A NOTE ON CONVERGENCE IN THE SPACES OF L^p-DISTRIBUTIONS

J. ALEKSIĆ, S. PILIPOVIĆ, AND I. VOJNOVIĆ

ABSTRACT. We investigate convergence properties in weighted spaces of distributions $\mathcal{D}'_{L^p}$ and their test spaces $\mathcal{D}_{L^q}$, $\frac{1}{p}+\frac{1}{q}=1$. Also we give characterization of weak limits of weakly convergent sequences of L^p-distributions.

1. Introduction and preliminaries

L^p-distributions (also known as *distributions of L^p growth* or *weighted spaces of distributions*), are introduced in [12], further developed in [3] and widely investigated and used, cf. [1, 8–11] and references given there. These spaces, denoted by $\mathcal{D}'_{L^p}(\mathbb{R}^d)$, are dual spaces of $\mathcal{D}_{L^q}(\mathbb{R}^d)$, $1 \le q < \infty$ which consists of smooth functions whose derivatives belong to $L^q(\mathbb{R}^d)$. In particular, $\mathcal{D}'_{L^1}(\mathbb{R}^d) := \left(\dot{\mathcal{B}}(\mathbb{R}^d)\right)'$, where $\dot{\mathcal{B}}(\mathbb{R}^d) \subset \mathcal{D}_{L^\infty}(\mathbb{R}^d) = \{\phi \in C^\infty(\mathbb{R}^d) \,|\, \partial^\alpha\phi \in L^\infty(\mathbb{R}^d),\ \alpha \in \mathbb{N}_0^d\}$ is the closure of the space of smooth functions with compact support in the topology generated by the sequence of seminorms $\|\cdot\|_{m,\infty}$:

$$\|\phi\|_{m,\infty} = \sup_{|\alpha|\le m} \|\partial^\alpha \phi\|_{L^\infty}, \quad m \in \mathbb{N}_0. \tag{1}$$

Space $\dot{\mathcal{B}}(\mathbb{R}^d)$ contains functions from $\mathcal{D}_{L^\infty}(\mathbb{R}^d)$ with all derivatives vanishing at infinity.

Precisely, $\mathcal{D}_{L^q}(\mathbb{R}^d)$, $1 \le q < +\infty$, denotes the space of smooth functions ϕ, such that $\partial^\alpha\phi \in L^q(\mathbb{R}^d)$, for all multi-indices $\alpha \in \mathbb{N}_0^d$, with the topology generated by the sequence of seminorms

$$\|\phi\|_{m,q} = \left(\sum_{|\alpha|\le m} \|\partial^\alpha\phi\|_{L^q}^q\right)^{1/q}, \quad m \in \mathbb{N}_0, \tag{2}$$

cf. [3, Sect. 6.1] or [12, VI.§8]. It is known that $\mathcal{D}_{L^q}(\mathbb{R}^d)$ are Fréchet spaces (locally convex spaces which are metrizable and complete with respect to this metric) and that the space of smooth functions with compact support $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{D}_{L^q}(\mathbb{R}^d)$, $1 \le q < +\infty$. For $q = \infty$, instead of $\mathcal{D}_{L^\infty}(\mathbb{R}^d)$ we consider its subspace $\dot{\mathcal{B}}(\mathbb{R}^d)$. In the sequel, we will use the notation p for the conjugate number of q, $p = \frac{q}{q-1}$, $q \ge 1$ (for $q = 1$, $p = \infty$).

Date: September 8, 2014.

Key words and phrases. L^p - distributions, precompactness, duals

MSC2010: 46F05 .

The research is supported by Ministry of education and science of Republic of Serbia, project no. 174024.

Since $\mathcal{D}_{L^q}(I\!R^d)$, $1 \leq q < +\infty$, and $\dot{\mathcal{B}}(I\!R^d)$ are Fréchet spaces, the Banach-Steinhaus theorem holds on the duals. Namely, for a subset $H \subset \mathcal{D}'_{L^p}(I\!R^d)$, H is weakly - star bounded (i.e. in the topology $\sigma(\mathcal{D}'_{L^p}, \mathcal{D}_{L^q})$) if and only if H is strongly bounded (in the topology $\beta(\mathcal{D}'_{L^p}, \mathcal{D}_{L^q})$) if and only if H is equicontinuous if and only if H is relatively compact in the weak dual topology. For the properties of these topologies cf. [13, Chpt. 33] and [6].

