Defect distributions related to weakly convergent sequences in Bessel type spaces $H_\Lambda^{-s,p}$

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Abstract. We consider microlocal defect distributions associated to a weakly convergent sequences u_n in $H_\Lambda^{-s,p}$ and v_n in $H_\Lambda^{s+m,q}$ through the space of pseudo-differential operators with the symbols in $(s_\Lambda^{m,N+1})_0$. Symbols correspond to a weight function Λ determining a quasi-elliptic symbol. Results are applied to partial differential equations with symbols related to weights of the type Λ.

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1. Introduction

Our first aim in this paper is to study the defect distributions which correspond to the space of quasi-elliptic symbols which are determined by related weight functions, for example of the form

$$\Lambda := \sqrt{1 + \sum_{i=1}^{d} \xi_i^{2m_i}},\ \xi \in \mathbb{R}^d, \tag{1.1}$$

where $m = (m_1, \dots, m_d) \in \mathbb{N}^d$ and $\min_{1\leq i\leq d} m_i \geq 1$. In particular, $\langle\xi\rangle = \Big(1 + \sum_{i=1}^{d} \xi_i^2\Big)^{\frac{1}{2}}$ is a weight function of this form. We recall the properties of the spaces of symbols $M_{\rho,\Lambda}^m$, the spaces of multipliers $s_{\rho,\Lambda}^m$ and consider such symbols with the finite order of regularity and those which vanish at infinity. Then, by testing weakly convergent sequences in the corresponding weighted Bessel potential spaces $H_\Lambda^{-s,p}(\mathbb{R}^d), s \in \mathbb{R}, p \in (1,\infty)$, and their duals, we present consequences related to the introduced defect distributions.

Our second aim is the use of defect distributions in the analysis of a class of linear differential equations involving appropriate weights as symbols and prove the existence of strong distributional solutions for such equations.

Microlocal defect distributions (also called H-distributions) are introduced in [6] as an extension of H-measures (introduced in [12] and [19]) and further developed in [2], for weakly convergent sequences in Sobolev spaces, $W^{-k,p}(\mathbb{R}^d) = H^p_{-k}(\mathbb{R}^d)$, $k \in \mathbb{N}_0$, $p \in (1,\infty)$. Always, the motivation has been the existence of a solution for an equation with a sequence of weak solutions which corresponds to the sequence of approximating equations.

H-measures were applied to hyperbolic problems, in [1] as well as to parabolic problems in [4]. Fractional H-measures were introduced in [15] in order to treat problems with fractional derivatives. Classical H-measures were adapted for problems where all partial derivatives are of the same order. Parabolic variants are applicable to problems where the ratio between derivatives is a rational number, for example $1:2$ in [4] and $1:4$ in [8]. In [9] fractional H-measures with orthogonality property were introduced and application of localisation principle to fractional equation was presented.

Among many applications of the microlocal tools we emphasize possibility of testing strong convergence of weakly convergent sequences. Recall [2], Theorem 3.2:

Let $u_n \rightharpoonup 0$ in $W^{-k,p}(\mathbb{R}^d)$, $k \in \mathbb{N}_0, 1 < p < \infty$ and $q = \frac{p}{p-1}$. If for every sequence $v_n \rightharpoonup 0$ in $W^{k,q}(\mathbb{R}^d)$ the corresponding H-distribution is zero, then for every $\theta \in \mathcal{S}(\mathbb{R}^d)$, $\theta u_n \to 0$ strongly in $W^{-k,p}(\mathbb{R}^d)$.

(In the sequel, we skip "$n \to \infty$". Moreover, recall, $\mathcal{S}(\mathbb{R}^d)$ is the space of rapidly decreasing functions.) Similar theorem can be found in [3], for sequences in Bessel potential spaces. Recall [2] that an H-distribution μ is associated to a pair of sequences (u_n, v_n) in dual pairing $W^{-k,p} - W^{k,q}, k \in \mathbb{N}_0$, and acts on test functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\psi \in C^\kappa(\mathbb{S}^{d-1})$ in a sense that, up to a subsequences, for all test functions we obtain the following limit

$$\langle \mu, \varphi\psi \rangle := \lim_{n\to\infty} \langle \varphi_1 u_n, \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_n)} \rangle,$$

where $\mathcal{A}_\psi$ is a Fourier multiplier operator with symbol ψ and $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$. We have used the fact that any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ can be written in the form $\varphi = \varphi_1\varphi_2$, $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, cf. [18]. We have shown in [2] that a strong convergence of a weakly convergent sequence in $W^{-k,p}, k \in \mathbb{N}_0, p \in (1,\infty)$ can be tested on all weakly convergent sequences in the dual space $W^{k,q}$. Moreover, such a sequence can be tested on $W^{k+m,q} \subset W^{k,q}$, for $m \in \mathbb{N}$, but with the use of pseudo-differential operators of higher order m (cf. [3]). Also, in [3], results were given for sequences in $H^p_s(\mathbb{R}^d), s \in \mathbb{R},\ 1 < p < \infty$. These results were applied in [3] to solutions $u_n \in H^p_{-s}(\mathbb{R}^d)$ of linear equations of the type

$$\sum_{|\alpha|\le k} A_\alpha(x)\partial^\alpha u_n(x) = g_n(x),$$

with assumption that $\varphi g_n \to 0$ in $H^p_{-s-k}(\mathbb{R}^d)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Recent developments in hypoellipticity theory, cf. [7, 10, 16, 17], suggest the use of more general weight functions $\Lambda(\xi)$, instead of the usual one $\langle\xi\rangle$. Such weight functions are useful in various applications, they can be chosen in appropriate manner in order to get better estimates for solutions of Schrödinger type differential operator, cf. [7, 17].

In this paper we associate a microlocal defect distribution to a pair of sequences $u_n \in L^p(\mathbb{R}^d)$ and $v_n \in H^{s,q}_\Lambda(\mathbb{R}^d)$, where $H^{s,q}_\Lambda(\mathbb{R}^d)$ denotes the weighted Bessel space:

$$H^{s,q}_\Lambda(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) \mid \mathcal{F}^{-1}(\Lambda^s(\xi)\mathcal{F}u) \in L^q(\mathbb{R}^d)\}, \quad s \in \mathbb{R},\ q \in (1,\infty),$$

with a general weight function Λ given in Definition 2.1. It is a Banach space with respect to the norm

$$\|u\|_{H^{s,q}_\Lambda} := \|\mathcal{F}^{-1}(\Lambda(\xi)^s\mathcal{F}u)\|_{L^q}.$$

Here $\mathcal{F}$ denotes Fourier transform, i.e. $\mathcal{F}f(\xi) := \int_{\mathbb{R}^d} e^{-ix\xi} f(x)dx, \xi \in \mathbb{R}^d,\ f \in \mathcal{S}(\mathbb{R}^d)$.

An associated distribution, denoted by μ and called H_Λ-distribution, acts on $\mathcal{S}(\mathbb{R}^d)\hat{\otimes}(s^m_{\Lambda,N+1})_0$, the completion of the tensor product of spaces of test functions in the Schwartz space (regarding to the space variable x) and Hörmander type symbol classes $(s^m_{\Lambda,N+1})_0, m \in \mathbb{R}$, which will be introduced below, adapted to L^p boundedness property (regarding to the frequency variable ξ). Since $\mathcal{S}(\mathbb{R}^d)$ is nuclear the completion is the same for the π and the ε topologies and therefore we use notation $\mathcal{S}(\mathbb{R}^d)\hat{\otimes}(s^m_{\Lambda,N+1})_0$. Our main interest in this paper is to apply results to linear partial differential equations.

The paper is organized as follows. In Section 2 we introduce notation and definition of weight function. Symbol classes and multipliers with finite regularity are introduced and results regarding boundedness of pseudo-differential operators on $H^{s,p}_\Lambda(\mathbb{R}^d)$ are given, where $s \in \mathbb{R}$, $1 < p < \infty$. In Section 3 we prove compactness of commutator, then in Section 4 existence of H_Λ-distributions. In Section 4 we also analyze possible strong convergence of weakly convergent sequence in Theorem 4.5 and in Corollary 4.6. Finally, Section 5 is devoted to applications of previous results to linear partial differential equations.

2. Weight functions, symbols, multipliers

In this section all the definitions and assertions are taken from [7, 10, 11, 16, 17]. Only, we consider symbols and multipliers with the properties of their derivatives up to N, that is, with a limited regularity. Recall the definition of weight function:

Definition 2.1. ([7]) Positive function $\Lambda \in C^\infty(\mathbb{R}^d)$ is a weight function if the following assumptions are satisfied:

1. There exist positive constants $1 \le \mu_0 \le \mu_1$ and $c_0 < c_1$ such that
$$c_0\langle\xi\rangle^{\mu_0} \le \Lambda(\xi) \le c_1\langle\xi\rangle^{\mu_1}, \quad \xi \in \mathbb{R}^d;$$

2. There exists $\omega \geq \mu_1$ such that for any $\alpha \in \mathbb{N}_0^d$ and $\gamma \in \mathbb{K} \equiv \{0,1\}^d$
$$|\xi^\gamma \partial^{\alpha+\gamma}\Lambda(\xi)| \leq C_{\alpha,\gamma}\Lambda(\xi)^{1-\frac{1}{\omega}|\alpha|}, \quad \xi \in \mathbb{R}^d.$$
Constant ω is called the order of Λ.

