STRONG TRACES FOR AVERAGED SOLUTIONS OF HETEROGENEOUS ULTRA-PARABOLIC TRANSPORT EQUATIONS

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ABSTRACT. We prove that if traceability conditions are fulfilled then a weak solution $h \in L^\infty(I\!R^+ \times I\!R^d \times I\!R)$ to the ultra-parabolic transport equation

$$\partial_t h + \operatorname{div}_x (F(t,x,\lambda)h) = \sum_{i,j=1}^k \partial^2_{x_i x_j} (b_{ij}(t,x,\lambda)h) + \partial_\lambda \gamma(t,x,\lambda),$$

is such that for every $\rho \in C^1_c(I\!R)$, the velocity averaged quantity $\int_{I\!R} h(t,x,\lambda)$ $\rho(\lambda)d\lambda$ admits the strong $L^1_{\rm loc}(I\!R^d)$-limit as $t \to 0$, i.e. there exist $h_0(x,\lambda) \in L^1_{\rm loc}(I\!R^d \times I\!R)$ and set $E \subset I\!R^+$ of full measure such that for every $\rho \in C^1_c(I\!R)$,

$$L^1_{\rm loc}(I\!R^d) - \lim_{t\to 0,\ t \in E} \int_{I\!R} h(t,x,\lambda)\rho(\lambda)d\lambda = \int_{I\!R} h_0(x,\lambda)\rho(\lambda)d\lambda.$$

As a corollary, under the traceability conditions, we prove existence of strong traces for entropy solutions to ultraparabolic equations in heterogeneous media.

1. Introduction

The aim of this paper is to investigate behavior of the averaged quantity $\int_{I\!R} \rho(\lambda)$ $h(t,x,\lambda)d\lambda$, $\rho \in C^1_c(I\!R)$, as $t \to 0$, where $h \in L^\infty(I\!R^+ \times I\!R^d \times I\!R)$ is a weak solution to the equation

$$\partial_t h + \operatorname{div}_x (F(t,x,\lambda)h) = \sum_{i,j=1}^k \partial^2_{x_i x_j} (b_{ij}(t,x,\lambda)h) + \partial_\lambda \gamma(t,x,\lambda), \tag{1}$$

where $(t,x) \in I\!R^d_+ := I\!R^+ \times I\!R^d \equiv (0,\infty) \times I\!R^d$, $k \le d$, $k, d \in I\!N$, and $\lambda \in I\!R$.

The coefficients of the equation satisfy the following assumptions:

- $F \in C^1(I\!R^d_+ \times I\!R; I\!R^d)$ and $b_{ij} \in C^1(I\!R^d_+ \times I\!R; I\!R)$, $i,j = 1, ...k$;
- The matrix $b(t,x,\lambda) = [b_{ij}(t,x,\lambda)]_{i,j=1,\dots,k}$ is nonnegative definite in the sense that

$$\langle b(t,x,\lambda)\xi, \xi\rangle \ge c(\lambda)|\xi|^2, \quad \xi \in I\!R^k,\ \lambda \in I\!R, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $I\!R^k$. The nonnegative continuous function c fulfills the following: there exists increasing sequence of real

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numbers $\{\lambda_i\}_{i\in\mathbb{Z}}$, such that

$$c(\lambda) > 0, \quad \text{for} \quad \lambda \in \bigcup_{i=-\infty}^{+\infty} (\lambda_i, \lambda_{i+1}); \tag{3}$$

Moreover, the elements of the matrix b are of the following form,

$$b_{ij}(t,x,\lambda) = \sum_{l=1}^{k} \sigma^b_{il}(t,x,\lambda)\sigma^b_{lj}(t,x,\lambda), \quad i,j = 1,..,k, \tag{4}$$

where $\sigma^b_{il}(t,x,\lambda) = \sigma^b_{li}(t,x,\lambda)$.

- $\gamma \in \mathcal{M}(I\!R^d_+ \times I\!R)$ is a locally finite Borel measure.

We consider bounded solutions h being equal to zero if $\lambda \notin (-M, M)$ for some $M > 0$, and such that $\partial_{x_i} h \in L^2(I\!R^d_+; \mathcal{M}(I\!R))$, $i = 1,\dots,k$, i.e. we assume:

$$\begin{split} h \in L^\infty(I\!R^{d+1}_+), \quad (\exists M > 0)\ \lambda \notin (-M, M) \implies h(t,x,\lambda) = 0 \text{ and} \\ (\forall i = 1,\dots,k)\ \|\partial_{x_i} h\|_{\mathcal{M}(I\!R_\lambda)} \in L^2_{\rm loc}(I\!R^d_+). \end{split} \tag{5}$$

In Section 5 we will see that (5) is fulfilled for kinetic functions corresponding to entropy solutions to ultra-parabolic equations.

We shall prove that, under traceability conditions (see Definition 8 and Definition 12) which essentially provide us to somehow remove non-degenerate heterogeneity from the flux (see (14)), h has the following property:

Definition 1. We say that the function h admits an averaged trace if there exists a function $h_0 \in L^\infty(I\!R^d \times I\!R)$ and a set $E \subset I\!R^+$ of full measure such that

$$\lim_{\substack{t\to 0\\ t\in E}} \int_K \Big| \int_{I\!R} \left(h(t,x,\lambda) - h_0(x,\lambda)\right) \rho(\lambda)\, d\lambda \Big|\, dx = 0, \tag{6}$$

for any function $\rho \in C^1_c(I\!R)$ and any relatively compact $K \subset\subset I\!R^d$.

Equation (1) describes transport processes in heterogeneous media in which the diffusion (represented by the second order terms) can be neglected in certain directions, cf. [10]. It is linear, but since it has derivatives of a measure on the right-hand side, it describes entropy solutions to ultra-parabolic equations (this is a special case of [3]). Such equations were firstly considered by Graetz [6], and Nusselt [14], in the investigations concerning the heat transfer. Moreover, equations of type (1) describe processes in porous media (cf. [19]), such as oil extraction or CO_2 sequestration which typically occur in highly heterogeneous surroundings. One can also find applications in sedimentation processes, traffic flow, radar shape-from-shading problems, blood flow, gas flow in a variable duct and so on.

The question of existence of traces was firstly raised in the context of limit of hyperbolic relaxation toward a scalar conservation laws (see e.g. [12, 20]). After that, a few interesting papers appeared from the viewpoint of obtained results and from the viewpoint of developed techniques [8, 16, 17, 21]. Proofs were based on reducing scalar conservation laws on a transport equation (kinetic formulation [11]), and then using velocity averaging results. Here, we shall start with a transport equation and prove that such equations satisfy the trace property in the sense of (6). In the basis of our procedure are the classical blow up techniques, [5, 16, 21], velocity averaging results [9], and induction with respect to the space dimension [16].

All previous results on traces were given for the scalar conservation laws in homogeneous media (see e.g. [8, 17] and references therein). However, heterogenous framework is much more natural. In natural porous media, it is typical to find significant variations in permeability, even in sands and other materials which would otherwise be considered homogeneous [13]. That means that fluxes in equations describing phenomena in such media must depend on the space variable.

The paper is organized as follows.

First, in Section 3, we consider the case when the components $F_{k+1},...,F_d$ of the flux function $F=(F_1,...,F_d)$ depend only on the variable λ (without explicit dependence on t and x), i.e. we consider the equation

$$\partial_t h+\sum_{i=k+1}^{d}\partial_{x_i}(F_i(\lambda)h)-\sum_{i,j=1}^{k}\partial^2_{x_ix_j}(b_{ij}(t,x,\lambda)h)= \tag{7}$$
$$=\partial_\lambda\gamma(t,x,\lambda)-\sum_{i=1}^{k}\partial_{x_i}(F_i(t,x,\lambda)h),$$

in $\mathcal{D}'(I\!R^d_+\times I\!R)$. We prove that bounded weak solution h of (7) that satisfies (5) admits an averaged trace in the sense of Definition 1.