Schwartz [12, Theorem VI.25] provided the following representation: if $p \in [1, \infty]$, then

a) For every distribution $T \in \mathcal{D}'_{L^p}(I\!R^d)$ there exists $n \in I\!N_0$ such that T can be represented as a finite sum of derivatives of functions $f_\alpha \in L^p(I\!R^d)$,

$$T = \sum_{|\alpha| \leq n} \partial^\alpha f_\alpha, \tag{3}$$

where f_α are bounded continuous functions in $L^p(I\!R^d)$ and, moreover, for $p \neq \infty$ each f_α vanishes at infinity.

b) Also, a distribution $T \in \mathcal{D}'_{L^p}(I\!R^d)$ if and only if

$$T * \psi \in L^p(I\!R^d), \text{ for all } \psi \in \mathcal{D}(I\!R^d), \tag{4}$$

where $*$ denotes convolution, i.e. $\langle T*\psi, \varphi\rangle = \langle \psi(y), \langle T(x), \varphi(x+y)\rangle\rangle$ for $\varphi \in \mathcal{D}(I\!R^d)$.

Remark 1. Notice that (4) is equivalent to:

$$\text{there exists } m \in I\!N, \text{ such that for all } \psi \in C_c^m(I\!R^d),\ T * \psi \in L^p(I\!R^d). \tag{5}$$

In the above, $C_c^m(I\!R^d)$ denotes the space of continuous differentiable functions with compact support whose all derivatives up to order m are continuous. Namely, (5) implies (4) because $\mathcal{D}(I\!R^d) \subset C_c^m(I\!R^d)$. Conversely, if (4) holds, then we know that there exists $m \in I\!N$ such that $T = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha$ for $f_\alpha \in L^p(I\!R^d)$ so for every $\psi \in C_c^m(I\!R^d)$ we have that $T*\psi = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha * \psi$ and $\partial^\alpha f_\alpha * \psi = (-1)^{|\alpha|} f_\alpha * \partial^\alpha \psi \in L^p(I\!R^d)$. So, $T * \psi$ is finite sum of L^p functions and therefore $T * \psi \in L^p(I\!R^d)$.

2. Test spaces and their duals

Regarding L^q spaces it is known that every bounded sequence in $L^q(I\!R^d)$, $1 < q < \infty$, has a weakly convergent subsequence. The same assertion is true for $L^\infty(I\!R^d)$ when weak convergence is replaced by weak - star convergence. Only $L^1(I\!R^d)$ does not have this property. These assertions are proved in [5], where this property is called *precompactness*, i.e. we say that space is precompact (with respect to its topology) if and only if every bounded sequence has a weakly converging subsequence. Our aim is to see if weakly (or weakly - star) bounded sequences in $\mathcal{D}_{L^q}(I\!R^d)$ spaces have weakly (resp. weakly - star) convergent subsequences .

Lemma 2 (Weak compactness of $\mathcal{D}_{L^q}(I\!R^d)$)**.**

a) $\mathcal{D}_{L^q}(I\!R^d)$ *is weakly precompact for* $1 < q < \infty$.
b) $\dot{\mathcal{B}}(I\!R^d)$ *is weakly precompact.*
c) $\mathcal{D}_{L^\infty}(I\!R^d)$ *is weakly star precompact.*
d) $\mathcal{D}_{L^1}(I\!R^d)$ *is not weakly precompact.*

Proof: a) For Fréchet spaces the next theorem holds: the Fréchet space E is reflexive if and only if every bounded set in E is relatively weakly compact (meaning that it has a compact closure in weak topology, for proof see [7, Proposition 23.24, p. 276]). This immediately implies that spaces $\mathcal{D}_{L^q}(I\!R^d)$ are weakly compact for $1 < q < \infty$. But instead of using this theorem we will give here constructive proof.

