}_\psi(\varphi v_n)\|_{L^q} = \Big(\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} e^{ix\xi}\psi(\xi)\mathcal{F}(\varphi v_n)(\xi)d\xi\Big|^q dx\Big)^{\frac{1}{q}} =$$

$$\le c|\psi|_{s^m_{q,N}} \sum_{|r|=0}^{N} |a_r| \sum_{\alpha+\beta=r} \binom{r}{\alpha} \|(1+|\xi|^2)^{\frac{m-|\alpha|}{2}}\mathcal{F}((-ix)^\beta \varphi v_n)(\xi)\|_{L^p},$$

$$\le c|\psi|_{s^m_{q,N}} \Big\|\mathcal{F}(h^\beta)(\xi)(1+|\xi|^2)^{\frac{|m-|\alpha||}{2}}\Big\|_{L^1} < \infty.$$

Therefore,

$$|\mu_n(\varphi,\psi)| := \Big|\int_{\mathbb{R}^d} u_n \overline{\mathcal{A}_{\overline{\psi}}(\varphi v_n)} dx\Big| \le \|u_n\|_{L^p}\|\mathcal{A}_{\overline{\psi}}(\varphi v_n)\|_{L^q} \le$$

$$\le c|\psi|_{s^m_{q,N}} \Big\|\mathcal{F}(h^\beta)(\xi)(1+|\xi|^2)^{\frac{|m-|\alpha||}{2}}\Big\|_{L^1} \le c|\psi|_{s^m_{q,N}} |\mathcal{F}(h^\beta)(\xi)(1+|\xi|^2)^{\frac{|m-|\alpha||}{2}}|_{d+1,\mathcal{S}}$$

$$\le c_1|\psi|_{s^m_{q,N}}|\varphi|_{k_0,\mathcal{S}},$$

for some $k_0 \in \mathbb{N}$. Further on, the proof is analogous to the proof of Theorem 10 and we use commutation result, i.e. Theorem 15. ❑

Corollary 18. *Let $u_n \rightharpoonup 0$ in $H^p_{-s}(\mathbb{R}^d)$, $v_n \rightharpoonup 0$ in $H^q_{m+s}(\mathbb{R}^d)$, $1 < q \le 2$, $m, s \in \mathbb{R}$. Then, up to subsequences, there exists $\mu \in (\mathcal{S}(\mathbb{R}^d)\hat{\otimes} s^m_{q,N})'$ such that for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and all $\psi \in s^m_{q,N}$,*

$$\lim_{n\to\infty} \langle \varphi_1 u_n, \mathcal{A}_{\overline{\psi}}(\varphi_2 v_n)\rangle = \langle \mu, \varphi_1\overline{\varphi_2} \otimes \psi\rangle.$$

Proof: We define a sequence of functionals $\mu_n(\varphi,\psi) = \langle \mathcal{A}_{\langle\xi\rangle^{-s}}(u_n), \overline{\mathcal{A}_{\overline{\langle\xi\rangle^s\psi}}(\varphi v_n)}\rangle$. Then the claim follows from Theorem 17 and Corollary 16 . ❑

5. Applications

Let $u_n \rightharpoonup 0$ in $H^p_{-s}(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 < p < \infty$ such that the following sequence of equations is satisfied

$$\sum_{|\alpha|\le k} A_\alpha(x)\partial^\alpha u_n(x) = g_n(x), \tag{13}$$

where $A_\alpha \in \mathcal{S}(\mathbb{R}^d)$ and $(g_n)_n$ is a sequence of temperate distributions such that

$$\varphi g_n \to 0 \text{ in } H^p_{-s-k}(\mathbb{R}^d), \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{14}$$

The following theorem holds.

Theorem 19. *Let $u_n \rightharpoonup 0$ in $H^p_s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 < p < \infty$. Then, $\varphi u_n \to 0$ strongly in $H^p_{s-\varepsilon}(\mathbb{R}^d)$ for any $\varepsilon > 0$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.*

Proof: Denote by B_l the open ball centered at the origin with radius $l \in \mathbb{N}$. Rellich lemma implies that $H^p_s(B_l)$ is compactly embedded in $H^p_{s-\varepsilon}(B_l)$, for any $\varepsilon > 0$. Since $u_n \rightharpoonup 0$ in $H^p_s(B_l)$, by the diagonalization procedure, we can extract a subsequence (not relabeled) such that for all $l \in \mathbb{N}$

$$\varphi u_n \to 0 \text{ in } H^p_{s-\varepsilon}(B_l), \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{15}$$