If Λ is a weight function, then (cf. [7], p. 30) there exists $C > 0$ such that
$$\Lambda(z)^m \leq C\Lambda(\zeta)^m \langle z-\zeta\rangle^{|m|\omega}, \quad m \in \mathbb{R},\ z,\zeta \in \mathbb{R}^d. \tag{2.1}$$
It is well-known, if Λ is a weight function, then for any $m \in \mathbb{R}$, $\alpha \in \mathbb{N}_0^d, \gamma \in \mathbb{K}$,
$$|\xi^\gamma \partial_\xi^{\gamma+\alpha}\Lambda(\xi)^m| \leq C_{\alpha,\gamma}\Lambda(\xi)^{m-\frac{1}{\omega}|\alpha|}, \quad \xi \in \mathbb{R}^d.$$

We recall the well-known examples. Quasi-elliptic smooth functions $P_m = \Lambda$ and their powers, where Λ is given by (1.1), are examples of weight functions which satisfy conditions of Definition 2.1 (cf. [16]).

More general weights are defined by (cf. [11, 17])
$$\Lambda_{\mathcal{P}}(\xi) = \Big(\sum_{\alpha \in V(\mathcal{P})} \xi^{2\alpha}\Big)^{\frac{1}{2}},\ \xi \in \mathbb{R}^d,$$
where $\mathcal{P}$ is a given complete polyhedron with the set of vertices $V(\mathcal{P})$. Recall that a complete polyhedron is a convex polyhedron $\mathcal{P} \subset (\mathbb{R}_+ \cup \{0\})^d$ with the following properties: $V(\mathcal{P}) \subset \mathbb{N}_0^d$, $0 \in V(\mathcal{P})$, $V(\mathcal{P}) \neq \{0\}$, $N_0(\mathcal{P}) = \{e_1,\dots,e_d\}$ and $N_1(\mathcal{P}) \subset \mathbb{R}_+^d$. Here
$$\mathcal{P} = \{z \in \mathbb{R}^d : \nu\cdot z \geq 0,\ \ \forall \nu \in N_0(\mathcal{P})\} \cap \{z \in \mathbb{R}^d : \nu\cdot z \leq 1,\ \ \nu \in N_1(\mathcal{P})\},$$
and $N_0(\mathcal{P})$ and $N_1(\mathcal{P}) \subset \mathbb{R}^d$ are finite sets such that for all $\nu \in N_0(\mathcal{P})$, $|\nu| = 1$. We have that $\langle\xi\rangle^{\mu_0} \leq C\Lambda(\xi) \leq C_1\langle\xi\rangle^{\mu_1},\ \ \xi \in \mathbb{R}^d$, with $\mu_0 = \min_{\alpha \in V(\mathcal{P})\setminus\{0\}} |\alpha|$ and $\mu_1 = \max_{\alpha \in V(\mathcal{P})} |\alpha|$. The formal order of $\mathcal{P}$ is given by $\omega = \max\Big\{\frac{1}{\nu_j} : j = 1\dots d, \nu \in N_1(\mathcal{P})\Big\}$. Notice that $1 \leq \mu_0 \leq \mu_1 \leq \omega$.

2.1. Symbols

First, we recall the classical notions and assertions. Then we list the definitions of the symbols with finite regularity for which the same estimates hold, but with the careful choice of the regularity. Such results, concerning the commutator lemma are given in the next section.

Let Λ be a weight function of order ω, $m \in \mathbb{R}$ and $\rho \in (0,1/\omega]$. Spaces $S^m_{\rho,\Lambda}$, $\rho \in (0,1/\omega]$ and $S^m_\Lambda = S^m_{1/\omega,\Lambda}$ were defined in quoted papers (cf. [10, p. 88]). We recall that the spaces of Λ-symbols are connected with standard Hörmander's spaces $S^m_{\tilde\rho,\delta}$, $m \in \mathbb{R},\ 0 \leq \delta \leq \tilde\rho \leq 1$:
$$S^h_{\rho\mu_1,0} \subset S^m_{\rho,\Lambda} \subset S^k_{\rho\mu_0,0},$$
where $h := \min\{m\mu_0, m\mu_1\}$, $k := \max\{m\mu_0, m\mu_1\}$ and $\rho \in (0,1/\omega]$. When $\mathcal{P}$ is the polyhedron with set of vertices $\{0\} \cup \{e_i : 1 \leq i \leq d\}$, then $\omega = 1$ and $S^m_{\rho,\Lambda} = S^m$, where $S^m = S^m_{1,0}$ is the standard Hörmander's space of symbols. In this case we have that $\Lambda_{\mathcal{P}}(\xi) = \langle\xi\rangle$.

Pseudo-differential operator T_a with a symbol $a \in S^m_{\rho,\Lambda}$ is defined in a usual manner,

$$T_a u(x) := \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\, đ\xi, \; x \in \mathbb{R}^d, \quad u \in \mathcal{S}(\mathbb{R}^d),$$

where $đ\xi = (2\pi)^{-d} d\xi$. In general, these operators are unbounded on $L^p(\mathbb{R}^d)$, $p \neq 2$. Namely, considering $S^m_{\tilde{\rho},\delta}$ spaces for $0 \leq \delta < \tilde{\rho} \leq 1$ it is known that pseudo-differential operators of order zero are L^2 bounded and the same is true when $\delta = \tilde{\rho} \neq 1$. In the case $\delta = \tilde{\rho} = 1$, L^2 continuity does not hold in general. When $m = 0$, $\tilde{\rho} = 1$ and $0 \leq \delta < 1$ we have L^p boundedness for $1 < p < \infty$. If $\tilde{\rho} \neq 1$, i.e. if $\tilde{\rho} < 1$ we do not have L^p boundedness in general (for more details see [10]).

Since, $S^0_{\rho,\Lambda} \subset S^0_{\rho\mu_0,0}$ and $\rho\mu_0 \leq \mu_0/\omega \leq 1$, operators with symbols in $S^0_{\rho,\Lambda}$ can be unbounded on L^p. The L^p-boundedness holds with the use of symbols $M^m_{\rho,\Lambda}$ (cf. [10, p. 88]). One has [11, Proposition 5.3]: Let $a \in M^m_{\rho,\Lambda}$, $m, s \in \mathbb{R}$, $\rho \in (0, 1/\omega]$ and $1 < p < \infty$. Then,

$$T_a : H^{s+m,p}_\Lambda(\mathbb{R}^d) \to H^{s,p}_\Lambda(\mathbb{R}^d)$$

is a linear, continuous operator.

Now we recall the definitions but with the differentiation up to $N \in \mathbb{N}$.

Definition 2.2. Let $m \in \mathbb{R}$, $\rho \in (0, 1/\omega]$ and $N \in \mathbb{N}_0$. We denote by $S^{m,N}_{\rho,\Lambda}$ the space of functions $a \in C^N(\mathbb{R}^{2d})$ such that for all $|\alpha|, |\beta| \leq N$,

$$|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)| \leq C_{\alpha,\beta} \Lambda(\xi)^{m-\rho|\alpha|}, \; x, \xi \in \mathbb{R}^d.$$

We denote by $M^{m,N}_{\rho,\Lambda}$ the space of functions $a \in C^N(\mathbb{R}^{2d})$ such that for every $\gamma \in \mathbb{K}$ and for all $|\alpha|, |\beta| \leq N$,

$$\xi^\gamma \partial^\gamma_\xi a(x,\xi) \in S^{m,N}_{\rho,\Lambda}. \tag{2.2}$$

As before, when $\rho = 1/\omega$ we denote $S^{m,N}_\Lambda = S^{m,N}_{1/\omega,\Lambda}$ and $M^{m,N}_\Lambda = M^{m,N}_{1/\omega,\Lambda}$.

Condition (2.2) is equivalent to

$$|\xi^\gamma \partial^{\gamma+\alpha}_\xi \partial^\beta_x a(x,\xi)| \leq C_{\alpha,\beta,\gamma} \Lambda(\xi)^{m-\rho|\alpha|}, \; x, \xi \in \mathbb{R}^d,$$

for all $|\alpha|, |\beta| \leq N$, $\gamma \in \mathbb{K}$. Then:

$$|a|_{M^{m,N}_{\rho,\Lambda}} := \max_{|\gamma|,\gamma\in\mathbb{K}} \max_{|\alpha|,|\beta|\leq N} \sup_{x,\xi\in\mathbb{R}^d} |\xi^\gamma \partial^{\gamma+\alpha}_\xi \partial^\beta_x a(x,\xi)| \Lambda(\xi)^{-m+\rho|\alpha|}$$

is the norm on $M^{m,N}_{\rho,\Lambda}$.

One can prove, as in Theorem 13 in [3], that T_a is a bounded operator if $N > 2d$.