Let $1 < q < \infty$ and $(u_n)_n$ be bounded sequence in $\mathcal{D}_{L^q}(I\!R^d)$. We have to prove that $(u_n)_n$ has a weakly convergent subsequence. If $(u_n)_n$ has a constant subsequence the proof is done, so we assume the opposite. Since $(u_n)_n$ is bounded sequence in $\mathcal{D}_{L^q}(I\!R^d)$, i.e. with respect to seminorms (2), then for every $n \in I\!N$ functions u_n and all their derivatives are bounded in $L^q(I\!R^d)$.

Since $(u_n)_n$ is bounded in $L^q(I\!R^d)$, and $L^q(I\!R^d)$ is weakly precompact, it contains a weakly convergent subsequence in $L^q(I\!R^d)$, denoted by

$$\phi_n \rightharpoonup \phi_0 \in L^q(I\!R^d).$$

The sequence $(\partial_{x_1}\phi_n)_n$ is also bounded in $L^q(I\!R^d)$, so there exists its subsequence $(\partial_{x_1}\phi_{(1,0,\dots,0),n})_n$ and function $\phi_{(1,0,\dots,0)} \in L^q(I\!R^d)$ with the following two properties

$$\partial_{x_1}\phi_{(1,0,\dots,0),n} \rightharpoonup \phi_{(1,0,\dots,0)}, \text{ but also } \phi_{(1,0,\dots,0),n} \rightharpoonup \phi_0.$$

Moreover, $\partial_{x_1}\phi_0 = \phi_{(1,0,\dots,0)}$.

In the same manner we obtain sequences of other derivatives. So, for every $\alpha \in I\!N_0^d$ there exists $(\phi_{\alpha,n})_n$ which is subsequence of $(\phi_n)_n$ such that

$$\begin{aligned} \phi_{\alpha,n} &\rightharpoonup \phi_0 \\ \partial_{x_1}\phi_{\alpha,n} &\rightharpoonup \phi_{(1,0,\dots,0)} \\ &\dots \\ \partial^\alpha\phi_{\alpha,n} &\rightharpoonup \phi_\alpha \end{aligned} \tag{6}$$

and $\phi_\alpha = \partial^\alpha\phi_0$.

Now, let $A : I\!N_0 \to I\!N_0^d$ be a bijection $A(k) = \alpha_k$, $k \in I\!N_0$, and choose the sequence $(\phi_{\alpha_k,k})_{k\in I\!N}$. Notice that the sequence $(\phi_{\alpha_k,k})_k$ is subsequence of $(\phi_k)_k$ (so does not contain any constant subsequence), sequence $(\partial_{x_1}\phi_{\alpha_k,k})_k$ is subsequence of $(\partial_{x_1}\phi_{(1,0,\dots,0),k})_k$ and so on. Since limit of weakly convergent sequence is unique for $\alpha \in I\!N_0^d$ we have that

$$\partial^\alpha\phi_{\alpha_k,k} \rightharpoonup \phi_\alpha \text{ in } L^q(I\!R^d).$$

Now we can conclude that $(\phi_{\alpha_k,k})_k$ is subsequence of given sequence $(u_n)_n$ which weakly converges in $\mathcal{D}_{L^q}(I\!R^d)$. To show this, take a test function $\theta \in \mathcal{D}'_{L^p}(I\!R^d)$. Since $\theta \in \mathcal{D}'_{L^p}(I\!R^d)$ we know that θ can be represented as a finite sum of derivatives of $f_\beta \in L^p(I\!R^d)$,

$$\theta = \sum_{|\beta|\le p} \partial^\beta f_\beta. \tag{7}$$

We have that

$$\langle \phi_{\alpha_k,k}, \theta\rangle = \langle \phi_{\alpha_k,k}, \sum_{|\beta|\le p} \partial^\beta f_\beta\rangle = \sum_{|\beta|\le p} (-1)^{|\beta|}\langle \partial^\beta\phi_{\alpha_k,k}, f_\beta\rangle$$

and when $k \to \infty$

$$\langle \phi_{\alpha_k,k}, \theta\rangle \to \sum_{|\beta|\le p} (-1)^{|\beta|}\langle \partial^\beta\phi_0, f_\beta\rangle = \langle \phi_0, \theta\rangle.$$