Choose smooth cutoff functions χ_l such that $\chi_l(x) = 1$ for $x \in B_l$ and $\chi_l(x) = 0$ for $x \in \mathbb{R}^d \backslash B_{l+1}$. Then, $\varphi = \chi_l \varphi + (1 - \chi_l)\varphi$ and

$$\|\varphi u_n\|_{H^p_{s-\varepsilon}} \leq \|\chi_l \varphi u_n\|_{H^p_{s-\varepsilon}} + \|(1-\chi_l)\varphi u_n\|_{H^p_{s-\varepsilon}} \leq$$
$$\leq \|\chi_l \varphi u_n\|_{H^p_{s-\varepsilon}} + \sup_{|x|>l} |\varphi|_{k_0} \|u_n\|_{H^p_s},$$

where by $|\varphi|_{k_0}$ is denoted semi-norm for function $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Notice that (15) implies $\|\chi_l \varphi u_n\|_{H^p_{s-\varepsilon}} \to 0$ as $n \to \infty$. Since $u_n \rightharpoonup 0$ in $H^p_s(\mathbb{R}^d)$, there is a constant $M > 0$ such that $\|u_n\|_{H^p_s} \leq M$. Let $\varepsilon > 0$. Since $\varphi \in \mathcal{S}(\mathbb{R}^d)$, there exists $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$ we have estimate $\sup_{|x|>l} |\varphi|_{k_0} < \varepsilon/2M$. Hence, for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|\varphi u_n\|_{H^p_{s-\varepsilon}} < \varepsilon$, for $n > n_0$, i.e. $\varphi u_n \to 0$ in $H^p_{s-\varepsilon}(\mathbb{R}^d)$, which completes the proof. ❑

Lemma 20. *Let* (13) *and* (14) *hold. Then, there exists a sequence* (f_n) *in* $H^p_{-s-k}(\mathbb{R}^d)$ *such that*

$$\sum_{|\alpha|=k} \partial^\alpha (A_\alpha(x) u_n(x)) = f_n(x) \tag{16}$$

and

$$\varphi f_n \to 0 \textit{ in } H^p_{-s-k}(\mathbb{R}^d), \textit{ for every } \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{17}$$

Proof: We rewrite equation (13) in the divergence form,

$$\sum_{|\alpha|=k} \partial^\alpha (A_\alpha(x) u_n(x)) = g_n(x) - \sum_{|\alpha|<k} A_\alpha(x) \partial^\alpha u_n(x) + \sum_{|\alpha|=k} \sum_{\beta<\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} A_\alpha \partial^\beta u_n(x).$$

Put

$$f_n(x) := g_n(x) - \sum_{|\alpha|<k} A_\alpha(x) \partial^\alpha u_n(x) + \sum_{|\alpha|=k} \sum_{\beta<\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} A_\alpha \partial^\beta u_n(x).$$

Since $u_n \rightharpoonup 0$ in $H^p_{-s}(\mathbb{R}^d)$, then $\partial^\alpha u_n \rightharpoonup 0$ in $H^p_{-s-|\alpha|}(\mathbb{R}^d)$. Therefore, by Theorem 19, $\varphi \partial^\alpha u_n \to 0$ in $H^p_{-s-|\alpha|-\varepsilon}(\mathbb{R}^d)$, for any $\varepsilon > 0$. Choosing $\varepsilon = k - |\alpha|$, which is possible because $|\alpha| < k$, we conclude that $\varphi \partial^\alpha u_n \to 0$ in $H^p_{-s-k}(\mathbb{R}^d)$. Hence, $\varphi f_n \to 0$ in $H^p_{-s-k}(\mathbb{R}^d)$, as we claimed. ❑

Theorem 21. *Let* $u_n \rightharpoonup 0$ *in* $H^p_{-s}(\mathbb{R}^d)$, $s \in \mathbb{R}$, *satisfies (13), (14) and* $\psi \in s^m_{\infty,N}$. *Then, for any* $v_n \rightharpoonup 0$ *in* $H^q_{s+m}(\mathbb{R}^d)$ *, the corresponding distribution* $\mu_\psi \in \mathcal{S}'(\mathbb{R}^d)$ *satisfies*

$$\sum_{|\alpha|\leq k} A_\alpha(x) \frac{\xi^\alpha}{\langle \xi \rangle^k} \mu_\psi = 0 \quad \textit{in } \mathcal{S}'(\mathbb{R}^d). \tag{18}$$

Moreover, if $\psi = \langle \xi \rangle^m$ *and (18) implies* $\mu_{\langle \xi \rangle^m} = 0$ *we have the strong convergence* $\theta u_n \to 0$, *in* $H^p_{-s}(\mathbb{R}^d)$, *for every* $\theta \in \mathcal{S}(\mathbb{R}^d)$.