In Section 4, under additional assumptions on the flux which we called traceability, we shall use appropriate change of variables to reduce equation (1) on an equation of type (7) in a neighborhood of every point where the existence of traces could be lost.

In Section 5, under the traceability conditions, we shall prove existence of traces for entropy solutions to ultra-parabolic equations. In particular, this includes entropy solutions to some examples of heterogeneous scalar conservation laws.

2. Auxiliary results

First, we prove existence of a weak trace.

Proposition 2. *If $h\in L^\infty(I\!R^d_+\times I\!R)$ is a distributional solution to (1) satisfying (5), then there exists $h_0\in L^\infty(I\!R^{d+1})$, such that*

$$h(t,\cdot,\cdot)\rightharpoonup h_0,\ \text{weakly-}\star\text{ in } L^\infty(I\!R^{d+1}),\ \text{as } t\to 0,\ t\in E,$$

where $E:=\{t>0\,|\,(t,x,\lambda)\text{ is a Lebesgue point to } h(t,x,\lambda)\text{ for a.e. } (x,\lambda)\in I\!R^{d+1}\}$.

Moreover, there exists a zero sequence ε_m such that for almost every $y\in I\!R^d$, it holds $h_0(\sqrt{\varepsilon_m}\bar x+\varepsilon_m\tilde x+y,\lambda)\to h_0(y,\lambda)$, strongly in $L^1_{\rm loc}(I\!R^d\times I\!R)$ as $m\to\infty$, where $\bar x=(x_1,...,x_k,0,...,0)$ and $\tilde x=(0,...,0,x_{k+1},...,x_d)$.

Proof: Since $h\in L^\infty(I\!R^d_+\times I\!R)$, the family $\{f(t,\cdot,\cdot)\}_{t\in E}$ is bounded in $L^\infty(I\!R^{d+1})$. Due to weak-$\star$ precompactness of $L^\infty(I\!R^{d+1})$, there exists a sequence $\{t_m\}_{m\in I\!N}$, $t_m\to 0$, as $m\to\infty$, and $h_0\in L^\infty(I\!R^{d+1})$, such that

$$h(t_m,\cdot,\cdot)\rightharpoonup h_0(\cdot,\cdot),\ \text{weakly-}\star\text{ in } L^\infty(I\!R^{d+1}),\text{ as } m\to\infty. \tag{8}$$

For $\phi\in C^\infty_c(I\!R^d)$, $\rho\in C^1_c(I\!R)$, denote

$$I(t):=\int_{I\!R^{d+1}}h(t,x,\lambda)\rho(\lambda)\phi(x)\,dxd\lambda,\quad t\in E.$$

With this notation, (8) means that

$$\lim_{m\to\infty} I(t_m) = \int_{I\!R^{d+1}} h_0(x,\lambda)\rho(\lambda)\phi(x)\,dxd\lambda =: I(0). \tag{9}$$

Now, fix $\tau \in E$ and $m_0 \in I\!N$, such that $t_m \leq \tau$, for $m \geq m_0$, to obtain

$$\begin{aligned}
I(\tau) - I(t_m) &= \int_{t_m}^{\tau} I'(t)\,dt = \int_{t_m}^{\tau}\int_{I\!R^{d+1}} \partial_t h(t,x,\lambda)\rho(\lambda)\phi(x)\,dxd\lambda\,dt \\
&= \sum_{i=1}^{d}\int_{(t_m,\tau]\times I\!R^{d+1}} h(t,x,\lambda)F_i(t,x,\lambda)\rho(\lambda)\partial_{x_i}\phi(x)\,dxd\lambda dt \\
&+ \sum_{i,j=1}^{k}\int_{(t_m,\tau]\times I\!R^{d+1}} h(t,x,\lambda)b_{ij}(t,x,\lambda)\rho(\lambda)\partial_{x_ix_j}\phi(x)\,dxd\lambda dt \\
&- \int_{(t_m,\tau]\times I\!R^{d+1}} \phi(x)\rho'(\lambda)\,d\gamma(t,x,\lambda).
\end{aligned}$$

Passing to the limit as $m \to \infty$, and having in mind (9), we obtain

$$\begin{aligned}
I(\tau) - I(0) &= \sum_{i=1}^{d}\int_{(0,\tau]\times I\!R^{d+1}} h(t,x,\lambda)F_i(t,x,\lambda)\rho(\lambda)\partial_{x_i}\phi(x)\,dx\,d\lambda\,dt \\
&+ \sum_{i,j=1}^{k}\int_{(0,\tau]\times I\!R^{d+1}} b_{ij}(t,x,\lambda)\rho(\lambda)h(t,x,\lambda)\partial_{x_ix_j}\phi(x)\,dx\,d\lambda\,dt \\
&- \int_{(0,\tau]\times I\!R^{d+1}} \rho'(\lambda)\phi(x)\,d\gamma(t,x,\lambda) \underset{\tau\to 0}{\longrightarrow} 0.
\end{aligned}$$

Thus, for all $\phi \in C_c^\infty(I\!R^d)$, $\rho \in C_c^1(I\!R)$,

$$\lim_{\tau\in E,\tau\to 0}\int_{I\!R^{d+1}} h(\tau,x,\lambda)\rho(\lambda)\phi(x)\,d\lambda\,dx = \int_{I\!R^{d+1}} h_0(x,\lambda)\rho(\lambda)\phi(x)\,d\lambda\,dx.$$

Having in mind that $h(\tau,\cdot)$ is bounded for almost every $\tau \in I\!R$, and $C_c^\infty(I\!R^{d+1})$ is dense in $L^1(I\!R^{d+1})$, we complete the first part of the proof.

The second part of the proof is the same as the proof of [21, Lemma 2]. ❑

2.1. **Scaling.** Denote

$$\bar{x} = (x_1,...x_k,0,...0) \in I\!R^d, \quad \tilde{x} = (0,...,0,x_{k-1},...,x_d) \in I\!R^d, \quad \bar{x}+\tilde{x} = x \in I\!R^d,$$

and change the variables in the following way, $t = \varepsilon_m\hat{t}$, $x_1 = y_1 + \sqrt{\varepsilon_m}\hat{x}_1$, $\dots$, $x_k = y_k + \sqrt{\varepsilon_m}\hat{x}_k$, $x_{k+1} = y_{k+1} + \varepsilon_m\hat{x}_{k+1}$, $\dots$, $x_d = y_d + \varepsilon_m\hat{x}_d$, i.e.

$$(t,x,\lambda) = (\varepsilon_m\hat{t}, \sqrt{\varepsilon_m}\bar{\hat{x}} + \varepsilon_m\tilde{\hat{x}} + y, \lambda), \tag{10}$$

where $(\varepsilon_m)_{m\in I\!N}$ is a sequence of positive numbers converging to zero and $y \in I\!R^d$ is a fixed vector.