This implies that $\phi_{\alpha_k,k} \longrightarrow \phi_0$ in $\mathcal{D}_{L^q}(I\!R^d)$.

b) Let $(u_n)_n$ be a bounded sequence of functions in $\dot{\mathcal{B}}(I\!R^d)$. This means that every function u_n is bounded with respect to seminorms (1). So functions u_n and all their derivatives are bounded in $L^\infty(I\!R^d)$. Since $(u_n)_n$ is bounded in $L^\infty(I\!R^d)$, there is its weakly - star convergent subsequence in $L^\infty(I\!R^d)$, denoted by

$$\phi_n \longrightarrow \phi_0 \in L^\infty(I\!R^d).$$

In the same manner as in the part a) we obtain sequences of derivatives of functions ϕ_n and then we choose sequence $(\phi_{\alpha_k,k})_k$. To show that $(\phi_{\alpha_k,k})_k$ is weakly convergent in $\dot{\mathcal{B}}(I\!R^d)$ we take a test function $\theta \in \mathcal{D}'_{L^1}(I\!R^d)$ which is finite sum of derivatives of $L^1(I\!R^d)$ functions and we get that

$$\langle \phi_{\alpha_k,k}, \theta\rangle \to \langle \phi_0, \theta\rangle,\ k \to \infty.$$

c) It is known (see [4]) that $\left((\dot{\mathcal{B}}(I\!R^d))'\right)' = \mathcal{D}_{L^\infty}(I\!R^d)$. Since $\mathcal{D}_{L^\infty}(I\!R^d)$ is dual of topological vector space, we will consider weak - star topology on $\mathcal{D}_{L^\infty}(I\!R^d)$. This means that sequence $(u_n)_n$ converges weakly - star to u in $\mathcal{D}_{L^\infty}(I\!R^d)$ if for every $g \in (\dot{\mathcal{B}}(I\!R^d))' = \mathcal{D}'_{L^1}(I\!R^d)$ it holds that $\langle u_n, g\rangle \to \langle u, g\rangle,\ n \to \infty$.

Let $(u_n)_n$ be given sequence in $\mathcal{D}_{L^\infty}(I\!R^d)$, which means that all functions u_n are bounded with respect to seminorms given by (1). So we conclude that u_n and all their derivatives are bounded in $L^\infty(I\!R^d)$. Now we can apply the same procedure as in proofs of parts a) and b) to see that this sequence has weakly - star convergent subsequence.

d) We will construct sequence $\mathcal{D}_{L^1}(I\!R^d)$ which does not have convergent subsequence. Take $\psi \in \mathcal{D}(I\!R^d)$ such that $\operatorname{supp}\psi = \overline{B(0;1)}$ (closed ball of radius 1 with center at 0), $0 \le \psi \le 1$, $\psi(x) > 0$ for $|x| < 1$ and $\int_{I\!R^d} \psi(x)dx = 1$. Define sequence of functions $f_n(x) := n^d\psi(nx),\ n \in I\!N$. Notice that $f_n \in \mathcal{D}(I\!R^d) \subset \mathcal{D}_{L^1}(I\!R^d),\ n \in I\!N,\ f_n \ge 0,\ \operatorname{supp} f_n = \overline{B(0;1/n)}$ and $\|f_n\|_{L^1} = 1$. Sequence $(f_n)_n$ is bounded in $\mathcal{D}_{L^1}(I\!R^d)$, i.e. with respect to all seminorms in $\mathcal{D}_{L^1}(I\!R^d)$.

Suppose that $(f_n)_n$ has convergent subsequence $(f_k)_k$. Since $\operatorname{supp} f_k = \overline{B(0;1/k)}$, the weak limit of $(f_k)_k$ can only be zero. Using Schwartz characterization (3) of duals we see that $1 \in (\mathcal{D}_{L^1}(I\!R^d))'$ so we have that $\int_{I\!R^d} f_n \cdot 1\,dx \to 0,\ n \to \infty$, which contradicts the fact that $\|f_n\|_{L^1} = 1$.