Proof: Let $v_n \rightharpoonup 0$ in $H^q_{s+m}(\mathbb{R}^d)$, $\varphi_1 \in \mathcal{S}(\mathbb{R}^d)$, $\varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and $\psi \in s^m_{\infty,N}$. We have to prove that, up to a subsequence,

$$\lim_{n\to\infty} \sum_{|\alpha|<k} \left\langle u_n A_\alpha \varphi_1 \,,\, \mathcal{A}_{\bar\Psi_\alpha}(\varphi_2 v_n)\right\rangle + \lim_{n\to\infty} \sum_{|\alpha|=k} \left\langle u_n A_\alpha \varphi_1 \,,\, \mathcal{A}_{\bar\Psi_\alpha}(\varphi_2 v_n)\right\rangle, \tag{19}$$

$$= \sum_{|\alpha|<k} \left\langle \mu, A_\alpha(x)\varphi_1\varphi_2 \frac{\xi^\alpha}{\langle\xi\rangle^k}\psi(\xi)\right\rangle + \sum_{|\alpha|=k} \left\langle \mu, A_\alpha\varphi_1\varphi_2 \frac{\xi^\alpha}{\langle\xi\rangle^k}\psi(\xi)\right\rangle = 0,$$

where $\Psi_\alpha = \dfrac{\xi^\alpha}{\langle\xi\rangle^k}\psi(\xi)$. Since $\Psi_\alpha \in s^{m+|\alpha|-k}_{\infty,N}$, it follows from Theorem 6 that $\mathcal{A}_{\bar\Psi_\alpha}$ is a compact operator from $H^q_{s+m}(\mathbb{R}^d)$ into $H^q_{s-|\alpha|+k-\varepsilon}(\mathbb{R}^d)$, for any $\varepsilon > 0$. Then $\mathcal{A}_{\bar\Psi_\alpha}(\varphi_2 v_n) \to 0$ in $H^q_{s-|\alpha|+k-\varepsilon}(\mathbb{R}^d)$, for any $\varepsilon > 0$. Therefore, when $|\alpha| < k$ we can choose $\varepsilon = k - |\alpha|$ and this implies $\mathcal{A}_{\bar\Psi_\alpha}(\varphi_2 v_n) \to 0$ strongly in $H^q_s(\mathbb{R}^d)$. Since $u_n \rightharpoonup 0$ in $H^p_{-s}(\mathbb{R}^d)$, we conclude that

$$\lim_{n\to\infty} \sum_{|\alpha|<k} \left\langle u_n A_\alpha \varphi_1 \,,\, \mathcal{A}_{\bar\Psi_\alpha}(\varphi_2 v_n)\right\rangle = 0.$$

It remains to prove that $\lim\limits_{n\to\infty} \sum\limits_{|\alpha|=k} \left\langle u_n A_\alpha \varphi_1 \,,\, \mathcal{A}_{\bar\Psi_\alpha}(\varphi_2 v_n)\right\rangle = 0$. We will prove that

$$\lim_{n\to\infty} \sum_{|\alpha|=k} \left\langle u_n A_\alpha \,,\, \mathcal{A}_{\bar\Psi_\alpha}(\varphi v_n)\right\rangle = 0,$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Since we can write $\varphi = \varphi_1\bar\varphi_2$ for $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, Theorem 8 implies

$$\lim_{n\to\infty} \sum_{|\alpha|=k} \left\langle u_n A_\alpha \,,\, \mathcal{A}_{\bar\Psi_\alpha}(\varphi v_n)\right\rangle = \lim_{n\to\infty} \sum_{|\alpha|=k} \left\langle u_n A_\alpha \varphi_1 \,,\, \mathcal{A}_{\bar\Psi_\alpha}(\varphi_2 v_n)\right\rangle.$$

Since $\mathcal{A}_{\bar\Psi_\alpha} = \mathcal{A}_{\frac{\xi^\alpha}{\langle\xi\rangle^k}} \circ \mathcal{A}_\psi$ and $\mathcal{A}_{\frac{\xi^\alpha}{\langle\xi\rangle^k}} = \partial^\alpha \mathcal{A}_{\langle\xi\rangle^{-k}}$, it follows that

$$\lim_{n\to\infty} \sum_{|\alpha|=k} \left\langle u_n A_\alpha \,,\, \mathcal{A}_{\bar\Psi_\alpha}(\varphi v_n)\right\rangle = \lim_{n\to\infty} \sum_{|\alpha|=k} (-1)^{|\alpha|} \left\langle \partial^\alpha_x(u_n A_\alpha) \,,\, \mathcal{A}_{\langle\xi\rangle^{-k}\psi}(\varphi v_n)\right\rangle.$$

Then $\mathcal{A}_{\langle\xi\rangle^{-k}\psi}(\varphi v_n) \in H^q_{s+k}(\mathbb{R}^d)$, and because of the Lemma 20, i.e. (17), we have that

$$\lim_{n\to\infty} \sum_{|\alpha|=k} \left\langle u_n A_\alpha \varphi_1 \,,\, \mathcal{A}_{\bar\Psi_\alpha}(\varphi_2 v_n)\right\rangle = 0.$$

Therefore, we have proved (18).

If $\mu_{\langle\xi\rangle^m} = 0$, then Corollary 14 implies $\theta u_n \to 0$ in $H^p_{-s}(\mathbb{R}^d)$, for every $\theta \in \mathcal{S}(\mathbb{R}^d)$. ❏

Theorem 22. *Let $u_n \rightharpoonup 0$ in $H^p_{-s}(\mathbb{R}^d)$, $v_n \rightharpoonup 0$ in $H^q_{s+m}(\mathbb{R}^d)$ and $\sigma \in s^r_{\infty,N}$, $\psi \in s^m_{\infty,N}$, $s, m, r \in \mathbb{R}$. Assume that $\mathcal{A}_\sigma u_n = f_n \to 0$ strongly in $H^p_{-s-r}(\mathbb{R}^d)$. Then*

$$\mu_{\frac{\sigma(\xi)}{\langle\xi\rangle^r}\psi} = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{20}$$

Proof: The assumption implies that $\mathcal{A}_{\langle\xi\rangle^{-r}}(\mathcal{A}_\sigma u_n) = \mathcal{A}_{\langle\xi\rangle^{-r}}(f_n) \to 0$ in $H^p_{-s}(\mathbb{R}^d)$. It follows that

$$\langle \mathcal{A}_{\langle\xi\rangle^{-r}}(\mathcal{A}_\sigma u_n), \mathcal{A}_{\overline{\psi}}(\varphi v_n)\rangle \to 0,\ n\to\infty,\ \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Using the factorization property ($\varphi = \overline{\varphi_1}\varphi_2$) and Corollary 9, we have that for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$,

$$\lim_{n\to\infty} \langle \varphi_1 u_n, \mathcal{A}_{\overline{\sigma(\xi)\langle\xi\rangle^{-r}\psi}}(\varphi_2 v_n)\rangle = 0,$$

i.e for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ H-distribution $\mu_{\frac{\sigma(\xi)}{\langle\xi\rangle^r}\psi}$ satisfies $\langle \mu_{\frac{\sigma(\xi)}{\langle\xi\rangle^r}\psi}, \varphi_1\overline{\varphi_2}\rangle = 0$. Therefore, (20) holds. ❑

Appendix A. L^p-boundedness theorem

Theorem 23. *Let N be an integer such that $N > 2d$, $1 < p < \infty$ and $T : S^0_N \times L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ be defined by*

$$T(\sigma, u) = T_\sigma u.$$

Then, T is a continuous bilinear operator and there exists $c_N > 0$ such that the following estimate holds

$$\|T_\sigma u\|_{L^p} \le c_N |\sigma|_{S^0_N} \|u\|_{L^p}. \tag{21}$$

Proof: Steps of the proof are the same as in the proof of Theorem 10.7 from [16]. For the sake of completeness, we will focus ourselves on constants appearing in the estimates of Theorem 10.7, which will imply continuity with respect to σ. We represent $\mathbb{R}^d$ as a union of cubes, i.e. $\mathbb{R}^d = \bigcup_{l\in\mathbb{Z}^d} Q_l$, where Q_l is the cube with centre at l, with edges parallel to coordinate axes and of length one. Further on, we introduce $\eta \in C^\infty_c(\mathbb{R}^d)$ such that $\eta(x) = 1$ for $x \in Q_0$ and define $\sigma_l(x,\xi) = \eta(x-l)\sigma(x,\xi)$, $x, \xi \in \mathbb{R}^d$, $l \in \mathbb{Z}^d$. Then $T(\sigma_l, \cdot) = T_{\sigma_l} = \eta(x-l)T_\sigma$ and

$$\int_{Q_l} |(T_\sigma\varphi)(x)|^p dx \le \int_{\mathbb{R}^d} |(T_{\sigma_l}\varphi)(x)|^p dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{22}$$

Next,

$$(T_{\sigma_l}\varphi)(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\lambda}\Big[(2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\xi}\hat{\sigma}_l(\lambda,\xi)\hat{\varphi}(\xi)d\xi\Big]d\lambda, \tag{23}$$

where $\hat{\sigma}_l(\lambda,\xi) = \int_{\mathbb{R}^d} e^{-i\lambda x}\sigma_l(x,\xi)dx$ for $\lambda, \xi \in \mathbb{R}^d$. The proof of Lemma 10.9 in [16] gives that for all α, $\beta \in \mathbb{N}^d_0$,

$$|(-i\lambda)^\beta \partial^\alpha_\xi \hat{\sigma}_l(\lambda,\xi)| \le c_\beta \langle\xi\rangle^{-|\alpha|} \sup_{\gamma\le\beta, x,\xi\in\mathbb{R}^d} |\partial^\alpha_\xi \partial^\gamma_x \sigma(x,\xi)|\langle\xi\rangle^{|\alpha|}.$$

Moreover, for all $\alpha \in \mathbb{N}^d_0$ and for all positive integers n there is a $c_n > 0$ such that

$$|\partial^\alpha_\xi \hat{\sigma}_l(\lambda,\xi)| \le c_n\langle\xi\rangle^{-|\alpha|}(1+|\lambda|)^{-n}\Big(\sup_{|\beta|\le n, x,\xi\in\mathbb{R}^d} |\partial^\alpha_\xi\partial^\beta_x\sigma(x,\xi)|\langle\xi\rangle^{|\alpha|}\Big). \tag{24}$$

Hence, for any integer $N > d/2$ we conclude from (24) that $|\partial^\alpha_\xi \hat{\sigma}_l(\lambda,\xi)| \le B|\xi|^{-|\alpha|}$ for $|\xi| > \xi_0$ and $|\alpha| \le N$, where

$$B = c_N(1+|\lambda|)^{-N} \max_{|\alpha|,|\beta|\le N} \sup_{x,\xi\in\mathbb{R}^d} |\partial^\alpha_\xi \partial^\beta_x \sigma(x,\xi)|\langle\xi\rangle^{|\alpha|}.$$

Therefore, we can use Theorem 1 with $\psi(\xi) = \hat{\sigma}_l(\lambda, \xi)$ and $B = c_N(1+|\lambda|)^{-N}|\sigma|_{S^0_N}$ to conclude that the operator $(\tilde{T}_{l,\lambda}\varphi)(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\xi}\hat{\sigma}_l(\lambda, \xi)\hat{\varphi}(\xi)d\xi$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ can be extended to a bounded operator on $L^p(\mathbb{R}^d)$ so that with a suitable $c > 0$

$$\|\tilde{T}_{l,\lambda}\varphi\|_p \le cc_N(1 + |\lambda|)^{-N}|\sigma|_{S^0_N}\|\varphi\|_p,\ \varphi \in L^p(\mathbb{R}^d). \tag{25}$$

Then, by (23), there exists (new) $c > 0$ such that

$$\|T_{\sigma_l}\varphi\|_p \le cc_N|\sigma|_{S^0_N}\|\varphi\|_p\int_{\mathbb{R}^d}(1 + |\lambda|)^{-N}d\lambda. \tag{26}$$

Then, (26) and (22), for integer $N > d$, imply that there exists $c > 0$, independent on l, so that

$$\int_{Q_l}|(T_\sigma\varphi)(x)|^p dx \le cc_N^p(|\sigma|_{S^0_N})^p\|\varphi\|_p^p,\ \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{27}$$

According to [16], Lemma 10.10, for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ vanishing in a neighborhood of fixed $x \in \mathbb{R}^d$, we have that $(T_\sigma\varphi)(x) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} K(x, x - z)\varphi(z)dz$, where $K(x, z) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{iz\xi}\sigma(x, \xi)d\xi, x, z \in \mathbb{R}^d$, in the sense of distributions. Following the proof of Lemma 10.10, we have that for every integer $k > d$ there exists $C_k > 0$ such that

$$|K(x, z)| \le C_k|z|^{-k}|\sigma|_{S^0_k},\ z \ne 0. \tag{28}$$

Next, we construct cubes Q^*_l and Q^{**}_l as in the proof of Theorem 10.7 in [16]. More precisely, Q^{**}_l is the double of Q_l and Q^*_l has the same center l as Q_l and Q^{**}_l and $Q_l \subset Q^*_l \subset Q^{**}_l$. Then $\psi \in C^\infty_c(\mathbb{R}^d)$ is introduced so that its support is in Q^{**}_l, $0 \le \psi(x) \le 1$ and $\psi(x) = 1$ in a neighborhood of Q^*_l. Then we write $T_\sigma\varphi = T_\sigma\varphi_1 + T_\sigma\varphi_2$, where $\varphi_1 = \psi\varphi$ and $\varphi_2 = (1 - \psi)\varphi$. We introduce notation $I_l = \int_{Q_l}|(T_\sigma\varphi)(x)|^p dx$ and $J_l = \int_{Q_l}|(T_\sigma\varphi_2)(x)|^p dx$. Using (27) we get

$$I_l \le c2^p c_N^p(|\sigma|_{S^0_N})^p\|\varphi_1\|_p^p + 2^p J_l. \tag{29}$$

By (28) we have that for every integer $k > d$ there is a $C > 0$ such that

$$|(T_\sigma\varphi_2)(x)| \le CC_k|\sigma|_{S^0_k}\int_{\mathbb{R}^d\backslash Q^*_l}|x - z|^{-k}|\varphi_2(z)|dz,\ x \in Q_l, z \in \mathbb{R}^d\backslash Q^*_l.$$

Next, following [16] ((10.13), (10.14) and (10.15) , Theorem 10.7) and taking $1/p + 1/q = 1$, we obtain, with a new constant $C > 0$:

$$|(T_\sigma\varphi_2)(x)| \le CC_k|\sigma|_{S^0_k}\int_{\mathbb{R}^d\backslash Q^*_l}\frac{(\mu + |x - z|)^{-k/2}|\varphi_2(z)|}{(\mu + |l - z|)^{\frac{k}{2}(\frac{1}{p}+\frac{1}{q})}}dz,$$

where $x \in Q_l,\ z \in \mathbb{R}^d\backslash Q^*_l,\ \mu = \sqrt{d}/2 + 1$. Then, by Minkowski's and Hölder's inequality:

$$\Big(\int_{Q_l}|T_\sigma\varphi_2|^p dx\Big)^{\frac{1}{p}} \le CC_k|\sigma|_{S^0_k}\Big(\int_{\mathbb{R}^d\backslash Q^*_l}\frac{dz}{(\mu + |l - z|)^{k/2}}\Big)^{\frac{1}{q}}\Big(\int_{\mathbb{R}^d\backslash Q^*_l}\frac{|\varphi_2(z)|^p dz}{(\mu + |l - z|)^{k/2}}\Big)^{\frac{1}{p}}$$

We conclude, with a new constant C and for $k/2 > d$, that

$$J_l \le CC_k^p(|\sigma|_{S^0_k})^p\int_{\mathbb{R}^d\backslash Q^*_l}\frac{|\varphi_2(z)|^p dz}{(\mu + |l - z|)^{k/2}}. \tag{30}$$

By (29) and (30), there exists $C_1 > 0$ such that:

$$\int_{Q_l} |(T_\sigma\varphi)(x)|^p dx \leq C_1 C_k^p (|\sigma|_{S_k^0})^p \Big(\int_{Q_l^{**}} |\varphi(x)|^p dx + \int_{\mathbb{R}^d \setminus Q_l^*} \frac{|\varphi_2(z)|^p dz}{(\mu + |l-z|)^{k/2}} \Big).$$

Summing over all $l \in \mathbb{Z}^d$, we get:

$$\int_{\mathbb{R}^d} |(T_\sigma(\varphi)(x))|^p dx \leq C_2 (|\sigma|_{S_k^0})^p \Big(1 + \sum_{l \in \mathbb{Z}^d} \frac{1}{(1+|l|)^{k/2}} \Big) \int_{\mathbb{R}^d} |\varphi(x)|^p dx.$$

Therefore, with $k = N > 2d$, we obtain the desired estimate:

$$\int_{\mathbb{R}^d} |(T_\sigma(\varphi)(x))|^p dx \leq C_N (|\sigma|_{S_N^0})^p \int_{\mathbb{R}^d} |\varphi(x)|^p dx.$$

Extending by density both sides to $u \in L^p(\mathbb{R}^d)$, we obtain (21). ❑

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