If we prove that for any $\rho \in C_c^1(I\!R)$, the sequence

$$\int h^m(\hat{t},\hat{x},\lambda)\rho(\lambda)d\lambda := \int h(\varepsilon_m\hat{t}, \sqrt{\varepsilon_m}\bar{\hat{x}} + \varepsilon_m\tilde{\hat{x}} + y, \lambda)\rho(\lambda)d\lambda, \tag{11}$$

converges pointwise almost everywhere along a subsequence (the same subsequence for almost every $y \in I\!R^d$), we will obtain that function h admits an averaged trace in the sense of Definition 1. To this end, rewrite equation (7) in terms of variables $(\hat{t},\hat{x})$ given by (10):

$$\begin{aligned} L_{h^m} :=&\partial_{\hat{t}} h^m + \sum_{i=k+1}^{d} \partial_{\hat{x}_i}\left(F_i(\lambda)h^m\right) - \sum_{i,j=1}^{k} \partial^2_{\hat{x}_i\hat{x}_j}(b^m_{ij}h^m) = \\ &= \varepsilon_m \partial_\lambda \gamma^m - \sqrt{\varepsilon_m}\sum_{i=1}^{k}\partial_{\hat{x}_i}\left(F^m_i h^m\right) =: \partial_\lambda \hat{\gamma}^m, \end{aligned} \tag{12}$$

where $b^m_{ij}(\hat{t},\hat{x},\lambda) = b_{ij}(\varepsilon_m \hat{t}, y + \sqrt{\varepsilon_m}\bar{\hat{x}} + \varepsilon_m \tilde{\hat{x}}, \lambda)$ and the same for F^m_i and γ^m.

Let us remark that, considering the right hand side of (12) and due to assumption (5), we have that for any $\rho \in C^1_c(I\!R)$, $\int \rho(\lambda) L_{h^m} d\lambda \in \mathcal{M}(I\!R^d_+)$. Moreover, we have the following lemma whose proof is the same as the one from [16, Lemma 3.2].

Lemma 3. *After a possible extraction of a subsequence, for a.e $y \in I\!R^d$ and any $\rho \in C^1_c(I\!R)$,*

$$\int_{I\!R} \rho(\lambda) L_{h^m}(\hat{t},\hat{x},\lambda) d\lambda \to 0, \ \textit{as } m\to\infty, \ \textit{in } \mathcal{M}(I\!R^d_+) \ \ \textit{strongly.}$$

3. Averaged traces in the homogeneous case

In this section, we consider equation (7). We shall prove the following theorem.

Theorem 4. *Assume that $h \in L^\infty(I\!R^d_+ \times I\!R)$ is a weak solution to (7) satisfying (5). Then, there exists a function $h_0 \in L^\infty(I\!R^{d+1})$ such that (6) holds.*

In order to prove the theorem, we need a corollary of the result from [9] (see also Remark 16 and Section 5 there).

Theorem 5. [9, Theorem 7] *Denote by $P = \{\xi \in I\!R^{d+1} : \xi_0^2 + \xi_1^4 + ... + \xi_k^4 + \xi_{k+1}^2 + .. + \xi_d^2 = 1\}$ (the ultra-parabolic manifold).*

Assume that $h_n \rightharpoonup 0$ weakly in $L^\infty_{\rm loc}(I\!R^+ \times I\!R^d \times I\!R)$, where h_n represent weak solutions to

$$\partial_t h_n + {\rm div}_x\left(\tilde{F}(t,x,\lambda)h_n\right) = \sum_{s,j=1}^{k} \partial^2_{x_s x_j}\left(\tilde{b}_{sj}(t,x,\lambda)h_n\right) + \partial_\lambda \gamma_n(t,x,\lambda), \tag{13}$$

where $\tilde{F}$ and $\tilde{b} = (\tilde{b}_{sj})_{s,j=1,\dots,k}$ are continuous functions, the matrix $\tilde{b} = (\tilde{b}_{sj})_{s,j=1,\dots,k}$ is positively definite almost everywhere, and the sequence $(\gamma_n)_n$ is strongly precompact in $L^2_{\rm loc}(I\!R; W^{-1,q}_{\rm loc}(I\!R^d_+))$.

Assume that for every $\xi \in P$ and almost every $(t,x) \in I\!R^d_+$ (i is the imaginary unit below)

$$i\left(\xi_0 + \sum_{j=k+1}^{d} \tilde{F}_j(t,x,\lambda)\xi_j\right) + \sum_{s,j=1}^{k} \tilde{b}_{sj}(t,x,\lambda)\xi_s\xi_j \neq 0 \quad \text{a.e. } \lambda \in I\!R. \tag{14}$$

Then, for any $\rho \in C^1_c(I\!R)$,

$$\int_{I\!R} h_n(t,x,\lambda)\rho(\lambda)d\lambda \to 0 \quad \textit{strongly in } L^1_{\rm loc}(I\!R^+ \times I\!R^d).$$

Remark 6. Since the matrix b is positively-definite almost everywhere, it is enough to assume that for almost all $x \in I\!R^d$ and every $\tilde{\xi} \in I\!R^{d-k}$, $\tilde{\xi} \neq 0$, the function $\lambda \mapsto \sum\limits_{j=k+1}^{d} \tilde{F}_j \xi_j$ is not constant on a set of positive measure. Remark that we can also use somewhat weaker assumptions given in [18, Definition 2].

Remark also that, due to linearity of equation (13), the condition $h_n \rightharpoonup 0$ in Theorem 5 can be replaced by boundedness of $(h_n)_{n\in I\!N}$. In that case, there exists a subsequence of $(h_n)_{n\in I\!N}$ (not relabeled here) such that for some $h \in L^\infty_{\rm loc}(I\!R^d_+ \times I\!R)$, it holds $h_n \rightharpoonup h$. Then, the sequence $(h_n - h)_{n\in I\!N}$ satisfies equation of type (13), and we have

$$\int_{I\!R^m} h_n(t,x,\lambda)\rho(\lambda)d\lambda \to \int_{I\!R^m} h(t,x,\lambda)\rho(\lambda)d\lambda \quad \text{strongly in } L^1_{\rm loc}(I\!R^+ \times I\!R^d).$$

Next, we need the following proposition.

Proposition 7. *Let h be a distributional solution to* (7) *satisfying* (5)*, and suppose that, in* (7)*, the component F_d of the flux vector F is absent, i.e. equation* (7) *has the form*

$$\begin{aligned} &\partial_t h + \sum_{i=1}^{k} \partial_{x_i}(F_i(t,x,\lambda)h) + \sum_{i=k+1}^{d-1} \partial_{x_i}(F_i(\lambda)h) \\ &= \sum_{i,j=1}^{k} \partial_{x_i x_j}(b_{ij}(t,x,\lambda)h) + \partial_\lambda \gamma(t,x,\lambda), \quad \text{in } \mathcal{D}'(I\!R^+ \times I\!R^d \times I\!R). \end{aligned} \tag{15}$$

Then for a.e. $x_d \in I\!R$, $\tilde{h}(t,x',\lambda) := h(t,x',x_d,\lambda)$, $x' = (x_1,...,x_{d-1})$, is a weak solution to the (reduced) equation (15)*, and x_d is treated like a parameter.*

Proof: Take $\psi \in L^\infty_c(I\!R)$, $\phi \in C^2(I\!R^+ \times I\!R^{d-1})$, and $\rho \in C^1_c(I\!R)$ and test (7) on $\phi(t,x')\psi(x_d)\rho(\lambda)$ to obtain

$$\Big| \int_{I\!R^+ \times I\!R^d \times I\!R} \psi(x_d)\phi(t,x')\rho'(\lambda)d\gamma(t,x',x_d,\lambda) \Big| \leq C(\|\rho\|_{C^1}, \|\phi\|_{C^2})\|\psi\|_{L^\infty}. \tag{16}$$

Furthermore, it is not difficult to see that $\int_{I\!R^+ \times I\!R^{d-1} \times I\!R} \phi(t,x')\rho'(\lambda)d\gamma(t,x',x_d,\lambda) \in \mathcal{M}(I\!R_d)$ is absolutely continuous with respect to the Lebesgue measure. Thus, for every fixed ρ and ϕ, the function $x_d \mapsto \int_{I\!R^+ \times I\!R^{d-1} \times I\!R} \phi(t,x')\rho'(\lambda)d\gamma(t,x',x_d,\lambda)$ belongs to $L^1_{\rm loc}(I\!R)$. Therefore, there exists a set E of full measure such that (15) is satisfied for $x_d \in E$ for the taken functions ρ and ϕ.