❑

Remarks about duals and reflexivity:

- The spaces $\mathcal{D}_{L^q}(I\!R^d)$, $1 < q < \infty$, are reflexive, i.e.

$$\left((\mathcal{D}_{L^q}(I\!R^d))'\right)' = (\mathcal{D}'_{L^p}(I\!R^d))' = \mathcal{D}_{L^q}(I\!R^d), \quad 1 < q < \infty,\ p = q/q-1.$$

- Space $\mathcal{D}_{L^1}(I\!R^d)$ is not reflexive. This space is Fréchet space and we have found bounded sequence in $\mathcal{D}_{L^1}(I\!R^d)$ which does not have weakly convergent subsequence. Then aforementioned [7, Proposition 23.24, p. 276] implies that $\mathcal{D}_{L^1}(I\!R^d)$ is not reflexive.
  We can also conclude that $(\mathcal{D}_{L^1}(I\!R^d))' = \mathcal{D}'_{L^\infty}(I\!R^d)$ is not reflexive.

- Since $\Big((\dot{\mathcal{B}}(\mathbb{R}^d))'\Big)' = \mathcal{D}_{L^\infty}(\mathbb{R}^d)$ it follows that $\mathcal{D}_{L^\infty}(\mathbb{R}^d)$ is not reflexive. Indeed, $\dot{\mathcal{B}}(\mathbb{R}^d)$ is closed in $\mathcal{D}_{L^\infty}(\mathbb{R}^d)$, and if $\mathcal{D}_{L^\infty}(\mathbb{R}^d)$ were reflexive, then this would imply that $\dot{\mathcal{B}}(\mathbb{R}^d)$ is reflexive, which is not true (closed subspace of reflexive Fréchet space is reflexive, see [7]).
  This also implies that $\mathcal{D}_{L^\infty}(\mathbb{R}^d)$ is not weakly precompact.

3. Duals

Recall that $\mathcal{D}_{L^q}(\mathbb{R}^d)$ and $\mathcal{D}'{}_{L^p}(\mathbb{R}^d)$ can also be presented as

$$\mathcal{D}_{L^q}(\mathbb{R}^d) = \bigcap_{k\in\mathbb{N}_0} W^{k,q}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{D}'{}_{L^p} = \bigcup_{k\in\mathbb{N}_0} W^{-k,p},$$

where $W^{k,q}(\mathbb{R}^d)$ are Sobolev spaces, for details see [3]. Properties of Sobolev spaces are sistematically studied in e.g. [2].

Let $\mathcal{A}(\mathbb{R}^d)$ be any of $W^{k,q}(\mathbb{R}^d)$ or $\mathcal{D}_{L^q}(\mathbb{R}^d)$. By $\mathcal{A}_{\text{loc}}(\mathbb{R}^d)$ we denote the space of all functions f such that $\varphi f \in \mathcal{A}(\mathbb{R}^d)$ for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$. We know that weak convergence of the sequence $(v_n)_n$ in $W^{k,q}(\mathbb{R}^d)$ implies the strong convergence in $W^{k-1,q}_{\text{loc}}(\mathbb{R}^d)$, i.e. for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $(\varphi v_n)_n$ converges strongly in $W^{k-1,q}(\mathbb{R}^d)$. Namely, $v_n \rightharpoonup v$ in $W^{k,q}(\mathbb{R}^d)$ implies that $\partial^\alpha v_n \rightharpoonup \partial^\alpha v$ in $L^q(\mathbb{R}^d)$, for all $|\alpha| \le k$. Then $\partial^\alpha v_n \rightharpoonup \partial^\alpha v$ in $W^{1,q}(\mathbb{R}^d)$, for all $|\alpha| \le k-1$ and also in $L^q_{\text{loc}}(\mathbb{R}^d)$, since $W^{1,q}(\mathbb{R}^d)$ is compactly embedded in $L^q_{\text{loc}}(\mathbb{R}^d)$, by the Rellich's lemma. So for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and all $|\alpha| \le k-1$ we have that $\partial^\alpha(\varphi v_n) \to \partial^\alpha(\varphi v)$ in $L^q(\mathbb{R}^d)$. Hence, $(\varphi v_n)_n$ strongly converges in $W^{k-1,q}(\mathbb{R}^d)$. This is the reason why weak convergence in $\mathcal{D}_{L^q}(\mathbb{R}^d)$ implies the strong convergence in $\mathcal{D}_{L^q,\text{loc}}(\mathbb{R}^d)$