Theorem 2.3. *Let $a \in M^{m,N}_{\rho,\Lambda}$, $N > 2d$, $s, m \in \mathbb{R}$, $\rho \in (0, 1/\omega]$ and $1 < p < \infty$. Then*

$$T_a : H^{s+m,p}_\Lambda(\mathbb{R}^d) \to H^{s,p}_\Lambda(\mathbb{R}^d)$$

is a linear, continuous operator and there exists $c_N > 0$ such that

$$\|T_a u\|_{H^{s,p}_\Lambda(\mathbb{R}^d)} \leq c_N |a|_{M^{m,N}_{\rho,\Lambda}} \|u\|_{H^{s+m,p}_\Lambda(\mathbb{R}^d)}.$$

As in [3], following [13], we define $M^{m,N}_{\rho,\Lambda,0}$ space.

Definition 2.4. Let $m \in \mathbb{R}, \rho \in (0, 1/\omega]$ and $N \in \mathbb{N}_0$. We denote by $M^{m,N}_{\rho,\Lambda,0}$ the space of functions $a \in C^N(\mathbb{R}^{2d})$ such that $a \in M^{m,N}_{\rho,\Lambda}$ and for every $\gamma \in \mathbb{K}$ and for all $|\alpha|, |\beta| \le N$,

$$|\xi^\gamma \partial_\xi^{\gamma+\alpha} \partial_x^\beta a(x,\xi)| \le c_{\alpha,\beta,\gamma}(x) \Lambda(\xi)^{m-\rho|\alpha|}, \; x, \xi \in \mathbb{R}^d \tag{2.3}$$

where $c_{\alpha,\beta,\gamma}(x)$ is a bounded function and $\lim_{|x|\to\infty} c_{\alpha,\beta,\gamma}(x) = 0$.

We introduce spaces of multipliers, following definitions of $S^{m,N}_{\rho,\Lambda}$ and $M^{m,N}_{\rho,\Lambda}$ spaces.

Definition 2.5. Let $m \in \mathbb{R}, \rho \in (0, 1/\omega]$, $N \in \mathbb{N}_0$. Then $s^{m,N}_{\rho,\Lambda}(\mathbb{R}^d)$ is the space of all $\psi \in C^N(\mathbb{R}^d)$ for which the norm

$$|\psi|_{s^{m,N}_{\rho,\Lambda}} := \max_{|\gamma|:\gamma\in\mathbb{K}} \max_{|\alpha|\le N} \sup_{\xi\in\mathbb{R}^d} |\xi^\gamma \partial_\xi^{\alpha+\gamma} \psi(\xi)| \Lambda(\xi)^{-m+\rho|\alpha|} < \infty.$$

If $\rho = 1/\omega$, then we denote $s^{m,N}_{\Lambda} = s^{m,N}_{1/\omega,\Lambda}$.

We will need the Lizorkin-Marcinkiewicz theorem:

Theorem 2.6. *([14]) Let ψ be a continuous function such that $\partial^\gamma \psi(\xi)$, $\xi \in \mathbb{R}^d$ are also continuous for all $\gamma \in \mathbb{K}$. If there exists $B > 0$ such that*

$$|\xi^\gamma \partial^\gamma \psi(\xi)| \le B, \quad \xi \in \mathbb{R}^d, \quad \gamma \in \mathbb{K},$$

then for every $1 < p < \infty$ there exists $C > 0$ depending only on p, B and d such that

$$\|\mathcal{A}_\psi u\|_{L^p} \le C\|u\|_{L^p}.$$

If $\psi \in s^{0,N}_{\rho,\Lambda}(\mathbb{R}^d)$, then $|\xi^\gamma \partial^\gamma \psi(\xi)| \le B$, $\gamma \in \mathbb{K}, \xi \ne 0$. Therefore, by Theorem 2.6, we have the following result.

Corollary 2.7. *Let $\psi \in s^{0,N}_{\rho,\Lambda}(\mathbb{R}^d)$,$\rho \in (0, 1/\omega]$, $N > d$ and $1 < p < \infty$. Then, $\mathcal{A}_\psi$ is a continuous linear operator on $L^p(\mathbb{R}^d)$ and*

$$\|\mathcal{A}_\psi(u)\|_{L^p} \le C|\psi|_{s^{0,N}_{\rho,\Lambda}} \|u\|_{L^p}. \tag{2.4}$$

3. Compactness of a commutator

First we prove a version of the Rellich theorem, which will be used in the sequel.

Lemma 3.1. *Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, $u_n \rightharpoonup 0$ in $L^q(\mathbb{R}^d)$ and Λ be a weight function. Then $\varphi u_n \to 0$ strongly in $H^{-\varepsilon,q}_\Lambda(\mathbb{R}^d)$, for any $\varepsilon > 0$.*

Proof. We have to show that $\mathcal{A}_{\Lambda(\xi)^{-\varepsilon}}(\varphi u_n) \to 0$, in $L^q(\mathbb{R}^d)$, where Λ is a weight function satisfying properties from the Definition 2.1. Applying the Rellich theorem for the weight $\langle\xi\rangle$ it follows that $\mathcal{A}_{\langle\xi\rangle^{-\varepsilon}}(\varphi u_n) \to 0$ in $L^q(\mathbb{R}^d)$, for any $\varepsilon > 0$. Notice that $\mathcal{A}_{\Lambda(\xi)^{-\varepsilon}}(\varphi u_n) = \mathcal{A}_{\langle\xi\rangle^{\varepsilon}\Lambda(\xi)^{-\varepsilon}}\mathcal{A}_{\langle\xi\rangle^{-\varepsilon}}(\varphi u_n)$. In order to apply Theorem 2.6, we will show that there exists $B > 0$ such that

$$|\xi^\gamma \partial^\gamma(\langle\xi\rangle^\varepsilon \Lambda(\xi)^{-\varepsilon})| \le B,\ \xi \in \mathbb{R}^d,\ \gamma \in \mathbb{K}.$$

There holds:

$$|\xi^\gamma \partial^\gamma(\langle\xi\rangle^\varepsilon \Lambda(\xi)^{-\varepsilon})| = \Big|\xi^\gamma \sum_{\beta\le\gamma} \partial^{\gamma-\beta}\langle\xi\rangle^\varepsilon \partial^\beta \Lambda(\xi)^{-\varepsilon}\Big| =$$

$$\Big|\sum_{\beta\le\gamma} \xi^{\gamma-\beta}\partial^{\gamma-\beta}\langle\xi\rangle^\varepsilon \xi^\beta \partial^\beta \Lambda(\xi)^{-\varepsilon}\Big| \le \sum_{\beta\le\gamma} |\xi|^{|\gamma|-|\beta|}|\partial^{\gamma-\beta}\langle\xi\rangle^\varepsilon||\xi^\beta\partial^\beta\Lambda(\xi)^{-\varepsilon}| \le$$

$$C\langle\xi\rangle^\varepsilon \Lambda(\xi)^{-\varepsilon} \le \Lambda(\xi)^{\frac{\varepsilon}{\mu_0}}\Lambda(\xi)^{-\varepsilon} = \Lambda(\xi)^{\varepsilon(\frac{1}{\mu_0}-1)} \le B,$$

since $\beta \in \mathbb{K}$, Λ is weight function and $\mu_0 \ge 1$. Theorem 2.6 implies that $\mathcal{A}_{\langle\xi\rangle^{\varepsilon}\Lambda(\xi)^{-\varepsilon}}$ maps continuously $L^q(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$. Since $\mathcal{A}_{\langle\xi\rangle^{-\varepsilon}}(\varphi u_n) \to 0$, in $L^q(\mathbb{R}^d)$ it follows that $\mathcal{A}_{\langle\xi\rangle^{\varepsilon}\Lambda(\xi)^{-\varepsilon}}\mathcal{A}_{\langle\xi\rangle^{-\varepsilon}}(\varphi u_n) \to 0$ in $L^q(\mathbb{R}^d)$ and the proof is complete. □

In the sequel with $\langle D_x\rangle = \sqrt{1-\Delta_x}$ we denote the pseudo-differential operator with symbol $\langle\xi\rangle$, i.e. $\langle D_x\rangle f = \int e^{ix\xi}\langle\xi\rangle \hat{f}(\xi) đ\xi,\ x \in \mathbb{R}^d$. We use powers of $\langle D_x\rangle$ and the partial integration.

The proofs of the assertions of this section are similar to the ones that we have given in our paper [3]. For the sake of completeness of the paper, we give all the details.

Theorem 3.2. *Let $m \in \mathbb{R}, \rho \in (0, 1/\omega]$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\psi \in s^{m,N}_{\rho,\Lambda}(\mathbb{R}^d)$, $N \ge 3d + 3$. Then, $\mathcal{A}_\psi T_\varphi$ is a compact operator from $H^{m,q}_\Lambda(\mathbb{R}^d)$ into $H^{-\varepsilon,q}_\Lambda(\mathbb{R}^d)$, for any $\varepsilon > 0$.*

Proof. We will show that the symbol of the composition $\mathcal{A}_\psi T_\varphi$, denoted by σ, is in $M^{m,N-d-1}_{\rho,\Lambda,0}$, for odd d, or $M^{m,N-d-2}_{\rho,\Lambda,0}$ for even d.