Now, take test functions $\phi\rho \in S$, $\phi = \phi(t,x')$, $\rho = \rho(\lambda)$, where S is a countable dense subset of $C^2_c(I\!R^{d-1}_+) \times C^1_c(I\!R)$. Again, we can find a set E of full measure such that (15) is satisfied for $x_d \in E$ and all $\phi\rho \in S$. Using the density of S in $C^2_c(I\!R^{d-1}_+) \times C^1_c(I\!R)$, we conclude that for every $x_d \in E$, (15) holds for any $\eta = \phi\rho \in C^2_c(I\!R^{d-1}_+) \times C^1_c(I\!R)$, i.e. due to the density arguments again, for any $\eta \in C^2(I\!R^{d-1}_+ \times I\!R)$. ❑

Proof of Theorem 4: First assume that equation (7) is such that for almost every $y \in I\!R^d$ the flux and the diffusion matrix

$$(1, F(0,y,\lambda)) \ \text{ and } \ b(0,y,\lambda) = (b_{sj}(0,y,\lambda))_{s,j=1,\dots k} \tag{17}$$

respectively, are non-degenerate on an interval (a,b) in the sense of (14) (we replace there $\tilde F$ and $\tilde b$ by $F(0,y,\lambda)$ and $b(0,y,\lambda)$). Consider the sequence $h^m(\hat t,\hat x,y,\lambda)=h(\varepsilon_m\hat t,\sqrt{\varepsilon_m}\bar{\hat x}+\varepsilon_m\tilde{\hat x}+y,\lambda)$ defined in (11), that satisfies (12). For a fixed y, the sequence $(h^m)_m$ is bounded in $L^\infty(I\!R^{d+1}_+)$ and has a weakly-$\star$ convergent subsequence (not relabeled). Denote by $\tilde h_0(\hat t,\hat x,y,\lambda)$ its weak-$\star$ limit.

Notice that the sequence of functions $w_m=h^m-\tilde h_0$ fulfills assumptions of Theorem 5 (i.e. it satisfies equation (13) with $\tilde F=F(0,y,\cdot)$ and $\tilde b=b(0,y,\cdot)$), so there exists a subsequence of $(h^m)_m$ (not relabeled) such that for any $\rho\in C^1_c(a,b)$

$$\int_a^b h^m(\hat t,\hat x,y,\lambda)\rho(\lambda)d\lambda\to\int_a^b \tilde h_0(\hat t,\hat x,y,\lambda)\rho(\lambda)d\lambda \ \text{ in } \ L^1_{\rm loc}(I\!R^d_+\times I\!R). \tag{18}$$

According to Lemma 3, the function $\tilde h_0$ satisfies (we remind that y is fixed):

$$\partial_{\hat t}\tilde h_0+\sum_{i=k+1}^d\partial_{\hat x_i}\left(F_i(\lambda)\tilde h_0\right)=\sum_{i,j=1}^k\partial^2_{\hat x_i\hat x_j}\left(\hat b_{ij}(0,y,\lambda)\tilde h_0\right).$$

Since the last equation is linear, the function $\tilde h_0$ is also its isentropic solution (see e.g. [15]) and thus, repeating the proof of [16, Proposition 2], we conclude that $\tilde h_0$ admits the strong trace at $t=0$. From here and according to Proposition 2, we conclude that

$$\tilde h_0(0,\hat x,y,\lambda)=h_0(y,\lambda),$$

where h_0 is defined in Proposition 2. Details can be found in the proof of [16, Theorem 2].

Since a solution to a Cauchy problem for linear ultra-parabolic equations with regular coefficients must be unique, we conclude that, for almost every $y\in I\!R^d$, the limit from (18) does not depend on the choice of the subsequence. Moreover, we conclude that for almost every $(y,\lambda)\in I\!R^d\times I\!R$, it holds

$$\tilde h_0(\hat t,\hat x,y,\lambda)=h_0(y,\lambda).$$

Thus, for $t_m=\varepsilon_m\hat t$ ($\hat t>0$ is fixed)

$$\int_a^b h(t_m,\hat x,y,\lambda)\rho(\lambda)d\lambda\to\int_a^b h_0(y,\lambda)\rho(\lambda)d\lambda \ \text{ in } \ L^1_{\rm loc}(I\!R^d).$$

Since the sequence (t_m) is arbitrary, this proves Theorem 4 in the non-degenerate case.

Now assume that the non-degeneracy is lost in (a,b) for the flux and the diffusion from (17). In this case we use the method of mathematical induction with respect to $d-k\geq 0$.

Step 1. Assume that $d-k=0$. In this case, equation (7) reduces to the strictly parabolic equation which means that it satisfies the non-degeneracy condition from (14) for the flux and the diffusion from (17). Thus, the existence of traces follows from the first part of the proof.

Step 2. Assume that if $h\in L^\infty(I\!R^{d-1}_+\times I\!R)$, $h=h(t,x_1,...,x_{d-1},\lambda)$, is a weak solution to (15), then there exists $h_0\in L^\infty(I\!R^{d-1}\times I\!R)$ and a set $E\subset I\!R^+$ of full measure, such that for all zero sequences $t_m\in E$ and every $\rho\in C^1_c(I\!R)$

$L^1_{\rm loc}(I\!R^{d-1})-\lim_{m\to\infty}\int_{I\!R} h(t_m,x,\lambda)\rho(\lambda)d\lambda=\int_{I\!R} h_0(x,\lambda)\rho(\lambda)d\lambda$.

Step 3. Let $h\in L^\infty(I\!R^d_+\times I\!R)$ be a weak solution to (7). Since the non-degeneracy is lost in (a,b), there exists nonzero vector $(\xi_{k+1},...,\xi_d)\in I\!R^{d-k+1}$ and a constant c_d such that

$$\xi_{k+1}F_{k+1}(\lambda)+...+\xi_dF_d(\lambda)=-c_d,\quad \lambda\in(a,b). \tag{19}$$

Introduce the change of spatial variables $\tilde{x}\equiv(x_{k+1},...,x_d)\in I\!R^{d-k}\mapsto(z_{k+1},...,z_d)\equiv\tilde{z}\in I\!R^{d-k}$ as $\tilde{z}=ct+A\tilde{x}$, where $c=(c_{k+1},...,c_d)^\top$ and $A=[a_{ij}]_{i,j=k+1,...,d}\in I\!R^{d-k\times d-k}$, $a_{ij}=a_{ji}$. Other spatial variables will remain unchanged, i.e. $z_1=x_1$, ..., $z_k=x_k$. With this change, for $h=h(t,z,\lambda)$, equation (7) becomes

$$h_t+\sum_{i=1}^k\partial_{z_i}\left[F_i(t,z,\lambda)h\right]+\sum_{l=k+1}^d\partial_{z_l}\left[\left(c_l+\sum_{i=k+1}^d a_{li}F_i\right)h\right]=\sum_{i,j=1}^k\partial_{z_j}\left[b_{ij}\partial_{z_i}h\right]+\partial_\lambda\gamma.$$

Denote $\tilde{F}_l:=c_l+\sum_{i=k+1}^d a_{li}F_i$, $l=k+1,...,d$, and $\tilde{F}_i:=F_i$, $i=1,...,k$. According to (19), we choose $a_{d,k+1}:=\xi_{k+1}$, ..., $a_{d,d}:=\xi_d$ and obtain $\partial_{z_d}\tilde{F}_d(t,z,\lambda)=0$, for $\lambda\in(a,b)$. Thus, for $\lambda\in(a,b)$, the equation takes the following form,

$$\partial_t h+\sum_{i=1}^k\partial_{z_i}(F_i(t,z,\lambda)h)+\sum_{i=k+1}^{d-1}\partial_{z_i}(\tilde{F}_i(\lambda)h)=\sum_{i,j=1}^k\partial^2_{z_iz_j}(b_{ij}h)+\partial_\lambda\gamma. \tag{20}$$

According to Proposition 7, for a fixed (parameter) z_d, the function $h=h(t,z',z_d,\lambda)$, $z'\in I\!R^{d-1}$ is a weak solution to (20), i.e. $h(t,z',z_d,\lambda)$ is a weak solution to (20), for a.e. $z_d\in I\!R$.

Then, according to the inductive hypothesis, for a.e. $z_d\in I\!R$, there is $\tilde{h}_0(z',\lambda)\equiv\tilde{h}_0[z_d]\in L^\infty(I\!R^{d-1}\times I\!R)$ such that for all $\rho\in C^1_c(I\!R)$,

$$L^1_{\rm loc}(I\!R^{d-1})-\lim_{m\to\infty}\int_{I\!R}h(t_m,z',z_d,\lambda)\rho(\lambda)d\lambda=\int_{I\!R}h_0[z_d](z',\lambda)\rho(\lambda)d\lambda,$$

for a sequence (t_m) of positive numbers tending to zero.

To obtain the analogical assertion in $I\!R^d$, we need a special choice of (t_m,z_d), so we use the following construction (the same construction is used in [16]). Denote

$$\begin{aligned}
E:=&\{t>0\,|\,(t,x,\lambda)\text{ is a Lebesgue point to }h(t,x,\lambda)\text{ for a.e. }(x,\lambda)\in I\!R^d\times I\!R\},\\
\mathcal{M}:=&\{(t,z,\lambda)\equiv(t,z',z_d,\lambda)\,|\,(t,z,\lambda)\text{ is a Lebesgue point to }h\text{ and}\\
&\qquad\qquad(t,z',\lambda)\text{ is a Lebesgue point to }h(\cdot,z_d,\cdot)\},\\
\mathcal{M}_t:=&\{(z,\lambda)\,|\,(t,z,\lambda)\in\mathcal{M}\},
\end{aligned}$$

which are sets of full measure. There exists a subsequence $(t_r)_r$ from $E'=\{t>0:\mathcal{M}_t$ is a set of full measure$\}$ such that (t_r,z',λ) is a Lebesgue point to $h(\cdot,z_d,\cdot)$. Take

$$z_d\in\mathcal{Z}=\bigcap_r\mathcal{Z}_r,\ \text{ where }\mathcal{Z}_r:=\{s\in I\!R\,|\,(z',s,\lambda)\in\mathcal{M}_{t_r}\}.$$

Applying the inductional hypothesis to $h(t_r,z',z_d,\lambda)$ we obtain that there exists $\tilde{h}_0(\cdot,z_d,\lambda)\in L^\infty(I\!R^{d-1}\times I\!R)$ such that

$$L^1_{\rm loc}(I\!R^{d-1})-\lim_{r\to\infty}\int_{I\!R}h(t_r,z',z_d,\lambda)\rho(\lambda)d\lambda=\int_{I\!R}\tilde{h}_0(z',z_d,\lambda)\rho(\lambda)d\lambda.$$

With the choice $z=(z',z_d)$, we have that $\tilde h_0(z,\lambda)=\tilde h_0(z',z_d,\lambda)\in L^\infty(I\!R^d\times I\!R)$, and then apply the Lebesgue dominated convergence theorem to conclude that

$$L^1_{\rm loc}(I\!R^d)-\lim_{r\to\infty}\int_{I\!R} h(t_r,z,\lambda)\rho(\lambda)d\lambda=\int_{I\!R}\tilde h_0(z,\lambda)\rho(\lambda)d\lambda.$$

Now, the same limit relation follows for the original variable x, i.e.

$$L^1_{\rm loc}(I\!R^d)-\lim_{r\to\infty}\int_{I\!R} h(t_r,x,\lambda)\rho(\lambda)d\lambda=\int_{I\!R}\tilde h_0(x,\lambda)\rho(\lambda)d\lambda.$$

□

4. The heterogeneous case; proof of the main theorem

In this section we shall prove the main result of the paper. First, we introduce the notion of traceability.

Definition 8. We say that the coefficients of equation (1) satisfy *the strong traceability conditions* at the point $(x^0,\lambda^0)\in I\!R^d\times I\!R$ if one of the following two conditions are satisfied:

1) There exists a neighborhood of $(0,x^0,\lambda^0)$ in which the non-degeneracy condition (14) is fulfilled for the flux and the diffusion from (17), or

2) There exists a neighborhood U of $x^0\in I\!R^d$, $T>0$, an interval $\lambda_0\in(\alpha,\beta)\subset I\!R$, and a regular change of variables $(\hat t,\hat x):(0,T)\times U\to(0,\hat T)\times\hat U\subset I\!R^{d+1}$ such that

- For all $s=k+1,...,d$, $t\in(0,T)$, $x\in U$ and $\lambda\in(\alpha,\beta)$

$$\hat F_s+\sum_{i,j=1}^k b_{ij}\cdot\left(2\sum_{r=0}^d\partial_{\hat x_r}\left[\frac{\partial\hat x_r}{\partial x_j}\frac{\partial\hat x_s}{\partial x_i}\right]-\frac{\partial^2\hat x_s}{\partial x_i\partial x_j})\right)=:p_s(\lambda)\tag{21}$$

  is independent of $x\in U$, $t\in(0,T)$, where we denote $\hat F_s=\sum_{i=0}^d F_i\frac{\partial\hat x_s}{\partial x_i}$, for $s=0,...,d$, $F_0=1$, $x_0=t$ and $\hat x_0=\hat t$.
- The coefficient

$$\hat F_0=\sum_{j=0}^d F_j\frac{\partial\hat t}{\partial x_j}\equiv const\neq 0.\tag{22}$$

- The matrix

$$\left(\sum_{i,j=1}^k b_{ij}\frac{\partial\hat x_s}{\partial x_i}\frac{\partial\hat x_r}{\partial x_j}\right)_{s,r=0,\dots,d}=:(\hat b_{sr})_{s,r=0,\dots,d},\tag{23}$$

  has the same properties as the matrix b from (2) but with respect to the variables $(\hat x_0,\hat x_1,...,\hat x_d)$.

Remark 9. The matrix $\hat b$ defined in (23) should have null entries for $\max\{s,r\}>k$ and also for $\min\{s,r\}=0$. This implies that

$$\frac{\partial\hat x_s}{\partial x_i}=0\quad\text{if}\quad s\in\{0,k+1,k+2,...,d\}\text{ and }i\in\{1,...,k\},$$

i.e. that the variables $\hat{x}_s = \hat{x}_s(t, x_{k+1}, ..., x_d)$, $s \in \{0, k+1, k+2, ..., d\}$, do not depend on x_i, $i = 1, ..., k$. This fact reduces condition (21) to

$$\hat{F}_s = \frac{\partial \hat{x}_s}{\partial t} + \sum_{i=k+1}^{d} F_i \frac{\partial \hat{x}_s}{\partial x_i} = p_s(\lambda), \quad s = k+1, ..., d$$

and condition (22) to

$$\hat{F}_0 = \frac{\partial \hat{t}}{\partial t} + \sum_{j=k+1}^{d} F_j \frac{\partial \hat{t}}{\partial x_j} \equiv const.$$

We would like to thank to the referee for this remark.

Theorem 10. *Let $h \in L^\infty(I\!R^d_+ \times I\!R)$ be a weak solution to* (1)*. Moreover, assume that equation* (1) *satisfies the strong traceability conditions from Definition* 8 *on a set $E \times F \subset I\!R^d \times I\!R$ of full measure. Then the function h admits an averaged trace in the sense of Definition* 1*.*

Proof: Fix $(x^0, \lambda^0) \in I\!R^{d+1}$ and the neighborhood $U \times (\alpha, \beta)$, $U = U(x^0)$, so that the strong traceability property is fulfilled. If the flux $(1, F(0, x, \lambda))$ and diffusion matrix $b(0, x, \lambda)$ are non-degenerate in that neighborhood, then we repeat the first part of the proof of Theorem 4 to conclude that the function h admits an averaged trace on $U \times (\alpha, \beta)$.

Consider now the change of variables from the second condition from the strong traceability definition. Using the same notation as in Definition 8 we calculate

$$\sum_{i=0}^{d} \partial_{x_i}(F_i h) = \sum_{i=0}^{d}\sum_{s=0}^{d} \frac{\partial (F_i h)}{\partial \hat{x}_s} \frac{\partial \hat{x}_s}{\partial x_i} = \tag{24}$$
$$= \sum_{s=0}^{d} \partial_{\hat{x}_s}[\hat{F}_s h] - h \sum_{i,s=0}^{d} F_i \, \partial_{\hat{x}_s}\left[\frac{\partial \hat{x}_s}{\partial x_i}\right],$$

where $\hat{F} = (\hat{F}_0, ..., \hat{F}_d)$, and

$$\sum_{i,j=1}^{k} \partial^2_{x_i x_j}(b_{ij} h) = \sum_{i,j=1}^{k} \left(\sum_{s,r=0}^{d} \frac{\partial^2 (b_{ij} h)}{\partial \hat{x}_s \partial \hat{x}_r} \frac{\partial \hat{x}_s}{\partial x_i} \frac{\partial \hat{x}_r}{\partial x_j} + \sum_{s=0}^{d} \frac{\partial (b_{ij} h)}{\partial \hat{x}_s} \frac{\partial^2 \hat{x}_s}{\partial x_i \partial x_j} \right) = \tag{25}$$
$$= \sum_{s,r=0}^{d} \partial^2_{\hat{x}_s \hat{x}_r}[\hat{b}_{sr} h] + \sum_{s=0}^{d} \partial_{\hat{x}_s}\left[h \sum_{i,j=1}^{k} b_{ij} \cdot \left(\frac{\partial^2 \hat{x}_s}{\partial x_i \partial x_j} - 2 \sum_{r=0}^{d} \partial_{\hat{x}_r}\left[\frac{\partial \hat{x}_s}{\partial x_i} \frac{\partial \hat{x}_r}{\partial x_j}\right] \right) \right]$$
$$+ h \sum_{i,j=1}^{k} b_{ij} \left(\sum_{s,r=0}^{d} \partial^2_{\hat{x}_s \hat{x}_r}\left[\frac{\partial \hat{x}_s}{\partial x_i} \frac{\partial \hat{x}_r}{\partial x_j}\right] - \sum_{s=0}^{d} \partial_{\hat{x}_s}\left[\frac{\partial^2 \hat{x}_s}{\partial x_i \partial x_j}\right] \right)$$

By inserting (24) and (25) into (1), we get from the strong traceability conditions (keep in mind the fact that the matrix B is symmetric)

$$\partial_{\hat t}[\hat F_0 h] + \sum_{s=1}^k \partial_{\hat x_s}\left[h\cdot\left(\hat F_s - \sum_{i,j=1}^k b_{ij}\left(2\sum_{r=1}^k \partial_{\hat x_r}\left[\frac{\partial \hat x_s}{\partial x_i}\frac{\partial \hat x_r}{\partial x_j}\right] - \frac{\partial^2 \hat x_s}{\partial x_i \partial x_j}\right)\right)\right] \tag{26}$$
$$+\sum_{s=k+1}^d \partial_{\hat x_s}[h\cdot p_s(\lambda)] - \sum_{s,r=1}^k \partial^2_{\hat x_s \hat x_r}[h\cdot \hat b_{sr}] = h\cdot g(\hat t,\hat x,\lambda) + \partial_\lambda \gamma(\hat t,\hat x,\lambda),$$

for $(\hat t,\hat x,\lambda)\in(0,\hat T)\times\hat U\times(\alpha,\beta)$, and appropriate function $\hat g$. Equation (26) has the form of equation (7) and thus, according to Theorem 4, $\hat h := h(\hat t,\hat x,\lambda)$ admits averaged traces on $\hat U\times(\alpha,\beta)$. Since the transformation $(\hat t,\hat x)$ is regular, we conclude that the averaged traces exist on $U\times(\alpha,\beta)$ as well.

Since the averaged traces exist in a neighborhood of almost every point (x_0,λ_0), by a diagonalization procedure, we conclude that they exist globally. ❑

Remark 11. Notice that from the proof of Theorem 4 it follows that it is enough to demand here the strong traceability conditions for the flux and diffusion of the form

$$(1, F(0,x,\lambda)) \quad\text{and}\quad (b_{sj}(0,x,\lambda))_{s,j=1,\dots k}$$

which implies that in the change of variables given in the strong traceability conditions we can take $\hat t = t$. This implies that condition (22) can be omitted.

Again, we would like to thank to the referee for this remark.

The idea of reduction of the space dimension and the previous theorem can be used to weaken the strong traceability conditions.

Definition 12. We say that the coefficients of equation (1) satisfy weak traceability conditions at the point $(x^0,\lambda^0)\in I\!R^d\times I\!R$ if there exists a neighborhood $(0,T)\times U\times(\alpha,\beta)$ of the point $(0,x^0,\lambda^0)$, and a regular transformation $\hat x:(0,T)\times U\to(0,\hat T)\times\hat U$ which reduces equation (1) on

$$\partial_{\hat t}\hat h + \sum_{i=l}^d \partial_{\hat x_i}(\hat F_i(\hat t,\hat x,\lambda)\hat h) = \sum_{i,j=1}^k \partial^2_{\hat x_i\hat x_j}\left(\hat b_{ij}(\hat t,\hat x,\lambda)\hat h\right)$$
$$+\sum_{m=1}^k \partial_{x_m}(\hat F(\hat t,\hat x,\lambda)\hat h) + \hat g(\hat t,\hat x,\lambda)\hat h + \partial_\lambda\hat\gamma(\hat t,\hat x,\lambda),$$

where $k<l\le d$, and the last equation satisfies the strong traceability conditions on $(0,\hat T)\times\hat U\times(\alpha,\beta)$ with $\hat x_{k+1},\dots,\hat x_{l-1}$ considered as parameters. The functions $\hat F$, $\hat b$, and $\hat g$ are computed as in the proof of Theorem 10, while $\hat\gamma$ equals to $\gamma((t,x)(\hat t,\hat x),\lambda)$.

The following theorem is a consequence of the induction arguments and Proposition 7.

Theorem 13. *Let $h\in L^\infty(I\!R^+\times I\!R^d\times I\!R)$ be a weak solution to* (1). *Moreover, assume that equation* (1) *satisfies the weak traceability conditions from Definition* 12. *Then, there exists a function $h_0\in L^\infty(I\!R^d\times I\!R)$ such that* (6) *holds.*

5. Nonlinear ultra-parabolic equations

In this section, we shall consider the following equation

$$\partial_t u + \operatorname{div}_x f(t,x,u) - \sum_{i,j=1}^{k} \partial^2_{x_i x_j}(B_{ij}(t,x,u)) = 0, \tag{27}$$

where $-M \leq u \leq M$, $f \in C^1(I\!R^d_+ \times I\!R; I\!R^d)$, $F = \partial_u f \in C^1(I\!R^{d+1}_+)$ and $b_{ij} = \partial_u B_{ij} \in C^1(I\!R^{d+1}_+)$, $i,j = 1,...,k$ satisfy (2)-(4). The latter equation admits several weak solutions and it is usual to impose a conditions that solutions must satisfy in order to be physically relevant. Such conditions are called the entropy admissibility conditions [2, 3, 18], and they represent an extension of the Kruzhkov admissibility concept for scalar conservation laws [7], cf. [2, 3, 18]. Their natural generalization is quasi-solutions concept introduced in [16] for scalar conservation laws. We shall adapt it here to our situation. Before that, remark that since $f \in C^2(I\!R^d \times I\!R; I\!R^d)$, the first order derivatives of f are locally bounded and we have that

$$\sum_{i=1}^{d} D_{x_i} f_i(t,x,c) = \gamma_c \in \mathcal{M}(I\!R^d), \quad c \in I\!R,$$

where $\mathcal{M}$ stands for locally finite Borel measures.

Definition 14. We say that $u \in L^\infty(I\!R^d_+)$ is a quasi solution to (27) if it satisfies

$$\sum_{i=1}^{k} \sigma^b_{li}(u)\partial_{x_i} u \in L^2(I\!R^d_+), \quad l = 1, \dots, k, \tag{28}$$

and for almost every $c \in I\!R$, $(t,x) \in I\!R^d_+$, there exists a locally finite Borel measure $\mu_c \in \mathcal{M}(I\!R^d_+)$, such that

$$\begin{aligned} L^c_u(t,x) \equiv{}& \partial_t|u-c| + \operatorname{div}\left[\operatorname{sgn}(u-c)(f(t,x,u) - f(t,x,c))\right] \\ &- \sum_{i,j=1}^{k} \partial^2_{x_i x_j}\left(\operatorname{sgn}(u-c)(B_{ij}(t,x,u) - B_{ij}(t,x,c))\right) = -\mu_c(t,x), \end{aligned} \tag{29}$$

in $\mathcal{D}'(I\!R^d_+)$. The family of measures μ_c, $c \in I\!R$, is called the entropy defect-measure corresponding to u.

Putting $c > \|u\|_\infty$ in (29), it follows that there exists $\mu \in \mathcal{M}(I\!R^d_+)$, such that

$$\partial_t u + \operatorname{div}_x f(t,x,u) - \sum_{i,j=1}^{k} \partial^2_{x_i x_j}(B_{ij}(t,x,u)) = -\mu, \tag{30}$$

in $\mathcal{D}'(I\!R^d_+)$.

Lemma 15. *If u is a quasi solution to (27), then the cut-off function*

$$s_{a,b}(u)(t,x) = \max\{a, \min\{u(t,x), b\}\},$$

for $a, b \in I\!R$, $a < b$, is a quasi solution to (27) as well.

Proof: Denote, $v = v(t,x) = s_{a,b}(u(t,x))$ and $c' = \max\{a, \min\{c,b\}\}$. Notice that

$$s'_{a,b}(\lambda) = \begin{cases} 1, & a < \lambda < b \\ 0, & \lambda < a \text{ or } \lambda > b \end{cases}. \tag{31}$$

For a continuous function $F = F(t,x,\lambda)$, denote $S(v,c) = \text{sgn}(v-c)(F(t,x,v) - F(t,x,c))$. One can verify that

$$\begin{aligned} S(v,c) &= \begin{cases} S(u,c') - \frac{1}{2}\Big(S(u,a)+S(u,b)\Big) + \frac{1}{2}\Big(F(a)+F(b)\Big) - F(c), & c<a \\ S(u,c') - \frac{1}{2}\Big(S(u,a)+S(u,b)\Big) + \frac{1}{2}\Big(F(b)-F(a)\Big), & a<c<b \\ S(u,c') - \frac{1}{2}\Big(S(u,a)+S(u,b)\Big) - \frac{1}{2}\Big(F(a)+F(b)\Big) + F(c), & b<c \end{cases} \\ &= S(u,c') - \frac{1}{2}\Big(S(u,a)+S(u,b)\Big) + \frac{1}{2}\Big(S(a,c)+S(b,c)\Big) \end{aligned}$$

The same arguing holds for $\text{sgn}(v-c)(b(t,x,v)-b(t,x,c))$. This enables us to conclude that

$$\begin{aligned} L_v^c &= -\mu_{c'} + \frac{1}{2}\Big(\mu_a + \mu_b + \text{sgn}(a-c)(\gamma_a - \gamma_c) + \text{sgn}(b-c)(\gamma_b - \gamma_a)\Big), \\ &= \begin{cases} -\mu_{c'} + \frac{1}{2}(\mu_a+\mu_b+\gamma_b+\gamma_a) - \gamma_c, & c<a \\ -\mu_{c'} + \frac{1}{2}(\mu_a+\mu_b+\gamma_b-\gamma_a), & a<c<b \\ -\mu_{c'} + \frac{1}{2}(\mu_a+\mu_b-\gamma_b-\gamma_a) + \gamma_c, & b<c \end{cases} \end{aligned}$$

which proves that v is a quasi solution to (7). ❑

A simple consequence of (2) and (28) is the following lemma.

Lemma 16. *Let u be an quasi solution to* (27)*. Then, for every $j \in I\!N$ and any $a<b$ such that $(a,b) \subset (\lambda_j, \lambda_{j+1})$, $m \in I\!N$, (λ_j are given in* (3)*)*

$$\partial_{x_i} s_{a,b}(u) \in L^2_{\text{loc}}(I\!R^+_d), \quad i=1,\dots,k.$$

Proof: From (2), (28), and (31), we conclude,

$$\begin{aligned} \sum_{i=1}^k |\partial_{x_i} s_{a,b}(u)|^2 &\leq \frac{s'_{a,b}(u)}{c(u)} \sum_{i,j=1}^k b_{ij}(t,x,u) u_{x_i} u_{x_j} \\ &\leq \max_{a\leq\lambda\leq b} (c(\lambda))^{-1} \sum_{j=1}^k \left(\sum_{i=1}^k \sigma_{ij} \partial_{x_i} s_{a,b}(u) \right)^2 \in L^1_{\text{loc}}(I\!R^+_d), \end{aligned}$$

where the last relation follows from (2). This concludes the proof. ❑

We shall need the following characterization of entropy solutions to (27) (see also [4]).

Proposition 17. [3] *The function u represents a quasi solution to* (27) *if and only if the kinetic function*

$$h(t,x,\lambda) = \begin{cases} 1, & 0 \leq \lambda \leq u(t,x) \\ -1, & u(t,x) \leq \lambda \leq 0 \\ 0, & \textit{otherwise} \end{cases} \tag{32}$$

satisfies the following linear equation,

$$\begin{aligned} &\partial_t h(t,x,\lambda) + \operatorname{div}\left[\partial_\lambda f(t,x,\lambda) h(t,x,\lambda)\right] \\ &- \sum_{i,j=1}^{k} \partial^2_{x_i x_j}\left(b_{ij}(t,x,\lambda) h(t,x,\lambda)\right) = -\partial_\lambda m(t,x,\lambda), \quad \text{in } \mathcal{D}'(I\!R^d_+ \times I\!R), \end{aligned} \tag{33}$$

where $m \in \mathcal{M}(I\!R^d_+ \times I\!R)$ *and* $b_{ij} = \partial_\lambda B_{ij}$.

Now, we can state the main result of this section:

Theorem 18. *Assume that* u *is a quasi solution to* (27) *and that the coefficients of equation* (33) *satisfy the weak traceability property. Then, there exists a function* $u_0 \in L^\infty(I\!R^d)$*, such that*

$$L^1_{\rm loc}(I\!R^d) - \lim_{t\to 0} u(t,\cdot) = u_0.$$

Proof: Let F be dense countable subset of $I\!R$, such that for all $c \in F$, (29) holds. Remark that for any $g \in L^\infty(I\!R^d_+ \times I\!R)$ the function h defined in (32) satisfies

$$\int_{I\!R}\int_{I\!R^d_+} h(t,x,\lambda) g(t,x,\lambda)\, dt dx d\lambda = \int G(t,x,u(t,x))\, dt dx, \tag{34}$$

where $G(t,x,v) = \int_0^v g(t,x,\lambda)d\lambda$, (see e.g. [1, (1.2.5)]). If we take $g(t,x,\lambda) = \operatorname{sgn}(\lambda - c)$, from Proposition 2 we have that there exists a function $u_0 \in L^\infty(I\!R^d)$, and functions $v_c \in L^\infty(I\!R^d)$ such that for every $c \in I\!R$

$$u(t,\cdot) \rightharpoonup u_0 \quad \text{and} \quad |u(t,\cdot) - c| \rightharpoonup v_c, \tag{35}$$

weakly - $\star$ in $L^\infty(I\!R^d)$, as $t \to 0$, $t \in E$.

Furthermore, according to Theorem 13, there exists the averaged trace for the function h defined in (32). If we notice that

$$\int h(t,x,\lambda)\rho(\lambda)d\lambda = u,$$

if $\rho \equiv 1$ on $(-M,M)$, we conclude that $u(t,\cdot) \to u_0$ in $L^1_{\rm loc}(I\!R^d)$. $\Box$

Example 19. *Consider the one-dimensional scalar conservation law,*

$$\partial_t u + \partial_x(xu) = 0.$$

In this case, we simply take $\hat{x}(x) = \ln|x|$, $x \neq 0$, *and we infer that the strong traceability conditions are fulfilled locally almost everywhere (i.e. on the set* $(-\infty,0) \cup (0,\infty)$*) implying that the traces at* $t = 0$ *exist.*

Less trivial example is the two dimensional scalar conservation law which is linear in the direction of the first space variable (i.e. it is not non-degenerate),

$$\partial_t u + \partial_{x_1}(x_1 u) + \partial_{x_2}(x_1 u^2) = 0.$$

In this case, we first choose $\hat{t} = t$, $\hat{x}_1 = \ln|x_1|$, $x_1 \neq 0$, $\hat{x}_2 = x_2$. *Locally, this reduces equation to*

$$\partial_{\hat{t}}\hat{u} + \partial_{\hat{x}_1}\hat{u} + \partial_{\hat{x}_2}(e^{\hat{x}_1}\hat{u}^2) = \hat{u},$$

where $\hat{u} = u(\hat{t},\hat{x})$. *Then, we take* $\tilde{t} = \hat{x}_1 + \hat{t}$, $z_1 = \hat{x}_1 - \hat{t}$, $z_2 = \hat{x}_2$, *and we thus reduce the equation to*

$$2\partial_{\tilde{t}}\tilde{u} + \partial_{z_2}(e^{\frac{1}{2}(\tilde{t}+z_1)}\tilde{u}^2) = \tilde{u}.$$

This equation satisfies conditions from Definition 4.5. if we consider z_1 as a parameter (and, accordingly, its kinetic counterpart is also non-degenerate). Thus, the weak traceability conditions are fulfilled and the traces to u exist.

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References

[1] J. Brenier, *Averaged multivalued solutions for scalar conservation laws*, SIAM J. Numer. Anal. **21** (1984), 1013–1037.

[2] Chen, G-Q.; Karlsen, K. H. *Quasilinear anisotropic degenerate parabolic equations with time-space dependent diffusion coefficients.* Commun. Pure Appl. Anal. **4** (2005), no. 2, 241–266.

[3] Chen, G-Q.; Perthame, B. *Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations.* Ann. Inst. H. Poincaré Anal. Non Linéaire, **20** (2002), 645–668.

[4] Dalibard, Anne-Laure Kinetic formulation for a parabolic conservation law. Application to homogenization. SIAM J. Math. Anal. 39 (2007), no. 3, 891915 (electronic).

[5] Giga, Y.; Kohn, R. V. *Characterizing blowup using similarity variables.* Indiana Univ. Math. J., **36** (1987), 1–40.

[6] Graetz, L. *Uber die Warmeleitungsfahigkeit von flussigkeiten. P. 2.* Ann. Phys. Chem., **25** (1885), 337-357.

[7] Kružkov, S. N. *First order quasilinear equations with several independent variables.* (Russian) Mat. Sb. (N.S.) **81 (123)** (1970), 228–255.

[8] Kwon, Y. S.; Vasseur, A. *Strong traces for solutions to scalar conservation laws with general flux.* Arch. Ration. Mech. Anal. **185** (2007), 495–513.

[9] M. Lazar, D. Mitrovic, *Velocity averaging – a general framework*, Dynamics of PDE, **9** (2012), 239-260.

[10] Levich V. G., *Physicochemical Hydrodynamics.* [in Russian], Fizmatgiz, Moscow (1959).

[11] Lions, P. L.; Perthame, B.; Tadmor, E. *A kinetic formulation of multidimensional scalar conservation law and related equations.* J. Amer. Math. Soc. **7**, (1994), 169–191.

[12] Natalini, R. *Convergence to equilibrium for the relaxation approximations of conservation laws.* Comm. Pure Appl. Math., **49** (1996), 795–823.

[13] Jan M. Nordbitten, personal communication.

[14] Nusselt W., *Die abhangigkeit der warmeubergangszahl von der rohrlange.* Z. Ver. Deut. Ing., **54** (1910), 1154–1158.

[15] Panov, E. Yu., *Generalized solutions to the Cauchy problems for quasi-linear conservation laws*, PhD thesis, Moscow State University, 1991.

[16] Panov, E. Yu. *Existence of strong traces for generalized solutions of multidimensional scalar conservation laws.* J. Hyperbolic Differ. Equ. **2** (2005), 885–908.

[17] Panov, E. Yu. *Existence of strong trances for quasi-solutions of multidimensional conservation laws.* J. Hyperbolic Differ. Equ. **4** (2007), 729–770.

[18] Panov,E. Yu. *Ultra-parabolic equations with rough coefficients. Entropy solutions and strong precompactness property.* J. Math. Sci., 159(2) (2009), 180–228.

[19] Sazhenkov, S. A. *A Degenerate Hydrodynamic Dispersion Model.* J. Prime Res. in Math., **3** (2007), 140–153.

[20] Tzavaras, A. E. *Materials with internal variables and relaxation to conservation laws.* Arch. Ration. Mech. Anal., **146** (1999), 129–155.

[21] Vasseur, A. *Strong traces for solutions of multidimensional scalar conservation laws.* Arch. Ration. Mech. Anal. **160** (2001), 181–193.

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