But, in $\mathcal{D}'_{L^p}(\mathbb{R}^d)$ convergence is far more complicate. Bounded sets in $\mathcal{D}'_{L^p}(\mathbb{R}^d)$ are characterized in [1]. The characterization of bounded sets is important because f_n converges strongly to zero in $\mathcal{D}_{L^q}(\mathbb{R}^d)$ if and only if for all bounded sets $B' \subseteq \mathcal{D}'_{L^p}(\mathbb{R}^d)$, $\sup_{\phi\in B'}\langle f_n, \phi\rangle \to 0$, as $n \to 0$. Recall the following theorem.

Theorem 3. *[1, Theorem 1] Let $B' \subseteq \mathcal{D}'_{L^p}(\mathbb{R}^d)$, $1 \leqslant p \leqslant +\infty$. The following conditions are equivalent:*

(I) *B' is bounded;*

(II) *For every bounded $B \subseteq \mathcal{D}_{L^q}(\mathbb{R}^d)$ when $p \neq 1$ and for every bounded $B \subseteq \dot{\mathcal{B}}$ when $p = 1$ there exists $M > 0$ such that*

$$\sup\{|(T * \phi)(x)| : T \in B',\ \phi \in B,\ x \in \mathbb{R}^d\} < M;$$

(III) *For every bounded open set $\Omega \subseteq \mathbb{R}^d$ and for every $\phi \in \mathcal{D}_{L^q}(\mathbb{R}^d)$ when $p \neq 1$ and for every $\phi \in \dot{\mathcal{B}}$ when $p = 1$ there exists an $M_\phi > 0$ such that*

$$\sup\{|(T * \phi)(x)| : T \in B',\ x \in \Omega\} < M_\phi.$$

As a consequence of these results, we obtain the following two propositions.

Proposition 4. *If $T_n \rightharpoonup T$ in the sense of weak - star topology on $\mathcal{D}'_{L^p}(\mathbb{R}^d)$, then:*

(I) *the sequence $T_n * \theta$ is bounded in $L^p(\mathbb{R}^d)$ for every $\theta \in \mathcal{D}(\mathbb{R}^d)$,*

(II) *there exists large enough $m \in \mathbb{N}$ such that the sequence $T_n * \phi$ is bounded in $L^p(\mathbb{R}^d)$ for every $\phi \in C^m_c(\mathbb{R}^d)$.*

Proof: (I) Let $q \in [1, \infty)$; the case $\dot{\mathcal{B}}$ can be treated in a similar way. Since $\{T_n : n \in \mathbb{N}\}$ is bounded in $\mathcal{D}'_{L^p}(\mathbb{R}^d)$ by Theorem 3 (II),

$$\sup_{n \in \mathbb{N}; \phi \in B} |(T_n * \phi)(x)| \le M,$$

for any bounded set B in $\mathcal{D}_{L^q}(\mathbb{R}^d)$.

Let $B_1 = B \cap \mathcal{D}(\mathbb{R}^d)$ where B is the unit ball in $L^q(\mathbb{R}^d)$. Denote $\breve{\phi}(x) = \phi(0-x)$. For any $\theta \in \mathcal{D}(\mathbb{R}^d)$ we have

$$\sup_{n \in \mathbb{N}; \varphi \in B_1} |\langle T_n * \theta, \varphi \rangle| = \sup_{n \in \mathbb{N}; \varphi \in B_1} |\langle T_n * \breve{\varphi}, \breve{\theta} \rangle| = \sup_{n \in \mathbb{N}; \varphi \in B_1} |\left(T_n * (\theta * \breve{\varphi})\right)(0)| \le M,$$

since $\{\theta * \breve{\varphi} : \varphi \in B_1\}$ is a bounded set in $\mathcal{D}_{L^q}(\mathbb{R}^d)$. B_1 is dense in B, so we have that

$$\sup_{n \in \mathbb{N}; \varphi \in B} |\langle T_n * \theta, \varphi \rangle| \le M.$$

This implies that $\{T_n * \theta : n \in \mathbb{N}\}$ is a bounded set in $L^p(\mathbb{R}^d)$.