We need to prove that for $\psi \in s^{m,N}_{\rho,\Lambda}(\mathbb{R}^d)$ and for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the symbol of the composition σ, given by

$$\sigma(x,\xi) = \iint e^{-iy\eta}\psi(\xi+\eta)\varphi(x+y)dy\, đ\eta,\ x,\xi \in \mathbb{R}^d,$$

belongs to the space $M^{m,N}_{\Lambda,0}$. Using (2.1) it follows:

$$\Lambda(\xi+\eta)^m \le c\Lambda(\xi)^m\langle\eta\rangle^{|m|\omega},\ x,\xi \in \mathbb{R}^d,\ m \in \mathbb{R}.$$

We estimate ($x, \xi \in \mathbb{R}^d$):

$$|\sigma(x,\xi)| = \Big|\iint e^{-iy\eta}\langle y\rangle^{-2k}\langle D_\eta\rangle^{2k}\Big(\langle\eta\rangle^{-2l}\psi(\xi+\eta)\langle D_y\rangle^{2l}\varphi(x+y)\Big)dy\, đ\eta\Big|$$

$$\le \iint \langle y\rangle^{-2k}\langle\eta\rangle^{-2l}\Lambda(\xi+\eta)^m|\langle D_y\rangle^{2l}\varphi(x+y)|dy\, đ\eta$$

$$\leq c\iint \langle y\rangle^{-2k}\langle\eta\rangle^{-2l}\Lambda(\xi)^m\langle\eta\rangle^{|m|\omega}|\langle D_y\rangle^{2l}\varphi(x+y)|dy\, đ\eta \leq C\Lambda(\xi)^m,$$

for $2k>d$ and $2l-|m|\omega>d$. In the case when d is odd, we choose $2k=d+1$. Since $\varphi\in\mathcal{S}(\mathbb{R}^d)$, it follows that for any $M>0$ there exists $c_M>0$ such that

$$\langle D_y\rangle^{2l}\varphi(x+y)\leq c_M\langle x+y\rangle^{-M}\leq C_M\langle x\rangle^{-M}\langle y\rangle^M, x,y\in\mathbb{R}^d.$$

Then,

$$|\sigma(x,\xi)|\leq c\Lambda(\xi)^m\langle x\rangle^{-M},\ x,\xi\in\mathbb{R}^d, \tag{3.1}$$

where we choose $0<M<1$, so that $2k-M>d$.

Next, we estimate $\xi^\gamma\partial_\xi^{\gamma+\alpha}\partial_x^\beta\sigma(x,\xi)$. We have

$$\Big|\iint e^{-iy\eta}\xi^\gamma\partial_\xi^{\alpha+\gamma}\psi(\xi+\eta)\partial_x^\beta\varphi(x+y)dy\, đ\eta\Big| =$$

$$\Big|\iint e^{-iy\eta}\langle y\rangle^{-2k}\langle D_\eta\rangle^{2k}\Big(\langle\eta\rangle^{-2l}\xi^\gamma\partial_\xi^{\alpha+\gamma}\psi(\xi+\eta)\Big)\langle D_y\rangle^{2l}\partial_x^\beta\varphi(x+y)dy\, đ\eta\Big| \leq$$

$$c\iint\frac{\Lambda(\xi)^{m-\rho|\alpha|}}{\langle y\rangle^{2k}\langle\eta\rangle^{2l}}\langle\eta\rangle^{(|m-\rho|\alpha||)\omega}|\langle D_y\rangle^{2l}\partial_x^\beta\varphi(x+y)|dy\, đ\eta\leq c\langle x\rangle^{-M}\Lambda(\xi)^{m-\rho|\alpha|},$$

where $0<M<1$. Therefore,

$$|\xi^\gamma\partial_\xi^{\alpha+\gamma}\partial_x^\beta\sigma(x,\xi)|\leq c\langle x\rangle^{-M}\Lambda(\xi)^{m-\rho|\alpha|},\ x,\xi\in\mathbb{R}^d,$$

for $2l-|m-\rho|\alpha||\omega>d$, $2k=d+1$. Hence, if we assume that $N-d-1>2d$ we can apply Theorem 2.3. We have proved that $\sigma\in M^{m,N-d-1}_{\rho,\Lambda,0}$ for odd d. If d is even we choose $2k=d+2$ and then we need to assume that $N-d-2>2d$. Therefore, in both cases, it is enough to assume that $N\geq 3d+3$.

In the rest of the proof we apply an idea used in the proof of Theorem 3.2 [20], following also steps from the proof of Theorem 4 in our paper [3]. Take $\phi\in C_c^\infty(\mathbb{R}^d)$ such that $\phi(x)=1$ for $|x|\leq 1$ and $\phi(x)=0$ for $|x|\geq 2$ and let $\sigma_\nu(x,\xi)=\phi\Big(\dfrac{x}{\nu}\Big)\sigma(x,\xi),\ x,\xi\in\mathbb{R}^d,\ \nu\in\mathbb{N}$. Then, $T_{\sigma_\nu}=\phi_\nu T_\sigma$, for $\phi_\nu(x)=\phi\Big(\dfrac{x}{\nu}\Big)$.

The operator T_{σ_ν} is compact because T_σ is bounded from $H^{m,q}_\Lambda(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$ and the operator of multiplication by ϕ_ν is compact from $L^q(\mathbb{R}^d)$ into $H^{-\varepsilon,q}_\Lambda(\mathbb{R}^d)$, for any $\varepsilon>0$ (Lemma 3.1).

If $v\in H^{m,q}_\Lambda(\mathbb{R}^d)$, $1<q<\infty$, then Theorem 2.3 implies that there exists $c>0$ such that

$$\|(T_{\sigma_\nu}-T_\sigma)v\|_{H^{-\varepsilon,q}_\Lambda}\leq\|(T_{\sigma_\nu}-T_\sigma)v\|_{L^q}\leq c|\sigma_\nu-\sigma|_{M^{m,N-d-1}_{\rho,\Lambda}}\|v\|_{H^{m,q}_\Lambda}.$$

We estimate:

$$|\sigma_\nu-\sigma|_{M^{m,N-d-1}_{\rho,\Lambda}}=\max_{|\gamma|,\gamma\in\mathbb{K}}\max_{|\alpha|,|\beta|\leq N-d-1}\sup_{x,\xi\in\mathbb{R}^d}\frac{|\partial_\xi^{\alpha+\gamma}\partial_x^\beta((\phi(\frac{x}{\nu})-1)\sigma(x,\xi))|}{\Lambda(\xi)^{m-\rho|\alpha|}}$$

$$\leq\max_{|\gamma|,\gamma\in\mathbb{K}}\max_{|\alpha|,|\beta|\leq N-d-1}\sup_{|x|\geq\nu,\xi\in\mathbb{R}^d}\frac{|\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\partial_x^{\beta-\gamma}(\phi(\frac{x}{\nu})-1)\partial_\xi^{\alpha+\gamma}\partial_x^\gamma\sigma(x,\xi)|}{\Lambda(\xi)^{m-\rho|\alpha|}}$$

$$\leq Cc_{\alpha,\gamma}(\nu).$$

Since $\sigma \in M^{m,N-d-1}_{\rho,\Lambda,0}$, it follows that $c_{\alpha,\gamma}(\nu) = o(1)$ as $\nu \to \infty$. We conclude that $\|T_{\sigma_\nu} - T_\sigma\|_{\mathcal{L}(H^{m,q}_\Lambda, H^{-\varepsilon,q}_\Lambda)} \to 0$ as $\nu \to \infty$, which implies that T_σ is also a compact operator.

□

Corollary 3.3. *Let $\varphi \in C^\infty_c(\mathbb{R}^d)$ and $u_n \rightharpoonup 0$ in $H^{m,q}_\Lambda(\mathbb{R}^d)$, $m \in \mathbb{R}$. Then $\varphi u_n \to 0$ strongly in $H^{m-\varepsilon,q}_\Lambda(\mathbb{R}^d)$, for any $\varepsilon > 0$.*

Proof. We have to show that $\mathcal{A}_{\Lambda(\xi)^{m-\varepsilon}}(\varphi u_n) \to 0$, $n \to \infty$ in $L^q(\mathbb{R}^d)$. Since $\Lambda(\xi)^m \in s^{m,N}_\Lambda$ and $u_n \rightharpoonup 0$ in $H^{m,q}_\Lambda(\mathbb{R}^d)$, Theorem 3.2 implies that $\mathcal{A}_{\Lambda(\xi)^m}(\varphi u_n) \to 0$ in $H^{-\varepsilon,q}_\Lambda$. This is equivalent with $\mathcal{A}_{\Lambda(\xi)^{-\varepsilon}}\mathcal{A}_{\Lambda(\xi)^m}(\varphi u_n) \to 0$ in $L^q(\mathbb{R}^d)$. The proof is completed.

□

Theorem 3.4. *Let $\psi \in s^{m,N}_{\rho,\Lambda}$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $m \in \mathbb{R}$ and $\rho \in (0, 1/\omega]$, $N \geq 3d+5$. Then the commutator $C = [\mathcal{A}_\psi, T_\varphi] = \mathcal{A}_\psi T_\varphi - T_\varphi \mathcal{A}_\psi$ is a compact operator from $H^{m,q}_\Lambda(\mathbb{R}^d)$ into $H^{\rho-\varepsilon,q}_\Lambda(\mathbb{R}^d)$, $\varepsilon > 0$. If p denotes the symbol of C, then $p \in M^{m-\rho}_{\rho,\Lambda,N-d-3,0}$, if d is odd, or $p \in M^{m-\rho}_{\rho,\Lambda,N-d-4,0}$, if d is even.*

Proof. The proof is analogous to the proof of Theorem 5 in [3]. Let $\psi \in s^{m,N}_{\rho,\Lambda}$, $N \geq 3d + 5$, d odd and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The symbol of the composition $\mathcal{A}_\psi T_\varphi$ is given by $\sigma(x, \xi) = \iint e^{-iy\eta}\psi(\xi + \eta)\varphi(x + y)dy\, đ\eta$, $x, \xi \in \mathbb{R}^d$. Using Taylor's expansion, we obtain that

$\sigma(x, \xi) = I_1(x, \xi) + I_2(x, \xi)$, where

$$I_1(x,\xi) = \sum_{|\alpha|\leq 1} \frac{1}{\alpha!} \iint e^{-iy\eta}\eta^\alpha \partial^\alpha_\xi \psi(\xi)\varphi(x+y)dy\, đ\eta$$

and $I_2(x, \xi) = 2\sum_{|\alpha|=2} \frac{1}{\alpha!} \iint e^{-iy\eta}\eta^\alpha\Big(\int_0^1 (1-\theta)^2\partial^\alpha_\xi\psi(\xi+\theta\eta)d\theta\Big)\varphi(x + y)dy\, đ\eta$. Then, $I_1(x,\xi) = \sum_{|\alpha|\leq 1} \frac{1}{\alpha!}\partial^\alpha_\xi\psi(\xi)D^\alpha_y\varphi(y)|_{y=x}$ and similarly,

$$I_2(x,\xi) = 2\sum_{|\alpha|=2}\frac{1}{\alpha!}\iint e^{-iy\eta}\Big(\int_0^1(1-\theta)^2\partial^\alpha_\xi\psi(\xi+\theta\eta)d\theta\Big)D^\alpha_y\varphi(x+y)dy\, đ\eta.$$

Since the symbol of $T_\varphi\mathcal{A}_\psi$ equals $\varphi(x)\psi(\xi)$, the symbol of commutator C is of the form $p(x, \xi) = \tilde{I}_1(x, \xi) + I_2(x, \xi)$, where

$$\tilde{I}_1(x,\xi) := \sum_{|\alpha|=1}\frac{1}{\alpha!}\partial^\alpha_\xi\psi(\xi)D^\alpha_y\varphi(y)|_{y=x}.$$

Therefore $\tilde{I}_1(x, \xi) \in M^{m-\rho,N-1}_{\rho,\Lambda,0}$. We need to estimate $I_2(x, \xi)$. Note that

$$I_2(x,\xi) = 2\sum_{|\alpha|=2}\frac{1}{\alpha!}\int_0^1(1-\theta)^2 I_3(x,\xi)d\theta,$$

where $I_3(x,\xi) = \iint e^{-iy\eta}\partial_\xi^\alpha \psi(\xi+\theta\eta) D_y^\alpha \varphi(x+y)dy\ đ\eta$. From the proof of Theorem 3.2 it follows that

$$|I_3(x,\xi)| \leq \iint \langle y\rangle^{-2k}\langle D_\eta\rangle^{2k}\Big(\langle\eta\rangle^{-2l}\partial_\xi^\alpha\psi(\xi+\theta\eta)\Big)\langle D_y\rangle^l\Big[D_y^\alpha\varphi(x+y)\Big]dy\ đ\eta$$
$$\leq C\Lambda(\xi)^{m-2\rho}\langle x\rangle^{-M},$$

for $2k = d+1$, $0 < M < 1$, $2l > d+|m-2\rho|\omega$. Also, from the proof of Theorem 3.2 it follows that $I_2 \in M^{m-2\rho,N-d-3}_{\rho,\Lambda,0}$. Since $\tilde{I}_1(x,\xi) \in M^{m-\rho,N-1}_{\rho,\Lambda,0} \subset M^{m-\rho,N-d-3}_{\rho,\Lambda,0}$ and $I_2 \in M^{m-2\rho,,N-d-3}_{\rho,\Lambda,0} \subset M^{m-\rho,,N-d-3}_{\rho,\Lambda,0}$ it follows that $p \in M^{m-\rho,N-d-3}_{\rho,\Lambda,0}$. Now we apply the proof of Theorem 3.2 to conclude that $C = T_p$ is a compact operator from $H^{m,q}_\Lambda(\mathbb{R}^d)$ into $H^{\rho-\varepsilon,q}_\Lambda(\mathbb{R}^d)$. The proof is analogous in the case when d is even. In order to apply Theorem 3.2 we assume that $N \geq 3d+5$. □

The proof of the next corollary is a direct consequence of Corollary 3.3 and Theorem 3.4.

Corollary 3.5. *Let $\psi \in s^{m,N}_{\rho,\Lambda}$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $m,s \in \mathbb{R}$ and $\rho \in (0,1/\omega]$, $N \geq 3d+5$. Then the commutator $C = [\mathcal{A}_\psi, T_\varphi] = \mathcal{A}_\psi T_\varphi - T_\varphi \mathcal{A}_\psi$ is a compact operator from $H^{m+s,q}_\Lambda(\mathbb{R}^d)$ into $H^{\rho+s-\varepsilon,q}_\Lambda(\mathbb{R}^d)$, $\varepsilon > 0$. If p denotes the symbol of C, then $p \in M^{m-\rho}_{\rho,\Lambda,N-d-3,0}$, if d is odd or $p \in M^{m-\rho}_{\rho,\Lambda,N-d-4,0}$, if d is even.*

4. Existence of H_Λ-distributions

We denote by $(s^{m,N}_{\rho,\Lambda})_0 \subset s^{m,N}_{\rho,\Lambda}$ the space of multipliers $\psi \in (s^{m,N}_{\rho,\Lambda})_0$ such that for all $|\alpha| \leq N$, $\gamma \in \mathbb{K}$

$$\lim_{n\to\infty}\sup_{|\xi|\geq n}\frac{|\xi^\gamma\partial^{\alpha+\gamma}\psi(\xi)|}{\Lambda(\xi)^{m-\rho|\alpha|}} = 0.$$

We need separability and completeness of the symbol spaces for the existence theorem of H_Λ-distributions. The following theorem holds since $\mathcal{S}(\mathbb{R}^d)$ is dense in $(s^{m,N}_{\rho,\Lambda})_0$ (for the proof see [3]).

Theorem 4.1. *Let $\rho \in (0,1/\omega]$, $m \in \mathbb{R}$. Then the space $((s^{m,N+1}_{\rho,\Lambda})_0, |\cdot|_{s^{m,N}_{\rho,\Lambda}})$ is separable.*

In order to obtain completeness we use completion of $(s^{m,N+1}_{\rho,\Lambda})_0$ with respect to the $|\cdot|_{s^{m,N}_{\rho,\Lambda}}$ norm. Completion is also denoted by $(s^{m,N+1}_{\rho,\Lambda})_0$. We assume that N is an integer such that $N > 2d$.

Theorem 4.2. *Let $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$ and $v_n \rightharpoonup 0$ in $H^{m,q}_\Lambda(\mathbb{R}^d)$, $m \in \mathbb{R}$, $\rho = 1/\omega$. Then, up to subsequences, there exists a distribution $\mu \in (\mathcal{S}(\mathbb{R}^d)\hat{\otimes}(s^{m,N+1}_\Lambda)_0)'$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and all $\psi \in (s^{m,N+1}_\Lambda)_0$,*

$$\lim_{n\to\infty}\langle u_n, \overline{\mathcal{A}_{\bar\psi}(\varphi v_n)}\rangle = \langle \mu, \bar\varphi\otimes\psi\rangle.$$

Proof. In the proof we follow the ideas given in the proofs of existence of H-distributions in [2, 3]. We consider a sequence of sesquilinear (linear in ψ and anti-linear in φ) functionals:

$$\mu_n(\varphi,\psi) = \int_{\mathbb{R}^d} u_n \overline{\mathcal{A}_{\overline{\psi}}(\varphi v_n)} dx, \ n \in \mathbb{N}.$$

Functionals μ_n are well defined because $\mathcal{A}_{\overline{\psi}}(\varphi v_n) \in L^q, n \in \mathbb{N}$.

Since $\psi(\xi) = \psi_1(\xi)\psi_2(\xi)$, $\psi_1(\xi) = \Lambda(\xi)^m \in s_\Lambda^{m,N+1}$, $\psi_2(\xi) = \Lambda(\xi\rangle^{-m}\psi(\xi) \in (s_\Lambda^{0,N+1})_0$. Using (2.4), it follows that

$$\|\mathcal{A}_{\overline{\psi}}(\varphi v_n)\|_{L^q} \le c|\psi_2|_{s_\Lambda^{0,N}} \|\mathcal{A}_{\overline{\psi_1}}(\varphi v_n)\|_{L^q} \le c_1 |\psi|_{s_\Lambda^{m,N}} \|\varphi v_n\|_{H_\Lambda^{m,q}},$$

where we use the estimate

$$|\psi_2|_{s_\Lambda^{0,N}} = |\Lambda(\xi)^{-m}\psi(\xi)|_{s_{\rho,\Lambda}^{0,N}} \le C|\Lambda(\xi)^{-m}|_{s_\Lambda^{-m,N}} |\psi|_{s_\Lambda^{m,N}} \le C_1 |\psi|_{s_\Lambda^{m,N}}.$$

Using inequality (2.1) and the exchange formula for the inverse Fourier transform of convolution, we have

$$\begin{aligned}
\|\varphi v_n\|_{H_\Lambda^{m,q}} &= \Big(\int_{\mathbb{R}^d} \Big|\mathcal{F}^{-1}(\Lambda(\xi)^m \hat{\varphi} * \hat{v}_n)\Big|^q dx\Big)^{\frac{1}{q}} \\
&= \Big(\int_{\mathbb{R}^d} \Big|\mathcal{F}^{-1}\Big(\Lambda(\xi)^m \int_{\mathbb{R}^d} \hat{\varphi}(\xi-\eta)\hat{v}_n(\eta)d\eta\Big)\Big|^q dx\Big)^{\frac{1}{q}} \\
&= \Big(\int_{\mathbb{R}^d} \Big|\mathcal{F}^{-1}(\int_{\mathbb{R}^d} \Lambda(\xi)^m \hat{\varphi}(\xi-\eta)\hat{v}_n(\eta)d\eta)\Big|^q dx\Big)^{\frac{1}{q}} \\
&\le c\Big(\int_{\mathbb{R}^d} \Big|\mathcal{F}^{-1}\Big(\int_{\mathbb{R}^d} \Lambda(\eta)^m (1+|\xi-\eta|^2)^{\frac{|m|\omega}{2}} \hat{\varphi}(\xi-\eta)\hat{v}_n(\eta)d\eta\Big)\Big|^q dx\Big)^{\frac{1}{q}} \\
&= c\Big(\int_{\mathbb{R}^d} \Big|\mathcal{F}^{-1}\Big(\hat{v}_n\Lambda(\cdot)^m * \hat{\varphi}(1+|\cdot|^2)^{\frac{|m|\omega}{2}}\Big)\Big|^q dx\Big)^{\frac{1}{q}} \\
&= c\Big(\int_{\mathbb{R}^d} \Big|\mathcal{F}^{-1}(\hat{v}_n\Lambda(\cdot)^m)\Big|^q \Big|(\mathcal{F}^{-1}(\hat{\varphi}(1+|\cdot|^2)^{\frac{|m|\omega}{2}})\Big|^q dx\Big)^{\frac{1}{q}} \\
&\le C \sup_{x\in\mathbb{R}^d} \Big|\mathcal{F}^{-1}((1+|\xi|^2)^{\frac{|m|\omega}{2}}\hat{\varphi})\Big| \|v_n\|_{H_\Lambda^{m,q}} \\
&\le C\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^{\frac{d+1}{2}}} \|\langle\xi\rangle^{d+1+|m|\omega}\hat{\varphi}\|_\infty d\xi \le C\|\langle\xi\rangle^{d+1+|m|\omega}\hat{\varphi}\|_\infty.
\end{aligned}$$

We choose the sequence of norms on $\mathcal{S}(\mathbb{R}^d)$: $|\varphi|_k = \sup_{|\alpha|\le k} \|\langle\xi\rangle^k \hat{\varphi}^{(\alpha)}(\xi)\|_\infty$, $k \in \mathbb{N}_0$. Therefore,

$$\Big|\int_{\mathbb{R}^d} u_n \overline{\mathcal{A}_{\overline{\psi}}(\varphi v_n)} dx\Big| \le C|\psi|_{s_\Lambda^{m,N}} |\varphi|_{d+1+\lceil |m|\omega\rceil}.$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be fixed. Then the mapping $\psi \mapsto \mu_n(\varphi,\psi) := \int_{\mathbb{R}^d} u_n \overline{\mathcal{A}_{\overline{\psi}}(\varphi v_n)} dx$ is linear and continuous, and similarly for fixed $\psi \in (s_\Lambda^{m,N+1})_0$, the mapping $\varphi \mapsto \mu_n(\varphi,\psi)$ is anti-linear and continuous. In the rest of the proof we follow the standard steps for proving the existence of H-distributions, as it was done in the proof of Theorem 3.1, in [2] and Theorem 6 in [3]. Repeating these steps we obtain that there exists $\mu \in (\mathcal{S}(\mathbb{R}^d)\hat{\otimes}(s_\Lambda^{m,N+1})_0)'$ defined as

$$\langle \mu(x,\xi), \varphi(x)\psi(\xi)\rangle = \lim_{\nu\to\infty}\int u_\nu \overline{\mathcal{A}_{\overline{\psi}}(\varphi v_\nu)}dx,\ \varphi \in \mathcal{S}(\mathbb{R}^d),\ \psi \in (s^m_{\Lambda,N+1})_0,$$

where u_ν is a subsequence of u_n and v_ν is a subsequence of v_n. Hence the proof is complete. □

Distribution obtained in Theorem 4.2 is called H_Λ-distribution. The following corollary follows from the proof of Theorem 4.2.

Corollary 4.3. *Let $u_n \rightharpoonup 0$ in $H_\Lambda^{-s,p}(\mathbb{R}^d)$ and $v_n \rightharpoonup 0$ in $H_\Lambda^{m+s,q}(\mathbb{R}^d)$, $s,m \in \mathbb{R}$, $\rho = 1/\omega$. Then, up to subsequences, there exists a distribution $\mu \in (\mathcal{S}(\mathbb{R}^d)\hat{\otimes}(s_\Lambda^{m,N+1})_0)'$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and all $\psi \in (s_\Lambda^{m,N+1})_0$,*

$$\lim_{n\to\infty}\langle u_n, \overline{\mathcal{A}_{\bar\psi}(\varphi v_n)}\rangle = \langle \mu, \bar\varphi\otimes\psi\rangle. \tag{4.1}$$

Remark 4.4. If we fix $\psi \in s_\Lambda^{m,N}$ in Theorem 4.2 we can consider a Schwartz distribution $\mu_\psi \in \mathcal{S}'(\mathbb{R}^d)$ defined via (4.1) as

$$\langle \mu_\psi, \varphi\rangle = \lim_{n\to\infty}\langle u_n, \mathcal{A}_\psi(\varphi v_n)\rangle.$$

In a similar manner as in [2] and [3] we prove the following theorem regarding strong convergence of a given weakly convergent sequence.

Theorem 4.5. *Let $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$. Assume that*

$$\lim_{n\to\infty}\langle u_n, \mathcal{A}_{\Lambda(\xi)^m}(\varphi v_n)\rangle = 0, \tag{4.2}$$

for every sequence $v_n \rightharpoonup 0$ in $H_\Lambda^{m,q}(\mathbb{R}^d)$, $m \in \mathbb{R}$. Then for every $\theta \in \mathcal{S}(\mathbb{R}^d)$, $\theta u_n \to 0$ strongly in $L^p(\mathbb{R}^d)$.

Proof. We will prove that for all $\theta \in \mathcal{S}(\mathbb{R}^d)$ and every bounded $B \subseteq L^q(\mathbb{R}^d)$,

$$\sup\{\langle \theta u_n, \phi\rangle \,:\, \phi \in B\} \to 0,\ n\to\infty.$$

Assume the opposite, i.e. that there exist $\theta \in \mathcal{S}(\mathbb{R}^d)$, a bounded set B_0 in $L^q(\mathbb{R}^d)$, an $\varepsilon_0 > 0$ and a subsequence θu_ν of θu_n such that

$$\sup\{|\langle \theta u_\nu, \phi\rangle| \,:\, \phi \in B_0\} \geq \varepsilon_0,\ \text{for every } \nu \in \mathbb{N}.$$

Choose $\phi_\nu \in B_0$ such that $|\langle \theta u_\nu, \phi_\nu\rangle| > \varepsilon_0/2$. Since $\phi_\nu \in B_0$ and B_0 is bounded in $L^q(\mathbb{R}^d)$, it follows that $\{\phi_\nu, \nu \in \mathbb{N}\}$ is weakly precompact in $L^q(\mathbb{R}^d)$, i.e. up to a subsequence, $\phi_\nu \rightharpoonup \phi_0$ in $L^q(\mathbb{R}^d)$. Moreover, since ϕ_0 is fixed, we have $\langle u_\nu, \phi_0\rangle \to 0$ and

$$|\langle \theta u_\nu, \phi_\nu - \phi_0\rangle| > \frac{\varepsilon_0}{4},\quad \nu > \nu_0. \tag{4.3}$$

Applying (4.2) on $u_\nu \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$ and $\mathcal{A}_{\Lambda(\xi)^{-m}}(\phi_\nu - \phi_0) \rightharpoonup 0$ in $H_\Lambda^{m,q}(\mathbb{R}^d)$, we obtain that for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\lim_{\nu\to\infty}\langle u_\nu, \mathcal{A}_{\Lambda(\xi)^m}(\varphi\mathcal{A}_{\Lambda(\xi)^{-m}}((\phi_\nu - \phi_0))\rangle = 0.$$

Choosing $\varphi = \theta$ and using Theorem 3.4, we get $\lim_{\nu\to\infty} \langle \theta u_\nu, \phi_\nu - \phi_0 \rangle = 0$, which contradicts (4.3). $\square$

The following corollary also holds.

Corollary 4.6. *Let* $u_n \rightharpoonup 0$ *in* $H^{-s,p}(\mathbb{R}^d)$. *If*

$$\lim_{n\to\infty} \langle u_n, \mathcal{A}_{\Lambda(\xi)^m}(\varphi v_n) \rangle = 0,$$

for every sequence $v_n \rightharpoonup 0$ *in* $H_\Lambda^{m+s,q}(\mathbb{R}^d)$, $m \in \mathbb{R}$, *then for every* $\theta \in \mathcal{S}(\mathbb{R}^d)$, $\theta u_n \to 0$ *strongly in* $H^{-s,p}(\mathbb{R}^d)$.

5. Applications

Let $u_n \rightharpoonup 0$ in $H_\Lambda^{-s,p}(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 < p < \infty$. We consider a sequence of linear equations

$$(T_p u_n)(x) = \int_{\mathbb{R}^d} e^{ix\xi} a(x) \sigma(\xi) \hat{u_n}(\xi)\, đ\xi = f_n(x), \tag{5.1}$$

where T_p is the operator with the symbol $p(x,\xi) = a(x)\sigma(\xi)$, $a \in C_b^\infty(\mathbb{R}^d)$ -the space of smooth, bounded functions with all derivatives also bounded, and $\sigma \in s_\Lambda^{r,N}$; recall that Λ is a weight function, $r \in \mathbb{R}$ and $\rho = 1/\omega$. Hence $p \in M_\Lambda^{r,N}$. For the right hand side of (5.1) we assume that $(f_n)_n$ is a sequence of temperate distributions such that

$$\varphi f_n \to 0 \text{ in } H_\Lambda^{-s-r,p}(\mathbb{R}^d), \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{5.2}$$

For fixed $\psi \in s_\Lambda^{m,N}$ we analyze Schwartz distribution $\mu_\psi \in \mathcal{S}'(\mathbb{R}^d)$, see Remark 4.4. We assume that $N \geq 3d + 5$ in order to apply Corollary 3.5, in the sequel. We obtain the following result.

Theorem 5.1. *Let* $u_n \rightharpoonup 0$ *in* $H_\Lambda^{-s,p}(\mathbb{R}^d)$, $s \in \mathbb{R}$ *satisfies* (5.1), (5.2) *and* $\psi \in s_\Lambda^{m,N}$. *Then, for any* $v_n \rightharpoonup 0$ *in* $H_\Lambda^{s+m,q}(\mathbb{R}^d)$ *the following equation is satisfied*

$$a(x)\mu_{\frac{\sigma(\xi)}{\Lambda(\xi)^r}\psi} = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{5.3}$$

Proof. Let $v_n \rightharpoonup 0$ in $H_\Lambda^{s+m,q}(\mathbb{R}^d)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\psi \in s_\Lambda^{m,N}$. We have to prove that, up to a subsequence,

$$\lim_{n\to\infty} \left\langle u_n, \mathcal{A}_{\frac{\sigma(\xi)}{\Lambda(\xi)^r}\psi}(\varphi a v_n) \right\rangle = 0,$$

Since

$$\lim_{n\to\infty} \left\langle u_n, \mathcal{A}_{\frac{\sigma(\xi)}{\Lambda(\xi)^r}\psi}(\varphi a v_n) \right\rangle = \lim_{n\to\infty} \left\langle \mathcal{A}_{\sigma(\xi)}(u_n), \mathcal{A}_{\Lambda(\xi)^{-r}\psi}(\varphi a v_n) \right\rangle$$

and $\mathcal{A}_{\Lambda(\xi)^{-r}\psi}(\varphi a v_n) \in H_\Lambda^{s+r,q}(\mathbb{R}^d)$, applying Corollary 3.5 and (5.2) we have that

$$\lim_{n\to\infty} \left\langle \mathcal{A}_{\sigma(\xi)}(u_n), \mathcal{A}_{\Lambda(\xi)^{-r}\psi}(\varphi_1 \varphi_2 a v_n) \right\rangle =$$

$$= \lim_{n\to\infty} \left\langle a\varphi_1 \mathcal{A}_{\sigma(\xi)}(u_n), \mathcal{A}_{\Lambda(\xi)^{-r}\psi}(\varphi_2 v_n)\right\rangle$$
$$= \lim_{n\to\infty} \left\langle \varphi_1 f_n, \mathcal{A}_{\Lambda(\xi)^{-r}\psi}(\varphi_2 v_n)\right\rangle = 0.$$

Therefore, we have proved (5.3). □

Remark 5.2. If $a(x) \neq 0$ and $\psi = \Lambda(\xi)^m$, $\sigma = \Lambda(\xi)^r$, equality (5.3) implies that $\mu_{\Lambda(\xi)^m} = 0$. Hence, in this case, $\varphi u_n \to 0$ in $H_\Lambda^{-s,p}(\mathbb{R}^d)$, according to Corollary 4.6.

5.1. Examples

1. Let $\mathcal{P}$ be complete polyhedron in $\mathbb{R}^d$ with set of vertices $V(\mathcal{P})$ and $\Lambda = \Lambda_{\mathcal{P}}$. Let $u_n \rightharpoonup 0$ in $H_\Lambda^{-s,p}(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 < p < \infty$, such that the following sequence of equations is satisfied

$$p(x, D)u(x) = \sum_{\alpha\in V(\mathcal{P})} a_\alpha(x) D^{2\alpha} u_n(x) = f_n(x), \tag{5.4}$$

where $a_\alpha(x) \in C_b^\infty(\mathbb{R}^d)$, and $(f_n)_n$ is a sequence of temperate distributions such that

$$\varphi f_n \to 0 \text{ in } H_\Lambda^{-s-2,p}(\mathbb{R}^d), \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{5.5}$$

The symbol of the given differential operator $p(x,\xi) = \sum_{\alpha\in V(\mathcal{P})} a_\alpha(x)\xi^{2\alpha}$ belongs to M_Λ^2.

Corollary 5.3. (of Theorem 5.1.)
Let $u_n \rightharpoonup 0$ in $H_\Lambda^{-s,p}(\mathbb{R}^d)$, $s \in \mathbb{R}$, satisfies (5.4), (5.5) and $\psi \in s_\Lambda^{m,N}$. Then, for any $v_n \rightharpoonup 0$ in $H_\Lambda^{s+m,q}(\mathbb{R}^d)$ and the corresponding distribution μ, there holds

$$\sum_{\alpha\in V(\mathcal{P})} a_\alpha(x)\mu_{\frac{\psi\xi^{2\alpha}}{\Lambda(\xi)^2}} = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{5.6}$$

Moreover, let $\psi = \Lambda(\xi)^m$ and the equality in (5.6) implies that $\mu_\psi = 0$. Then we have the strong convergence $\theta u_n \to 0$ in $H_\Lambda^{-s,p}(\mathbb{R}^d)$, for every $\theta \in \mathcal{S}(\mathbb{R}^d)$.

Proof. Let $v_n \rightharpoonup 0$ in $H_\Lambda^{s+m,q}(\mathbb{R}^d)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\psi \in s_\Lambda^{m,N}$. We have to prove that, up to a subsequence,

$$\lim_{n\to\infty} \sum_{\alpha\in V(\mathcal{P})} \left\langle u_n, \mathcal{A}_{\frac{\psi\xi^{2\alpha}}{\Lambda(\xi)^2}}(\varphi a_\alpha v_n)\right\rangle = 0.$$

Let $\mathcal{A}_{\psi_\alpha} = \mathcal{A}_{\frac{\psi\xi^{2\alpha}}{\Lambda(\xi)^2}}$. Since $\mathcal{A}_{\xi^{2\alpha}} \circ \mathcal{A}_{\frac{\psi(\xi)}{\Lambda^2(\xi)}} = D^{2\alpha}\mathcal{A}_{\Lambda(\xi)^{-2}\psi(\xi)}$, it follows that

$$\lim_{n\to\infty} \sum_{\alpha\in V(\mathcal{P})} \langle u_n, \mathcal{A}_{\psi_\alpha}(\varphi a_\alpha v_n)\rangle = \lim_{n\to\infty} \sum_{\alpha\in V(\mathcal{P})} \left\langle D_x^{2\alpha}(u_n), \mathcal{A}_{\Lambda(\xi)^{-2}\psi}(\varphi a_\alpha v_n)\right\rangle.$$

Then $\mathcal{A}_{\Lambda(\xi)^{-2}\psi}(\varphi a_\alpha v_n) \in H_\Lambda^{s+2,q}(\mathbb{R}^d)$, and Corollary 3.5 implies that

$$\lim_{n\to\infty} \sum_{\alpha\in V(\mathcal{P})} \left\langle D_x^{2\alpha}(u_n), \mathcal{A}_{\Lambda(\xi)^{-2}\psi}(\varphi a_\alpha v_n)\right\rangle =$$

$$\lim_{n\to\infty}\sum_{\alpha\in V(\mathcal{P})}\left\langle a_\alpha\varphi_1 D_x^{2\alpha}(u_n),\mathcal{A}_{\Lambda(\xi)^{-2}\psi}(\varphi_2 v_n)\right\rangle=0,$$

where we have used $\varphi=\varphi_1\varphi_2$ for $\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R}^d)$. Therefore, we have proved (5.6). □

2. Let

$$p(x,\xi)=\sum_{|\alpha|\le k}A_\alpha(x)\xi^\alpha \tag{5.7}$$

and let $\Lambda=\Lambda_{\mathcal{P}}$, where $\mathcal{P}$ is a complete polyhedron in $\mathbb{R}^d$ such that all multi-indices α that appear in equation (5.7) are contained in $\mathcal{P}$.

Now let $u_n\rightharpoonup 0$ in $H_\Lambda^{-s,p}(\mathbb{R}^d)$, $s\in\mathbb{R}$, $1<p<\infty$, such that the following sequence of equations is satisfied in $H_\Lambda^{-s-1,p}(\mathbb{R}^d)$:

$$\sum_{|\alpha|\le k}A_\alpha(x)D^\alpha u_n(x)=g_n(x), \tag{5.8}$$

where $A_\alpha\in C_b^\infty(\mathbb{R}^d)$, and $(g_n)_n$ is a sequence of tempered distributions such that

$$\varphi g_n\to 0 \text{ in } H_\Lambda^{-s-1,p}(\mathbb{R}^d),\ \text{for every } \varphi\in\mathcal{S}(\mathbb{R}^d). \tag{5.9}$$

According to [17], Example 2.7.9, we have that $p\in M_\Lambda^1$. Therefore $\sum_{|\alpha|\le k}A_\alpha(x)D^\alpha u_n(x)\in H_\Lambda^{-s-1,p}(\mathbb{R}^d)$.

Corollary 5.4. (of Theorem 5.1.)

Let $u_n\rightharpoonup 0$ in $H_\Lambda^{-s,p}(\mathbb{R}^d)$, $s\in\mathbb{R}$, satisfies (5.8), (5.9) and $\psi\in s_\Lambda^{m,N}$. Then, for any $v_n\rightharpoonup 0$ in $H_\Lambda^{s+m,q}(\mathbb{R}^d)$ and the corresponding distribution μ there holds that

$$\sum_{|\alpha|\le k}A_\alpha(x)\mu_{\frac{\psi(\xi)\xi^\alpha}{\Lambda(\xi)}}=0\quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{5.10}$$

Moreover, if $\psi=\Lambda(\xi)^m$ and (5.10) implies $\mu_{\Lambda(\xi)^m}=0$, then we have the strong convergence $\theta u_n\to 0$.

Proof. Let $v_n\rightharpoonup 0$ in $H_\Lambda^{s+m,q}(\mathbb{R}^d)$, $\varphi_1\in\mathcal{S}(\mathbb{R}^d)$, $\varphi_2\in\mathcal{S}(\mathbb{R}^d)$ and $\psi\in s_\Lambda^{m,N}$. We need to prove that, up to a subsequence,

$$\lim_{n\to\infty}\sum_{|\alpha|\le k}\left\langle u_n,\,\mathcal{A}_{\bar\Psi_\alpha}(A_\alpha\varphi v_n)\right\rangle=0,$$

where $\Psi_\alpha=\dfrac{\xi^\alpha}{\Lambda(\xi)}\psi(\xi)$. Since $\mathcal{A}_{\bar\Psi_\alpha}=\mathcal{A}_{\frac{\xi^\alpha}{\Lambda(\xi)}}\circ\mathcal{A}_\psi$ and $\mathcal{A}_{\frac{\xi^\alpha}{\Lambda(\xi)}}=\partial^\alpha\mathcal{A}_{\Lambda(\xi)^{-1}}$, it follows that

$$\lim_{n\to\infty}\sum_{|\alpha|\le k}\left\langle u_n,\,\mathcal{A}_{\bar\Psi_\alpha}(A_\alpha\varphi v_n)\right\rangle=\lim_{n\to\infty}\sum_{|\alpha|\le k}\left\langle D_x^\alpha(u_n)\,\mathcal{A}_{\Lambda(\xi)^{-1}\psi}(A_\alpha\varphi v_n)\right\rangle$$

$$=\lim_{n\to\infty}\sum_{|\alpha|\le k}\left\langle\varphi_1A_\alpha D_x^\alpha(u_n)\,\mathcal{A}_{\Lambda(\xi)^{-1}\psi}(\varphi_2 v_n)\right\rangle=0.$$

We have used Corollary 3.5 and (5.9), since $\mathcal{A}_{\Lambda(\xi)^{-1}\psi}(\varphi v_n)\in H_\Lambda^{s+1,q}(\mathbb{R}^d)$.

If $\mu_{\Lambda(\xi)^m} = 0$, then Corollary 4.6 implies $\theta u_n \to 0$ in $H_\Lambda^{-s,p}(\mathbb{R}^d)$ for every $\theta \in \mathcal{S}(\mathbb{R}^d)$. □

Remark 5.5. Equation (5.8) is considered in [3] with solutions in Bessel potential spaces $H^p_{-s}(\mathbb{R}^d)$ ($\Lambda(\xi) = \langle\xi\rangle$). In that case it was necessary to require convergence of type (5.9) in $H^p_{-s-k}(\mathbb{R}^d)$ to deduce result similar to (5.10) with $\Lambda(\xi)$ replaced by $\langle\xi\rangle^k$.

References

[1] Aleksić, J.; Mitrović, D.; Pilipović, S., *Hyperbolic conservation laws with vanishing nonlinear diffusion and linear dispersion in heterogeneous media*, J. Evol. Equ. 9 (2009), no. 4, 809-828.

[2] Aleksić, J.; Pilipović, S.; Vojnović, I., *H-distributions via Sobolev spaces*, Mediterr. J. Math. 13 (2016), no. 5, 3499-3512.

[3] Aleksić, J.; Pilipović, S.; Vojnović, I., *H-distributions with unbounded multipliers*, J. Pseudo-Differ. Oper. Appl. (2017) https://doi.org/10.1007/s11868-017-0200-5.

[4] Antonić N.; Lazar, M., *A parabolic variant of H-measures*, Ann. Univ. Ferrara Sez. VII Sci. Mat. 54 (2008), no. 2, 183-201.

[5] Antonić N.; Lazar, M., *Parabolic H-measures*, J. Funct. Anal. 265 (2013), no. 7, 1190-1239.

[6] Antonić, N.; Mitrović, D., *H-distributions: an extension of H-measures to an $L^p - L^q$ setting,* Abstr. Appl. Anal. 2011, Art. ID 901084, 12 pp.

[7] Boggiatto P., Buzano E. and Rodino L., *Global hypoellipticity and spectral theory*, Mathematical Research, 92, Akademie Verlag, Berlin, 1996, 187 pp.

[8] Erceg, M., Ivec, I., *Second commutation lemma for fractional H-measures*, J. Pseudo-Differ. Oper. Appl. (2017) https://doi.org/10.1007/s11868-017-0207-y.

[9] Erceg, M., Ivec, I., *On generalisation of H-measures*, Filomat, 31(16) (2017) 5027-5044.

[10] Garello, G., Morando, A., *L^p-Continuity for Pseudo-Differential Operators*, Pseudo-differential operators and related topics, 79-94, Oper. Theory Adv. Appl., 164, Birkhäuser, Basel, 2006.

[11] Garello, G., Morando, A., *L^p-Bounded Pseudodifferential Operators and Regularity for Multi-quasi-elliptic Equations*, Integral Equations Operator Theory 51 (2005), no. 4, 501-517.

[12] Gérard, P., *Microlocal defect measures*, Comm. Partial Differential Equations 16 (1991), no. 11, 1761-1794.

[13] Kumano-go, H., *Pseudodifferential operators*, MIT Press, Cambridge, Mass.-London, 1981. xviii+455 pp.

[14] Lizorkin, P.I., *(L^p, L^q)-multipliers of Fourier integrals. (Russian)*, Dokl. Akad. Nauk SSSR, 152 (1963), 808-811.

[15] Mitrović D., Ivec, I., *A generalization of H-measures and application on purely fractional scalar conservation laws*, Commun. Pure Appl. Anal. 10 (2011), no. 6, 1617-1627.

[16] Morando, A., *L^p-regularity for a class of pseudodifferential operators in $\mathbb{R}^n$*, J. Partial Differential Equations 18 (2005), no. 3, 241-262.

[17] Nicola F., Rodino L., *Global Pseudo-Differential Calculus on Euclidean Spaces*, Pseudo-Differential Operators. Theory and Applications, 4. Birkhäuser Verlag, Basel, 2010. x+306 pp.

[18] Petzeltová H., Vrbová P., *Factorization in the algebra of rapidly decreasing functions on* $\mathbb{R}^n$, Comment. Math. Univ. Carolin. 19 (1978), no. 3, 489-499.

[19] Tartar, L., *H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A 115 (1990), no. 3-4, 193-230.

[20] Wong, M. W., *Spectral theory of pseudo-differential operators*, Adv. in Appl. Math. 15 (1994), no. 4, 437-451.

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