(II) Let us show that $\{T_n * \theta : n \in \mathbb{N}\}$ is a bounded set in $L^p(\mathbb{R}^d)$ for every $\theta \in C_c^m(\mathbb{R}^d)$, for enough large m.

Let $\varphi \in \mathcal{D}_K(\mathbb{R}^d) = \{\varphi \in \mathcal{D}(\mathbb{R}^d) : \operatorname{supp} \varphi \subset K\}$, for compact set $K \subset \mathbb{R}^d$. Since $\{T_n * \varphi : n \in \mathbb{N}\}$ is a bounded set in $L^p(\mathbb{R}^d)$ it follows (with B_1 as above) that

$$\sup_{n \in \mathbb{N}; \psi \in B_1} |\langle T_n * \psi, \varphi \rangle| = \sup_{n \in \mathbb{N}; \psi \in B_1} |\langle T_n * \breve{\varphi}, \breve{\psi} \rangle| < \infty.$$

Thus $\{T_n * \psi : n \in \mathbb{N},\ \psi \in B_1\}$ is equicontinuous in $\mathcal{D}'_K(\mathbb{R}^d)$ and there exists a neighbourhood of zero in $\mathcal{D}_K(\mathbb{R}^d)$, $V_m(\varepsilon) := \{h \in \mathcal{D}_K(\mathbb{R}^d) : \|h\|_{K,m} \le \varepsilon\}$, where $\|h\|_{K,m} = \sup_{|\alpha| \le m} \|\partial^\alpha h\|_{L^\infty(K)}$, such that

$$h \in V_m(\varepsilon) \implies \sup_{n \in \mathbb{N}; \psi \in B_1} |\langle T_n * \breve{\psi}, \breve{h} \rangle| = \sup_{n \in \mathbb{N}; \psi \in B_1} |\langle T_n * h, \psi \rangle| \le 1.$$

This implies that $\sup_{n \in \mathbb{N}; \psi \in B} |\langle T_n * \breve{\psi}, \breve{h} \rangle| \le 1$ when $h \in V_m(\varepsilon)$ since B_1 is dense in B. The same holds for the closure of $V_m(\varepsilon)$ in

$$\mathcal{D}_{K,m}(\mathbb{R}^d) = \{\varphi \in C^m(\mathbb{R}^d) : \operatorname{supp} \varphi \subset K\} \text{ for compact set K} \subset \mathbb{R}^d.$$

Under the norm $\|h\|_{K,m}$ we have that $\mathcal{D}_{K,m}(\mathbb{R}^d)$ is a Banach space and for every $h \in \mathcal{D}_{K,m}(\mathbb{R}^d)$ it holds that

$$\sup_{n \in \mathbb{N}} |\langle T_n * h, \psi \rangle| \le c \|\psi\|_{L^q},\ \psi \in L^q(\mathbb{R}^d).$$

This implies that for every $h \in \mathcal{D}_{K,m}(\mathbb{R}^d)$, $\{T_n * h : n \in \mathbb{N}\}$ is bounded in $L^p(\mathbb{R}^d)$.

❑

Proposition 5. *If $T_n \rightharpoonup T$ in the sense of weak - star topology on $\mathcal{D}'_{L^p}(\mathbb{R}^d)$, then there exists $l \in \mathbb{N}$ and sequences $(S_{\alpha,n})_{n \in \mathbb{N}}$ converging weakly to S_α, $|\alpha| \le l$, in $L^p(\mathbb{R}^d)$, such that*

$$T_n = \sum_{|\alpha| \le l} \partial^\alpha S_{\alpha,n} \quad \text{and} \quad T = \sum_{|\alpha| \le l} \partial^\alpha S_\alpha.$$

Proof: Let $m \in I\!N$ be such that the sequence $T_n * \varphi$ is bounded in $L^p(I\!R^d)$ for every $\varphi \in C_c^m(I\!R^d)$ (existence of m is proven in Proposition 4). By (VI 6.22) in [12], there exists $k \in I\!N$, such that the parametrix of the operator Δ^k is in $C_c^m(I\!R^d)$, i.e. there exist $\theta \in \mathcal{D}(I\!R^d)$ and $\psi \in C_c^m(I\!R^d) \subseteq W^{m,q}(I\!R^d)$ such that $\delta = \Delta^k \psi + \theta$. Thus,

$$T_n = \Delta^k (T_n * \psi) + T_n * \theta, \quad T_n \in B'.$$

By Lemma 4 $\{T_n * \psi : n \in I\!N\}$ and $\{T_n * \theta : n \in I\!N\}$ are bounded sets in $L^p(I\!R^d)$. Moreover, they converge weakly in $L^p(I\!R^d)$, because for $\varphi \in \mathcal{D}(I\!R^d)$

$$\langle T_n * \psi, \varphi \rangle \to \langle T * \psi, \varphi \rangle,$$

since $\langle T_n, \breve{\psi} * \varphi \rangle \to \langle T, \breve{\psi} * \varphi \rangle$, and $\mathcal{D}(I\!R^d)$ is dense in $L^q(I\!R^d)$, $q \neq \infty$ and in $\dot{\mathcal{B}}$ for $q = \infty$. By the Banach Steinhaus theorem it follows that $T_n * \psi$ converges weakly in $\mathcal{D}'_{L^p}(I\!R^d)$. The same holds for $T_n * \theta$. We see that each T_n consists of two summands, the first one is derivative of L^p function of order k, i.e. it is a function $\Delta^k (T_n * \psi)$, and the second one is function $T_n * \theta$, which is in $L^p(I\!R^d)$. This summands are also weakly convergent, which proves the claim. ❑

References

[1] Abdullah, S.; Pilipović, S. *Bounded subsets in spaces of distributions of L^p-growth.* Hokkaido Math. J. 23 (1994), no. 1, 51–54.

[2] Adams, Robert A. *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.

[3] Barros-Neto, J. *An introduction to the theory of distributions.* Marcel Dekker, 1973.

[4] Dierolf P., Voigt J. *Calculation of the Bidual for Some Function Spaces. Integrable Distributions.*, Mathematische Annalen, 253, 63-87, Berlin, Gottingen, Heildeberg, 1980

[5] Evans, L. C. *Weak convergence methods for nonlinear partial differential equations.* CBMS Regional Conference Series in Mathematics, AMS Providence, RI, 1990.

[6] Horváth, J., *Topological vector spaces and distributions. Vol. I.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966.

[7] Meise, R., Vogt, D. *Introduction to Functional Analysis*, Oxford Graduate Texts in Mathematics, 1992.

[8] Ortner, N., Wagner, P. *Applications of weighted $\mathcal{D}_{L_p}$-spaces to the convolution of distributions.*, Bull. Polish Acad. Sci. Math. 37 (1989), no. 7-12, 579–595 (1990).

[9] Ortner, N., Wagner, P. *Explicit representations of L. Schwartz' spaces $\mathcal{D}_{L_p}$ and $\mathcal{D}'_{L_p}$ by the sequence spaces $s\hat{\otimes} l_p$ and $s'\hat{\otimes} l_p$, respectively, for* $1 < p < \infty$., J. Math. Anal. Appl. 404 (2013), no. 1, 1–10.

[10] Pahk, D. H. *On the convolution equations in the space of distributions of L^p-growth.* Proc. Amer. Math. Soc. 94 (1985), no. 1, 81–86.

[11] Ruzhansky, M. *L^p-distributions on symmetric spaces.* Results Math. 44 (2003), no. 1-2, 159–168.

[12] Schwartz, L. *Théorie des distributions.* Hermann, 1966.
Proc. Roy. Soc. Edinburgh Sect. A 115 (1990), no. 3-4, 193–230.

[13] Tréves, F. *Topological vector spaces, distributions and kernels.* Academic Press, 1967.

Jelena Aleksić
E-mail address: jelena.aleksic@dmi.uns.ac.rs

Stevan Pilipović
E-mail address: stevan.pilipovic@dmi.uns.ac.rs

Ivana Vojnović
E-mail address: ivana.vojnovic@dmi.uns.ac.rs

Department of Mathematics and Informatics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia