Title: Spectrum localizations for matrix operators on l^p spaces

Authors: J. Aleksić, V. Kostić, M. Žigić
(*Department of Mathematics, Faculty of science, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia*)

Abstract

We investigate localization of spectra of infinite matrices that can act as a linear operators from the Banach space l^p, $1 < p < \infty$, to itself. To that end, we propose method that is based on a generalization of a strict diagonal dominance of infinite matrices that is adapted to l^p and l^q norms, $p^{-1}+q^{-1} = 1$. Results are illustrated with several examples, and some interesting applications are discussed.

Keywords: Matrix operators, lp spaces, spectrum localization, diagonal dominance

MSC[2010] : 47A10 , 47B37

1 Introduction

Although spectra of matrices and operators play essential role in many applications, [18], effective computation of the spectrum of a (large) finite matrix, and especially of infinite matrices still remains a challenging task. Also, in some applications, instead explicitly computing the spectra, one often needs an estimates of their location in complex plane to deduce different operator properties. Due to that, in the literature on finite matrices, different methods were used to approximate the spectrum through various inequalities, bounds and the construction of localization sets in the complex plane, [19, 11].

From the end of IXX and beginning of XX century, when the concept of diagonal dominance for (finite) matrices first emerged in papers of Lévy (1881), Desplanques (1887), Minkowski (1900) and Hadamard (1903) (see [19] for detailed history on this subject) until now, there has been an extensive use of this matrix property and its many generalizations in various ways, in matrix theory as well as in its modern applications in the theory of dynamical systems, theory of neural networks, wi-fi communications and many others. The extensive use of these classes of matrices is, among other things, due to their nonsingularity, which made them a handy tool for obtaining different estimates of spectra of arbitrary matrices. In the matrix theory literature this is known as Geršgorin-type eigenvalue localization, [9, 19].

On the other hand, although infinite matrices considered as linear operators on Banach spaces appear in applications, they have, up to now, drawn less attention, [15]. Strict diagonal dominance as a property of infinite matrices (considered as linear operators on sequence spaces) has been investigated in [14]. There, an infinite matrix that was called diagonally dominant was necessary acting as an operator $l^\infty \to l^\infty$. So, contrary to the case of finite matrices, results of [14] could not be applied to infinite matrices that represent linear operators $l^p \to l^p$, for $p < \infty$. An extension of these results was obtained in [16] for $p = 1$ and, there, Geršgorin localization theorems for $p = 1$ and $p = \infty$ were introduced. In addition, in [5, 6] authors construct similar Geršgorin sets for operators on l^p, for $1 < p < \infty$.

Motivated by the need for simple localization methods of spectra of matrices and operators, we will generalize notion of strict diagonal dominance from the finite matrix case to the infinite case in a way that is applicable to operators on l^p spaces for all values $1 < p < \infty$. Consequently, we will obtain special classes of matrix operators with useful properties such as injectivity and boundedness, that extend results of [14]. Then, we construct localizations of spectra that generalize results from [5, 6, 16].

The paper is organized as follows. After short introduction of basic facts and notations, in the third section we introduce notion of SDD(p) infinite matrices and derive some of their properties. Then, in the fourth section we develop localization of spectra of matrix operators on l^p spaces.

2 Notation and basic properties

Denote by $\mathbf{A} = [a_{ij}]_{i,j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ an infinite matrix and define an operator $A : \mathcal{D}(A) \to \mathbb{C}^{\mathbb{N}}$ on $\mathcal{D}(A) \subset \mathbb{C}^{\mathbb{N}}$ as $A(x) = \mathbf{A}x$, $x \in \mathcal{D}(A)$. We are interested in case when $A : \mathcal{D}(A) \subseteq l^p \to l^p$, $1 < p < \infty$, where l^p is the Banach space of p-summable sequences with the p-norm $\|x\|_p^p = \sum_{i\in\mathbb{N}} |x_i|^p$. The cases $p = 1$ and $p = \infty$ are completely handled in [15].

Denote by $a_i = (a_{ij})_{j\in\mathbb{N}}$ the i-th row and by $a^j = (a_{ij})_{i\in\mathbb{N}}$ the j-th column of the matrix $\mathbf{A}$. Also , $\hat{a}_i := (\hat{a}_{ij})_{j\in\mathbb{N}} = ((1 - \delta_{ij})a_{ij})_{j\in\mathbb{N}}$, $i \in \mathbb{N}$, $\delta_{ij} =$

$\begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$, denotes the i-th deleted row of $\mathbf{A}$.

The operator A will be defined on whole l^p ($A : l^p \to \mathbb{C}^{\mathbb{N}}$) if and only if $a_i \in l^q$, $i \in \mathbb{N}$, $q = \frac{p}{p-1}$. If additionally the sequence of l^q norms or rows of $\mathbf{A}$, $(\|a_i\|_q)_{i\in\mathbb{N}} \in l^p$, then $A : l^p \to l^p$ is bounded linear operator. Also, in this case, all columns $a^j \in l^p$. This is illustrated in the following example.

Example 1. *Consider matrix*

$$\boldsymbol{A} = (a_{ij})_{i,j\in\mathbb{N}} = \left(\frac{1}{ij}\right)_{i,j\in\mathbb{N}} = \begin{bmatrix} 1 & 1/2 & 1/3 & \dots & 1/j & \dots \\ 1/2 & 1/4 & 1/6 & \dots & 1/2j & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1/i & 1/2i & 1/3i & \dots & 1/ij & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \quad (1)$$

Since $\|A\|_{op} = \sup_{\|x\|_p=1} \|Ax\|_p \leq \|(\|a_i\|_q)_{i\in\mathbb{N}}\|_p = \mu(p)$, *where*

$$\mu(p) := \zeta(p)^{1/p}\zeta(q)^{1/q}, \quad (2)$$

and $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ *is Riemann zeta function, for* $1 < p < \infty$, $\boldsymbol{A}$ *defines the bounded linear operator* $A : l^p \to l^p$, $1 < p < \infty$. *But, since* $Ae^1 = a^1 \notin l^1$, $A[l^1] \not\subset l^1$, *so* $\boldsymbol{A}$ *doesn't define linear operator* $l^1 \to l^1$.

If additionally to $a_i \in l^q$, which gives us just a linear operator $A : l^p \to \mathbb{C}^{\mathbb{N}}$, we have that $a^j \in l^p$, then $\mathcal{D}(A) := \{x \in l^p : \ Ax \in l^p\}$ is dense in l^p.

Lemma 2.

a) Let $a_i \in l^q$. *Then linear operator* $A : \mathcal{D}(A) \to l^p$ *is closed.*

b) If additionally $a^j \in l^p$ *then* $\mathcal{D}(A)$ *is dense in* l^p.

Proof: a) Take a sequence $x^n \in \mathcal{D}(A)$, $n \in \mathbb{N}$, such that $x^n \to x$ and $Ax^n \to y$ in l^p. Then

$$|(Ax)_i - y_i|^p \leq 2^p \left(\|a_i\|_q^p \|x - x^n\|_p^p + \|Ax^n - y\|_p^p\right) \to 0, \text{ as } n \to \infty,$$

which implies that $Ax = y$ and $x \in \mathcal{D}(A)$.

b) Since $Ae^j = a^j \in l^p$, then $Span\{e^j : j \in \mathbb{N}\} \subset \mathcal{D}(A)$, i.e. $\mathcal{D}(A)$ is dense in l^p. ❑

More about matrix operators on l^p can be found in [1, 2, 3, 8, 10, 12].

3 Diagonally dominant infinite matrices as operators on l^p spaces

Following the idea from [9], we introduce strictly diagonally dominant infinite matrices adapted to l^p spaces in the following definition.

Definition 3 ($\mathbb{S}_p$, the class of SDD(p) matrices)**.** An infinite matrix $\mathbf{A}$ is called SDD(p) matrix, i.e., $\mathbf{A} \in \mathbb{S}_p$, if and only if there exists $w \in \mathcal{B}_p^+ := \{x \in l^p : \|x\|_p < 1 \text{ and } x_i > 0,\ i \in \mathbb{N}\}$ such that for all $i \in \mathbb{N}$, $w_i|a_{ii}| > r_i^q(\mathbf{A}) := \|\hat{a}_i\|_q = \left(\sum_{j\neq i} |a_{ij}|^q\right)^{1/q}$.

Note that in the case of finite matrices for $p = 2$ this class was introduced in [7] as class of (finite) matrices that satisfy strong square sum criterion. In the following, we provide basic operator properties of infinite SDD(p) matrices and their two subclasses.

First, from the previous definition note that $\mathbf{A} \in \mathbb{S}_p$ implies that for all $i \in \mathbb{N}$, $|a_{ii}| > 0$, i.e. $a_{ii} \neq 0$, $0 < w_i < 1$, and all rows $a_i \in l^q$, with $\|a_i\|_q < \sqrt[q]{2}|a_{ii}|$. Moreover, $\mathbf{A} \in \mathbb{S}_p$ can be equivalently expressed as

$$s_p(\mathbf{A}) := \left[\sum_{i\in\mathbb{N}} \left(\frac{r_i^q(\mathbf{A})}{|a_{ii}|}\right)^p\right]^{\frac{1}{p}} < 1. \tag{3}$$

Lemma 4. *Every $\boldsymbol{A} \in \mathbb{S}_p$ defines linear injective operator $A : l^p \to \mathbb{C}^{\mathbb{N}}$.*

Proof: In previous remark we noticed that every SDD(p) matrix has l^q-rows which is enough for matrix $\mathbf{A}$ to define linear operator $A : l^p \to \mathbb{C}^{\mathbb{N}}$. To prove injectivity suppose that there is an $x \in l^p\backslash\{0\}$ such that $Ax = 0$. This implies that

$$|a_{ii}x_i| = |\hat{a}_i x| \leq r_i^q(\mathbf{A})\|x\|_p < w_i|a_{ii}|\|x\|_p, \qquad i \in \mathbb{N}.$$

Now, dividing by $\|x\|_p|a_{ii}| \neq 0$ we obtain that $\frac{|x_i|}{\|x\|_p} < w_i$, $i \in \mathbb{N}$, which contradicts the fact that $\|w\|_p < 1$. ❑

Corollary 5. *If matrix $\boldsymbol{A} \in \mathbb{S}_p$ has l^p-columns, then operator $A : \mathcal{D}(A) \to l^p$ is closed, injective and densely defined in l^p.*

Proof: The proof follows from Lemmas 2 and 4. ❑

Example 6. *Consider matrix $\boldsymbol{B} = (b_{ij})_{i,j\in\mathbb{N}}$, where*

$$b_{ij} = \begin{cases} \dfrac{1}{i^2} & , \quad i = j, \\[2ex] \dfrac{1}{i^3 j \mu(p)} & , \quad i \neq j, \end{cases}$$

where $\mu(p)$ is given by (2). Then, since

$$\sum_i \left(\sum_{j\neq i} \left| \frac{b_{ij}}{b_{ii}} \right|^q \right)^{p/q} = \zeta(p)^{-1} \zeta(q)^{-p/q} \sum_i \left(\sum_{j\neq i} \left| \frac{1}{ij} \right|^q \right)^{p/q} < 1.$$

we conclude that $\boldsymbol{B} \in \mathbb{S}_p$. Also, $\boldsymbol{B}$ has l^p-columns, so the operator $B : \mathcal{D}(B) \to l^p$ is closed, injective and densely defined. Moreover $\mathcal{D}(B) = l^p$ and B is bounded with $\|B\|_{op} \leq \mu(p)$.

As we shall soon see, diagonal elements of matrix $\mathbf{A} \in \mathbb{S}_p$ will determinate important properties of operator A. In that purpose denote by $d = \{a_{ii}\}_{i\in\mathbb{N}}$ the sequence whose elements are diagonal elements of $\mathbf{A}$, and by $D : \mathcal{D}(D) \to l^p$ the multiplication operator $Dx = (a_{ii}x_i)_{i\in\mathbb{N}}$, where $\mathcal{D}(D) := \left\{ \left(\frac{y_i}{a_{ii}} \right)_{i\in\mathbb{N}} : y \in l^p \right\}$. Also $d^{-1} := \left\{ \frac{1}{a_{ii}} \right\}_{i\in\mathbb{N}}$ and $D^{-1} : \mathcal{D}(D^{-1}) \to l^p$ will be the multiplication operator $D^{-1}x = \left(\frac{x_i}{a_{ii}} \right)_{i\in\mathbb{N}}$, where $\mathcal{D}(D^{-1}) := \left\{ (a_{ii}y_i)_{i\in\mathbb{N}} : y \in l^p \right\}$. Additionally, denote by $\mathcal{A}_d(\mathbf{A})$ the set of accumulation points of the sequence $d = \{a_{ii}\}_{i\in\mathbb{N}}$.

The operator D^{-1} will be defined and bounded on l^p if and only if $d^{-1} \in l^\infty$, i.e. if zero is not accumulation point of the sequence d. The operator D^{-1} is also injective. That is the reason why we introduce another class of matrices.

Definition 7. Let $\mathbb{S}_p^0$ denote the class of SDD(p) matrices such that zero is not accumulation point of their diagonal elements, i.e., an infinite matrix $\mathbf{A} \in \mathbb{S}_p^0$ if and only if $\mathbf{A} \in \mathbb{S}_p$ and $0 \notin \mathcal{A}_d(\mathbf{A})$.

Except SDD(p) property, matrices from $\mathbb{S}_p^0$ have property that $\inf\{|a_{ii}| : i \in \mathbb{N}\}$ is strictly positive, which provides that operator

$$D^{-1} : l^p \to \mathcal{R}(D^{-1}) = \mathcal{D}(D) = \left\{ \left(\frac{y_i}{a_{ii}} \right)_{i\in\mathbb{N}} \, : \, y \in l^p \right\}$$

is bounded and bijective. This leads us to the following operator property.

Theorem 8. *If $\boldsymbol{A} \in \mathbb{S}_p^0$, A has bounded inverse $A^{-1} : l^p \to \mathcal{D}(A)$, with*

$$\|A^{-1}\|_{op} \leq \frac{\|d^{-1}\|_\infty}{1 - s_p(\boldsymbol{A})},$$

where $s_p(\boldsymbol{A})$ is defined in (3). Moreover, if all $a^j \in l^p$ then $\mathcal{D}(A)$ is dense in l^p.

Proof: For any $x \in l^p$, $\|D^{-1}x\|_p^p \leq \left(\sup_{i\in\mathbb{N}} \frac{1}{|a_{ii}|} \right)^p \|x\|_p^p$. If $0 \notin \mathcal{A}_d(\mathbf{A})$, then $d^{-1} \in l^\infty$, and $\|D^{-1}\|_{op} \leq \|d^{-1}\|_\infty$.

Now rewrite $\mathbf{A} = \mathbf{D} + \mathbf{B}$, where $\mathbf{D} = [\delta_{ij} a_{ij}]_{i,j\in\mathbb{N}}$, and consider operator $D^{-1}B$, where $Bx = (\hat{a}_i \cdot x)_{i\in\mathbb{N}}$. For any $x \in l^p$

$$\|(D^{-1}B)x\|_p^p \leq \|x\|_p^p \sum_i \frac{\|\hat{a}_i\|_q^p}{|a_{ii}|^p} = \|x\|_p^p [s_p(\mathbf{A})]^p,$$

which implies that $s_p(\mathbf{A}) = \|D^{-1}B\|_{op} < 1$ and $\|(I+D^{-1}B)^{-1}\|_{op} \leq \dfrac{1}{1 - s_p(\mathbf{A})}$.

Now, $I + D^{-1}B : l^p \to l^p$ is bijection and operator $D : \mathcal{D}(D) \to l^p$, defined as $Dx = (a_{ii}x_i)_{i\in\mathbb{N}}$, where $\mathcal{R}(D^{-1}) = \mathcal{D}(D) = \left\{ \left(\frac{y_i}{a_{ii}} \right)_{i\in\mathbb{N}} : y \in l^p \right\}$, is onto, i.e. $\mathcal{R}(D) = l^p$. So the same conclusion is valid for the operator $A = D(I + D^{-1}B) : \mathcal{D}(A) \to l^p$, where now $\mathcal{D}(A) = \mathcal{D}[(I + D^{-1}B)^{-1}]\mathcal{D}(D)$. We can now define the inverse operator A^{-1} on $\mathcal{R}(A) = l^p$, and conclude that $\|A^{-1}\|_{op} \leq \|I + D^{-1}B\|_{op} \|D^{-1}\|_{op} \leq \dfrac{\|d^{-1}\|_\infty}{1 - s_p(\mathbf{A})} < \infty$.

Now suppose that $a^m = (a_i^m)_{i\geq 1} \in l^p$, $m \in \mathbb{N}$, and denote $z^m := \left(\dfrac{a_i^m}{a_{ii}} \right)_{i\in\mathbb{N}}$. One can see that $z^m \in \mathcal{D}(D)$, $m \in \mathbb{N}$ and also $z^m = (I+D^{-1}B)e^m$, $m \in \mathbb{N}$. Since $e^m = (I + D^{-1}B)^{-1} z^m \in \mathcal{D}(A)$, $\mathcal{D}(A)$ is dense in l^p. ❑

Therefore, better properties of the inverse of diagonal will give better properties of A^{-1}. Namely, if $d^{-1} \in c_0$ then D^{-1} is compact operator, and so is A^{-1}. The condition $d^{-1} \in c_0$ is fulfilled for example if $\lim_{i\to\infty} |a_{ii}| \to \infty$.

Example 9. *Given $1 < p < \infty$, let $\mu(p)$ be defined by (2) and consider matrix*

$$\boldsymbol{C} = (c_{ij})_{i,j\in\mathbb{N}} = \begin{pmatrix} 1\mu(p) & 1/2 & 1/3 & \dots \\ 1 & 2\mu(p) & 1/3 & \dots \\ 1 & 1/2 & 3\mu(p) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{4}$$

i.e., for $i, j \in \mathbb{N}$,

$$c_{ij} = \begin{cases} \dfrac{1}{j} & , \quad i \neq j \\ i\mu(p) & , \quad i = j. \end{cases}$$

The matrix $\boldsymbol{C} \in \mathbb{S}_p^0$, its columns are not in l^p and the sequence of diagonal elements diverge to infinity. Therefore, the operator $C : \mathcal{D}(C) \to l^p$ is bijective and C^{-1} is compact. Notice that $\mathcal{D}(C)$ is a small subset of l^p.

If, instead of $\boldsymbol{C}$, we consider symmetric matrix

$$\widetilde{\boldsymbol{C}} = \begin{pmatrix} 1\mu(p) & 1/2 & 1/3 & \dots \\ 1/2 & 2\mu(p) & 1/3 & \dots \\ 1/3 & 1/3 & 3\mu(p) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

i.e., for $i, j \in \mathbb{N}$,

$$\tilde{c}_{ij} = \begin{cases} \dfrac{1}{\max\{i,j\}} & , \quad i \neq j \\ i\mu(p) & , \quad i = j, \end{cases}$$

then additionally $\mathcal{D}(\tilde{C})$ is dense in l^p.

Boundedness of diagonal will provide operator on whole space l^p. Since $d \in l^\infty$ also means that infinity is not accumulation point of d we introduce another class of matrices.

Definition 10. Let $\mathbb{S}_p^\infty$ be the class of SDD(p) matrices with bounded diagonal, i.e., an infinite matrix $\mathbf{A} \in \mathbb{S}_p^\infty$ if and only if $\mathbf{A} \in \mathbb{S}_p$ and $d \in l^\infty$.

Lemma 11. *The matrix $\boldsymbol{A} \in \mathbb{S}_p^\infty$ defines a bounded injective operator $A : \mathcal{D}(A) = l^p \to \mathcal{R}(A) \subset l^p$.*

Proof: If $\mathbf{A} \in \mathbb{S}_p$ then $a_i \in l^q$, and for all $x \in l^p$, $a_i \cdot x \in \mathbb{C}$ is well defined. Moreover, rewrite $\mathbf{A} = \mathbf{D} + \mathbf{B}$, where $\mathbf{D} = [\delta_{ij} a_{ij}]_{i,j\in\mathbb{N}}$ to obtain

$$\begin{aligned} \|Ax\|_p &\leq \|Dx\|_p + \|Bx\|_p \leq \|x\|_p \left(\|d\|_\infty + \|\hat{a}_i\|_q\right) \\ &\leq \|x\|_p \|d\|_\infty \left(1 + \|w\|_p\right) < 2\|x\|_p \|d\|_\infty. \end{aligned}$$

Thus, $\mathcal{D}(A) = l^p$ and $\|A\|_{op} := \sup\limits_{\|x\|_p=1} \|Ax\|_p \leq 2\|d\|_\infty$. Injectivity of the operator A follows from Lemma 4. ❑

Corollary 12. *If $\boldsymbol{A} \in \mathbb{S}_p^0 \cap \mathbb{S}_p^\infty$ then $\boldsymbol{A}$ defines bounded linear bijective operator $A : l^p \to l^p$.*

Note that $\mathbf{A} \in \mathbb{S}_p^0 \cap \mathbb{S}_p^\infty$ does not imply the compactness of A^{-1}. These facts are illustrated in the following example.

Example 13. *Given $1 < p < \infty$, and $\mu(p)$ defined by (2), consider matrix*

$$\mathbf{M} = (m_{ij})_{i,j\in\mathbb{N}} = \begin{pmatrix} \mu(p) & 1/2 & 1/3 & \dots \\ 1/2 & \mu(p) & 1/6 & \dots \\ 1/3 & 1/6 & \mu(p) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{5}$$

i.e., for $i, j \in \mathbb{N}$,

$$m_{ij} = \begin{cases} \dfrac{1}{ij} & , \quad i \neq j \\ \mu(p) & , \quad i = j. \end{cases}$$

Then, since matrix $\mathbf{M}$ has constant sequence of diagonal entries, $\mathbf{M} \in \mathbb{S}_p^0 \cap \mathbb{S}_p^\infty$, and defined operator $M : l^p \to l^p$ is bijective with a bounded inverse.

On the other hand, matrix

$$\widetilde{\mathbf{M}} = (m_{ij})_{i,j\in\mathbb{N}} = \begin{pmatrix} \mu(p) & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 2\mu(p) & 1/6 & 1/8 & \dots \\ 1/3 & 1/6 & \mu(p) & 1/12 & \dots \\ 1/4 & 1/8 & 1/12 & 4\mu(p) & \dots \\ \dots & \dots & \dots & \dots & \end{pmatrix}, \tag{6}$$

i.e., for $i, j \in \mathbb{N}$,

$$\widetilde{m}_{ij} = \begin{cases} \dfrac{1}{ij} & , \quad i \neq j \\ i\mu(p) & , \quad i = j \equiv 0 \ (mod\ 2) \\ \mu(p) & , \quad i = j \equiv 1 \ (mod\ 2). \end{cases}$$

has a diagonal that consists of one constant sequence and one sequence that diverges to infinity. So, $\widetilde{\mathbf{M}} \in \mathbb{S}_p^0$, *implying that the operator* $\widetilde{M} : \mathcal{D}(\widetilde{M}) \to l^p$ *is bijective and* M^{-1} *is bounded. Furthermore,* $\widetilde{M}^{-1}$ *is not compact which is in accordance with the fact that* $\widetilde{\mathbf{M}} \notin \mathbb{S}_p^\infty$.

4 Spectrum localization for matrix operators on l^p spaces

In this section we provide a method to localize spectra of matrix operators on l^p spaces using classes of SDD(p) matrices. So, we recall basic definitions and properties.

First, denote $\mathcal{M}_q$ the class of matrix operators A whose corresponding matrix $\mathbf{A}$ has all rows in l^q. As we have seen, large class of operators on l^p space fulfils this condition. Then, let $\rho(A)$ denote the resolvent set of A, i.e.

$$\rho(A) = \{z \in \mathbb{C} \mid z - A : \mathcal{D}(A) \subseteq l^p \to l^p \text{ has bounded inverse}\},$$

and for $z \in \rho(A)$, operator $R_A(z) = (z - A)^{-1}$ denotes the resolvent operator of A. The complement of the the resolvent set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called the spectrum of A. Furthermore, let $\sigma_P(A)$ denote pointwise spectrum of A, i.e., $\sigma_P(A) = \{z \in \mathbb{C} \mid z - A : \mathcal{D}(A) \subseteq l^p \to l^p \text{ is not injective}\}$. Here the operator $z - A$ is understood as $zI - A$, where I is an identity operator on l^p. We know that the set $\rho(A)$ is open, the set $\sigma(A)$ is closed in $\mathbb{C}$, and the set $\sigma_P(A)$ consists of discrete points in $\mathbb{C}$. Furthermore, it is known that if an operator A is compact, then $\sigma(A) = \sigma_P(A)$.

Now, we prove theorems that provide regions in the complex plane that contain the spectrum of an operator $A \in \mathcal{M}_q$. For that purpose we consider following sets of complex numbers:

$$\Gamma_w^p(A) := \bigcup_{i\in\mathbb{N}} \Gamma\left(a_{ii}, \frac{r_i^q(\mathbf{A})}{w_i}\right), \quad \text{for } w \in \mathcal{B}_p^+, \tag{7}$$

and

$$\Gamma^p(A) := \bigcap_{w\in\mathcal{B}_p^+} \Gamma_w^p(A), \tag{8}$$

where $\Gamma(\xi, r)$ denotes a closed disc in the complex plane centered at $\xi \in \mathbb{C}$ with radius $r \geq 0$.

Theorem 14. *Given arbitrary $A \in \mathcal{M}_q$,*

$$\sigma_P(A) \subseteq \Gamma^p(A) \quad \text{and} \quad \sigma(A) \subseteq \Gamma^p(A) \cup \mathcal{A}_d(\boldsymbol{A}). \tag{9}$$

Moreover, the following characterization of (8) holds:

$$\Gamma^p(A) = \mathbb{C} \setminus \{z \in \mathbb{C} \ : \ s_p(z - \boldsymbol{A}) < 1\}, \tag{10}$$

where $s_p(\cdot)$ is defended in (3).

Proof: Suppose that $\lambda \in \sigma_P(A)$, i.e., that operator $z - A$ is not injective. Then, according to the Lemma 4, $\lambda\mathbf{I} - \mathbf{A} \notin \mathbb{S}_p$. Therefore, for every $w \in \mathcal{B}_p^+$, there exist $i \in \mathbb{N}$, such that $w_i|\lambda - a_{ii}| \leq r_i^q(\mathbf{A})$, i.e., $\lambda \in \Gamma\Big(a_{ii}, \frac{r_i^q(\mathbf{A})}{w_i}\Big)$.

To prove the second inclusion, we actually need to prove that $\sigma(A) \backslash \sigma_P(A) \subseteq \Gamma^p(A) \cup \mathcal{A}_d(\mathbf{A})$. If operator $\lambda - A$ do not have bounded inverse, then according to Theorem 8 the matrix $\lambda\mathbf{I} - \mathbf{A}$ can not be SDD(p) matrix with diagonal entries that have a zero as accumulation point, i.e., either $\lambda\mathbf{I} - \mathbf{A} \notin \mathbb{S}_p$ or $0 \in \mathcal{A}_d(\lambda\mathbf{I} - \mathbf{A})$. But this is equivalent to the fact that $\lambda \in \Gamma_w^p(A)$, for arbitrary $w \in \mathcal{B}_p^+$, or $\lambda \in \mathcal{A}_d(\mathbf{A})$.

Notise that $z \in \Gamma^p(A)$ if and only if $z - A \notin \mathbb{S}_p$. Thus, characterization (10) follows from (3). ❑

Since, obviously, sets $\Gamma_w^p(A)$ contain the sequence $d = \{a_{ii}\}_{i \in \mathbb{N}}$ for all $w \in \mathcal{B}_p^+$, if $\Gamma^p(A)$ would be closed set in $\mathbb{C}$, then $\Gamma^p(A) \cup \mathcal{A}_d(A) = \Gamma^p(A)$, and $\Gamma^p(A)$ would be the localisation set of the spectrum of A. The following example shows that this is not always the case, and that $\mathcal{A}_d(A)$ is a necessary part of (9).

Example 15. *Consider matrix*

$$\mathbf{N} = (n_{kj})_{k,j \in \mathbb{N}} = \begin{pmatrix} -2 & 1 & 1/2 & 1/3 & 1/4 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2e^{\mathrm{i}\frac{\pi}{3}} & 1/9 & 1/18 & \dots \\ 0 & 0 & 0 & 1/2 & 0 & \dots \\ 0 & 0 & 0 & 0 & 2e^{\mathrm{i}\frac{\pi}{5}} & \dots \\ \dots & \dots & \dots & \dots & \dots & \end{pmatrix}, \tag{11}$$

i.e., for $k, j \in \mathbb{N}$,

$$n_{kj} = \begin{cases} 0 & , \quad j < k \\ 2e^{\mathrm{i}\frac{\pi}{k}} & , \quad j = k \equiv 1 \ (mod\ 2) \\ \frac{2}{k} & , \quad j = k \equiv 0 \ (mod\ 2) \\ \frac{1}{(j-k)k^2} & , \quad j > k \equiv 1 \ (mod\ 2) \\ 0 & , \quad j > k \equiv 0 \ (mod\ 2) \end{cases}$$

which defines the operator $N : \mathcal{D}(N) \subseteq l^2 \to \mathcal{R}(N) \subseteq l^2$. *Since* $\mathbf{N}$ *is an upper triangular infinite matrix, it is easy to obtain that*

$$\sigma_P(N) = \{k^{-1} : k \in \mathbb{N}\} \ \cup \ \{2e^{\frac{\mathrm{i}\pi}{2k-1}} : k \in \mathbb{N}\},$$

and, for $w = \{\frac{\sqrt{6}}{k\pi}\}_{k\in\mathbb{N}}$,

$$\Gamma^2_w(N) = \{k^{-1} : k \in \mathbb{N}\} \ \cup \ \bigcup_{k\in\mathbb{N}} \Gamma\left(2e^{\frac{\mathrm{i}\pi}{2k-1}}, \frac{\pi^2}{6(2k-1)}\right).$$

So, $\mathcal{A}_d(N) = \{0, 2\} \subset \sigma(N)$, *but* $\mathcal{A}_d(N) \not\subset \Gamma^2(N)$ *since for all* $k \in \mathbb{N}$*:*

$$|0 - 2e^{\frac{\mathrm{i}\pi}{2k-1}}| = 2 > \frac{\pi^2}{6(2k-1)},$$

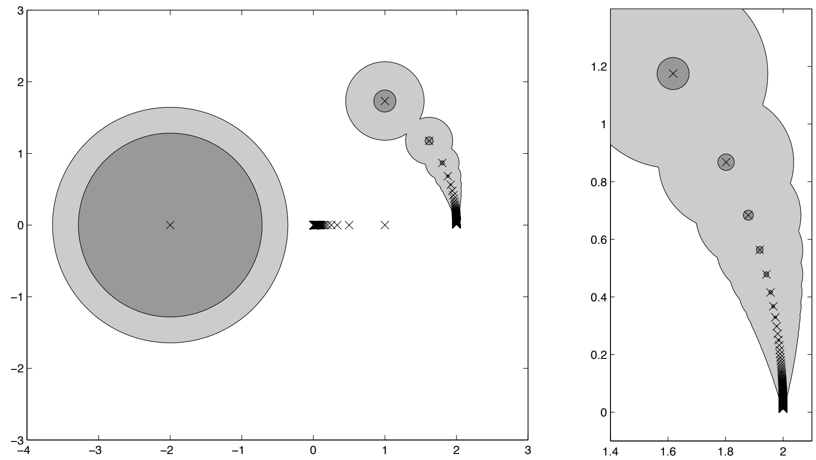
and

$$|2 - 2e^{\frac{\mathrm{i}\pi}{2k-1}}| = 2\sqrt{2}\sqrt{1 - \cos\left(\frac{\pi}{2k-1}\right)} > \frac{\pi^2}{6(2k-1)}.$$

This is illustrated in Figure 1, where the set $\Gamma^2_w(N)$ *is shown in light gray, while* $\Gamma^2(N)$ *is shown in dark gray and eigenvalues of* N *are marked by* $\times$.

Therefore, although sets $\Gamma\left(a_{ii}, \frac{r_i^q(\mathbf{A})}{w_i}\right)$, $i \in \mathbb{N}$ and $w \in \mathcal{B}_p^+$, are closed in $\mathbb{C}$, depending of the matrix operator A, $\Gamma^p_w(A)$ and $\Gamma^p(A)$ may or may not be closed. So, instead of working with $\Gamma^p_w(A)$ and $\Gamma^p(A)$, it may be more practical to use their closures $\overline{\Gamma^p_w(A)}$ and $\overline{\Gamma^p(A)}$ as localization sets for $\sigma(A)$.

We continue by investigating boundedness of (7) and (8) since this will determine geometric properties of these sets in the complex plane.



(a) Localization sets (b) Enlarged part

Figure 1: Sets $\Gamma^2(N)$ (dark gray) and $\Gamma^2_w(N)$, $w = \{\frac{\sqrt{6}}{k\pi}\}_{k\in\mathbb{N}}$, (light gray) for matrix $\mathbf{N}$ of (11). Elements of $\sigma_P(N)$ are marked by $\times$.

Theorem 16. *Let $A \in \mathcal{M}_q$, $d = \{a_{ii}\}_{i\in\mathbb{N}}$ and $r = \{r_i^q(A)\}_{i\in\mathbb{N}}$. The following statements hold:*

a) If $r \notin l^p$, then $\Gamma^p_w(A)$ is unbounded in $\mathbb{C}$, for all $w \in \mathcal{B}_p^+$.

b) If $r \notin l^p$ and $d \in l^\infty$, then $\Gamma^p(A) = \mathbb{C}$.

c) If $r \in l^p$, then there exist $w \in \mathcal{B}_p^+$ and $\gamma > 0$, depending of w, such that

$$\Gamma^p_w(A) = \bigcup_{i\in\mathbb{N}} \Gamma(a_{ii}, \gamma_i),$$

where $\gamma_i = \gamma$ or $\gamma_i = 0$, for $i \in \mathbb{N}$.

d) If $r \in l^p$ and $d \in l^\infty$, then $\Gamma^p(A)$ is bounded.

Therefore, $\Gamma^p(A)$ is bounded if and only if $r \in l^p$ and $d \in l^\infty$.

Proof: First, assume that $r \notin l^p$, for every $w \in \mathcal{B}_p^+$. Then, we have that for $\gamma_i := \frac{r_i^q(A)}{w_i}$, $i \in \mathbb{N}$, $\overline{\lim}_{i\to\infty}\gamma_i = +\infty$. Indeed, if we suppose that $\overline{\lim}_{i\to\infty}\gamma_i < +\infty$, then, there exist a constant $0 < M < \infty$ such that $r_i^q(A) < Mw_i$, for all $i \in \mathbb{N}$. But, then $\|r\|_p \leq M\|w\|_p < M < \infty$.

Therefore, we have that there exist a sequence $\{\gamma_{i_k}\}_{k\in\mathbb{N}}$, such that $\gamma_{i_k} \to \infty$, $k \to \infty$. But then,

$$\bigcup_{k\in\mathbb{N}} \Gamma(a_{i_k i_k}, \gamma_{i_k}) \subseteq \Gamma_w^p(A),$$

implies that $\Gamma_w^p(A)$ is unbounded in $\mathbb{C}$, and a) is proved. If in addition $d \in l^\infty$, then

$$\bigcup_{k\in\mathbb{N}} \Gamma(a_{i_k i_k}, \gamma_{i_k}) = \mathbb{C},$$

which implies b).

To prove c), let $r \in l^p$, and for arbitrary $\varepsilon > 0$ define the sequence w by

$$w_i := \begin{cases} \frac{r_i^q(A)}{\|r\|_p+\varepsilon}, & \text{for } r_i^q(A) > 0, \\ \\ \frac{v_i}{\|r\|_p+\varepsilon}, & \text{for } r_i^q(A) = 0, \end{cases}$$

where $i \in \mathbb{N}$, and v is an arbitrary sequence of positive numbers such that $\|v\|_p < \varepsilon$.

Then, $w \in \mathcal{B}_p^+$, and $z \in \Gamma_w^p(A)$ if and only if $|z - a_{ii}| \leq \|r\|_p + \varepsilon =: \gamma$, or $z = a_{ii}$, for some $i \in \mathbb{N}$. We conclude that for this particular w,

$$\Gamma_w^p(A) = \bigcup_{i\in\mathbb{N}} \Gamma(a_{ii}, \gamma_i),$$

where $\gamma_i = \gamma$ or $\gamma_i = 0$, for $i \in \mathbb{N}$.

Finally, since d) is a direct consequence of c), using a) we obtain that $\Gamma^p(A)$ is bounded if and only if $r \in l^p$ and $d \in l^\infty$. ❑

Results of the previous Theorem are illustrated in Figures 2 and 3. Namely, for the matrix $\mathbf{A}$ of Example 1, $||r||_p < \mu(p) < \infty$, and consequently set $\overline{\Gamma^p(A)}$ is compact in $\mathbb{C}$. Note that in this case $\Gamma^p(A)$ is closed, so $\sigma(A) \subset \Gamma^p(A)$. In Figure 2, the set $\Gamma^2(A)$ is shaded dark grey, while the set $\Gamma_w^2(A)$, for $w = \{\frac{\sqrt{6}}{k\pi}\}_{k\in\mathbb{N}}$, is light grey. To be precise, in order to obtain that the set $\Gamma_w^2(A)$ contains all eigenvalues of A, we need that $||w||_2 < 1$. But, for plotting purposes, $||w||_2 = 1$ doesn't make essential problem, since we can easily construct $\hat{w} = \{w_k - \delta_{k,1}\varepsilon\}_{k\in\mathbb{N}}$, for sufficiently small $\varepsilon > 0$, and obtain that $\Gamma_{\hat{w}}^2(A)$, which is arbitrarily close to $\Gamma_w^2(A)$, is a localization set for $\sigma_P(A)$. So, having this in mind, for plotting purposes, in the reminder of this section, we will use $w = \{\frac{\sqrt{6}}{k\pi}\}_{k\in\mathbb{N}}$.

On the other hand, for matrix $\mathbf{C}$ of Example 9, $r \notin l^p$, and, therefore, $\overline{\Gamma^p(C)}$ is unbounded in $\mathbb{C}$. This is shown in Figure 3. Again, set $\Gamma^2(C)$ is shaded dark gray and $\Gamma^2_w(C)$ light gray.

Here, we remark that $\overline{\Gamma^2_w(C)}$ is the closed right half-plane of $\mathbb{C}$, while $\Gamma^2_w(C)$ is the open right-half plane.

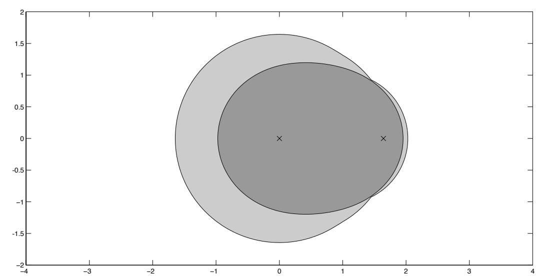


Figure 2: Sets $\Gamma^2(A)$ (dark gray) and $\Gamma^2_w(A)$, $w = \{\frac{\sqrt{6}}{k\pi}\}_{k\in\mathbb{N}}$, (light gray) for matrix $\mathbf{A}$ given in (1).

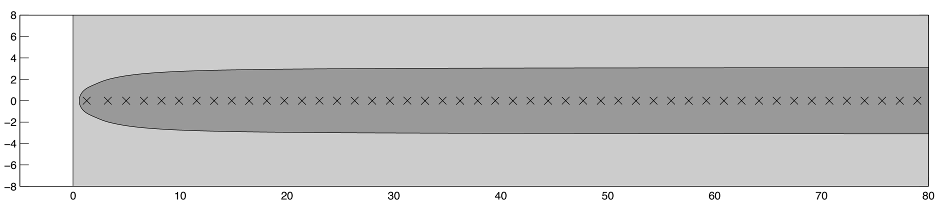


Figure 3: Sets $\Gamma^2(C)$ (dark gray) and $\Gamma^2_w(C)$, $w = \{\frac{\sqrt{6}}{k\pi}\}_{k\in\mathbb{N}}$, (light gray) for matrix $\mathbf{C}$ of (4).

Very useful property of Geršgorin-type localizations for finite matrices is that disjoint components reveal number of eigenvalues that are isolated in such a way, [19]. In [9] this property was proved for all eigenvalue localization sets that arise from subclasses of H-matrices. Similar results in the infinite case were obtained in [16] for l^∞ and l^1 matrix operators. Here we extend this isolation property to matrix operators on l^p space for $1 < p < \infty$. We will use well known result on bounded perturbations due to Rellich, see [4, Theorem 11.1.6]. Here, for the sake of completeness, we slightly reformulate it to better suit our application.

Lemma 17. *Let $U \subseteq \mathbb{C}$ be interior of a simple closed curve $\Omega \subset \mathbb{C}$. Given two closed operators A and B acting on a Banach space $(\mathcal{X}, \|\cdot\|)$, consider a family $A_t := A + tB$, for $t \in \mathbb{C}$. Suppose that R_A exists on Ω and that there exist $c_1, c_2 > 0$ such that $\|R_A(z)\| \leq c_1$ and $\|R_A(z)B\| \leq c_2$ for all $z \in \Omega$. Then, $\sigma(A) \cap U \neq \emptyset$ implies that $\sigma(A_t) \cap U \neq \emptyset$ for all $t \in \mathbb{C}$ such that $|t| < 1/c_2$, and, moreover, the spectral subspaces of A_t do not change dimensions.*

Proof: First, since $R_A(z)B$ is uniformly bounded on Ω, we have that for every $z \in \Omega$ and every $t \in \mathbb{C}$ such that $|t| < 1/c_2$, $(I - tR_A(z)B)^{-1}$ exists as a bounded operator and

$$(I - tR_A(z)B)^{-1} = \sum_{n\in\mathbb{N}} t^n (R_A(z)B)^n.$$

Therefore, for every $z \in \Omega$ and every $t \in \mathbb{C}$ such that $|t| < 1/c_2$, resolvent operator of A_t

$$R_{A_t}(z) = \left[I - tR_A(z)B\right]^{-1} R_A(z) = \sum_{n\in\mathbb{N}} t^n [R_A(z)B]^n R_A(z)$$

is jointly analytic function of (z, t), and for all relevant (z, t)

$$\|R_{A_t}(z)\| \leq \frac{c_1}{1 - c_2|t|}.$$

But, since $\sigma(A) \cap U \neq \emptyset$, this means that the Riesz projection operator (see Theorem 11.1.5 of [4])

$$P_{A_t} := \frac{1}{2\pi\mathrm{i}} \int_{\Omega} R_{A_t}(z) dz$$

is analytic in $t \in \mathbb{C}$ such that $|t| < 1/c_2$, and, therefore, according to Lemma 1.5.5 of [4], the space $\mathcal{R}(P_{A_t})$ has the same dimension for all $|t| < 1/c_2$. In another words, spectral subspaces of A_t do not change dimensions for $|t| < 1/c_2$, which completes the proof. ❑

Having this, for arbitrary matrix operator $A \in \mathcal{M}_q$ we prove that all disjoint parts of closures of localization sets (7) and (8) have to contain eigenvalues. Moreover, we prove that in bounded parts the number of eigenvalues has to coincide with the number of diagonal elements of A.

Theorem 18. *Given $A \in \mathcal{M}_q$, $w \in \mathcal{B}_p^+$ and $\emptyset \neq N \subseteq \mathbb{N}$, denote*

$$\mathcal{N}(A) := \overline{\bigcup_{i \in N} \Gamma(a_{ii}, \frac{r_i^q(A)}{w_i})}.$$

If N is such that $\mathcal{N}(A)$ is bounded and disjoint with $\overline{\Gamma_w^p(A) \setminus \mathcal{N}(A)}$, then $\mathcal{N}(A)$ contains exactly $card(N)$ eigenvalues of the operator A.

Proof: Let us denote $\mathcal{M}(A) := \overline{\Gamma_w^p(A) \setminus \mathcal{N}(A)}$. Since $\mathcal{N}(A)$ and $\mathcal{M}(A)$ are disjoint closed sets and $\mathcal{N}(A)$ is compact, we can construct a simple closed curve Ω in $\mathbb{C}$ such that $int(\Omega) \supset \mathcal{N}(A)$ and $\Omega \cap \overline{\Gamma_w^p(A)} = \emptyset$. But then, for every $z \in \Omega$, $w_i|z - a_{ii}| > r_i^q(A)$ holds for all $i \in \mathbb{N}$, and we obtain

$$\sum_{i \in \mathbb{N}} \left[\frac{r_i^q(A)}{|z - a_{ii}|}\right]^p \leq \|w\|_p < 1, \quad z \in \Omega. \tag{12}$$

Let $B := A - D$, $D : \mathcal{D}(D) \to l^p$ is a multiplication operator defined by $Dx = (a_{ii}x_i)_{i \in \mathbb{N}}$. Note that both, D and B, operators are closed. Then, since for arbitrary $z \in \Omega$ we have that $z \notin \{a_{ii} \ : \ i \in \mathbb{N}\} \cup \mathcal{A}_d(A)$, there exists $c_1 > 0$ such that for all $z \in \Omega$, $\|R_D(z)\| \leq c_1$.

Furthermore, since $r_i^q\left((z - D)^{-1}B\right) = \dfrac{r_i^q(B)}{z - a_{ii}}$, we obtain that for every $z \in \Omega$,

$$\|R_D(z)B\| \leq \sum_{i \in \mathbb{N}} \left[\frac{r_i^q(B)}{|z - a_{ii}|}\right]^p \leq c_2 = \|w\|_p < 1. \tag{13}$$

Finally, we construct a family of closed operators $A_t := D + tB$, $t \in [0, 1]$. Since for every $0 \leq t_2 \leq t_1 \leq 1$, we have that $\mathcal{N}(A_{t_2}) \subseteq \mathcal{N}(A_{t_1})$ and $\mathcal{M}(A_{t_2}) \subseteq \mathcal{M}(A_{t_1})$, which implies that $\Omega \subseteq \rho(A_t)$, i.e. the resolvent operator of A_t exists as a bounded operator in every point of Ω.

Now, taking $t = 0$ we have that $\sigma(A_0) = \sigma(D) = \{a_{ii} : i \in \mathbb{N}\} \cup \mathcal{A}_d(A)$, and according to Theorem 14, $\sigma(A_0) \subseteq \overline{\Gamma_w^p(A_0)} = \mathcal{N}(A_0) \dot{\cup} \mathcal{M}(A_0)$. Therefore, $\sigma(A_0) \cap \mathcal{N}(A_0) = \overline{\{a_{ii} \ : \ i \in N\}}$, and dimension of the space $\mathcal{R}(P_{A_0})$ has to be $card(N)$. But, since for all $t \in [0, 1]$, $tc_2 < 1$, Lemma 17 implies that dimension of the space $\mathcal{R}(P_{A_1}) = \mathcal{R}(P_A)$ is also $card(N)$, and, therefore, $int(\Omega)$ has to contain exactly $card(N)$ eigenvalues of the operator A. Since $\sigma(A) \cap int(\Omega) = \sigma(A) \cap \mathcal{N}(A)$, theorem is proved. ❑

Theorem 19. *Let $A \in \mathcal{M}_q$. If there exists a compact set $\mathcal{N}(A) \subseteq \mathbb{C}$ such that $\overline{\Gamma^p(A)} \setminus \mathcal{N}(A)$ is closed, then $card(\mathcal{N}(A) \cap \sigma(A)) = card(\mathcal{N}(A) \cap \overline{d})$, where $d = \{a_{ii}\}_{i\in\mathbb{N}}$.*

Proof: The proof of this theorem follows the same lines as the proof of the previous one. The only difference is in obtaining constant $c_2 < 1$ in (13). Namely, for a simple contour $\Omega \subset \mathbb{C} \setminus \overline{\Gamma^p(A)}$ such that $\mathcal{N}(A) \subset int(\Omega)$, from (10), we have that

$$s_p(z - A) = \sum_{i\in\mathbb{N}} \left[\frac{r_i^q(A)}{|z - a_{ii}|}\right]^p < 1.$$

Therefore, there exists $c_2 > 0$ such that $c_2 := \sup_{z\in\Omega} s_p(z - A) \leq 1$. So, it just remains to obtain that $c_2 \neq 1$. To that end, assume $c_2 = 1$, i.e., there exists sequence $\{z_n\}_{n\in\mathbb{N}} \subset \Omega$ such that $\lim_{n\to\infty} z_n = z$ and $s_p(z - A) = 1$. Since Ω is closed in $\mathbb{C}$, $z \in \Omega \subset \mathbb{C} \setminus \overline{\Gamma^p(A)} \subseteq \mathbb{C} \setminus \Gamma^p(A)$. But then, according to Theorem 14, $s_p(z - A) < 1$. ❏

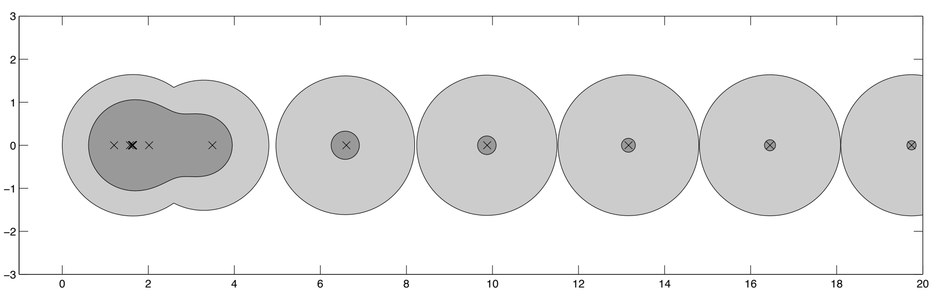


Figure 4: Sets $\Gamma^2(\widetilde{M})$ (dark gray) and $\Gamma^2_w(\widetilde{M})$, for $w = \{\frac{\sqrt{6}}{k\pi}\}_{k\in\mathbb{N}}$, (light gray) for matrix $\widetilde{\mathbf{M}}$ of (6).

Note here that if $r \in l^p$ and $d \in l^\infty$, according to Theorem 16, $\overline{\Gamma^p(A)}$ is compact, and taking $N = \mathbb{N}$, previous result states that $\sigma(A)$ is countable set.

As an illustration of this isolation property, consider matrix $\widetilde{\mathbf{M}}$ of Example 13. In Figure 4, set $\Gamma^2(\widetilde{M})$ is dark gray and set $\Gamma^2_w(\widetilde{M})$, for $w = \{\frac{\sqrt{6}}{k\pi}\}_{k\in\mathbb{N}}$, is light gray. According to Theorem 19 all dark gray compact disjoint sets, except the leftmost one, contain exactly one eigenvalue of $\widetilde{M}$, while the leftmost one contains $\aleph_0$ of them. The same holds for the light gray sets from Theorem 18, since

$$\Gamma_w^2(\widetilde{M}) = \bigcup_{k\in\mathbb{N}} \Gamma\left(\frac{\pi^2}{6}, \frac{\pi^2}{6} - \frac{\pi\sqrt{6}}{6(2k-1)}\right) \cup \bigcup_{k\in\mathbb{N}} \Gamma\left(\frac{\pi^2 k}{3}, \frac{\pi^2}{6} - \frac{\pi\sqrt{6}}{12k}\right),$$

and all circles

$$\Gamma\left(\frac{\pi^2 k}{3}, \frac{\pi^2}{6} - \frac{\pi\sqrt{6}}{12k}\right), \quad k \in \mathbb{N},$$

are mutually disjoint.

There are many possible applications of the results obtained in this section. For example, one can use them to determine sufficient conditions for an l^2 operator to be normal, or to determine if an infinite linear dynamical system has assimptoticaly stable equilibrium. To illustrate one such application, given $\omega \in \mathbb{C}$, let us consider Mathieu equation:

$$\frac{d^2y}{dx^2} + (\lambda - 2\omega\cos 2x)y = 0, \tag{14}$$

with the boundary condition $y(0) = y(\pi/2) = 0$. This well known equation arises in many applications - vibrations in an elliptic drum, the inverted pendulum, radio frequency quadrupole, frequency modulation, fixed-field alternating-gradient cyclotrons, the Paul trap for charged particles and mirror trap for neutral particles, stability of a floating body and others, [13].

One of the interesting problems when dealing with Mathieu equation is to determine when, depending on ω, there exists sequence of distinct complex values λ each one corresponding to a unique solution of (14) of the form $y(x) = \sum_{j=1}^{\infty} u_j \omega^{-j} \cos(2jx)$, [15], where $\{u_j\}_{j\in\mathbb{N}}$ are unknown coefficients. This problem can be reformulated as eigenvalue problem $Hu = \lambda u$, where $H = (h_{kj})_{k,j\in\mathbb{N}}$ is infinite tridiagonal matrix

$$h_{kj} = \begin{cases} \omega^2 & , \quad j = k-1, k \geq 2, \\ 4k^2 & , \quad j = k, \\ 1 & , \quad j = k+1, k \geq 1. \end{cases}$$

Then, the goal is to determine conditions on ω to assure that all eigenvalues of H are distinct. One way to do so is to use Theorem 18 and obtain sequence of disjoint disks. Namely, considering H as matrix operator on l^2, we have that all disks in (7) are disjoint if

$$|16 - 4| > \frac{1}{w_1} + \sqrt{1+\omega^4}\frac{1}{w_2}$$

and, for $k \geq 2$,

$$4|2k+1| = |4(k+1)^2 - 4k^2| > \sqrt{1+\omega^4}\left(\frac{1}{w_{k+1}} + \frac{1}{w_k}\right).$$

Therefore, taking $w = \{\frac{\sqrt{6}}{k\pi}\}_{k\in\mathbb{N}}$, we obtain that for $|\omega| < 1.7188$ all eigenvalues of H are simple. Off course, this is not the best possible value one can obtain using Theorems 18 and 19. In fact, the best upper bound would be $|\omega| < 3$, while the best possible bound $|\omega| < 6.9289$ can be computed using an iterative algorithm of [17] . So, as this example illustrates, one can be motivated to explore tighter localisation sets that are not computationally expensive, as it was done in the finite matrix case, see [9, 19]. This and another open questions and possible applications of SDD(p) matrices will be the subject of our future research.

Acknowledgement

The authors would like to thank Department of Mathematics and Computer Science, Faculty of Science, University of Novi Sad for the support. The work of J. Aleksić and M. Žigić is partially supported by the Ministry of Education and Science of Republic of Serbia, research project 174024 and Provincial Secretariat for Science of Vojvodina, research project 3605. The work of V. Kostić is partially supported by Ministry of Education and Science of Republic of Serbia, research project 174019, Provincial Secretariat for Science of Vojvodina, research projects 1850 and 2492, and the Einstein Fellowship Foundation of Berlin Mathematical School for 2013/14.

References

[1] Abramovich, Y. A.; Aliprantis, C. D.; Burkinshaw, O. Invariant subspaces of operators on l_p-spaces. J. Funct. Anal. 115 (1993), no. 2, 418–424.

[2] Borwein, D.; Jakimovski, A. Matrix operators on lp. Rocky Mountain J. Math. 9 (1979), no. 3, 463–477.

[3] Borwein, D.; Gao, X. Matrix operators on l^p to l^q. Canad. Math. Bull. 37 (1994), no. 4, 448–456.

[4] Davis, B. E. Linear Operators and their Spectra. Cambridge Studies in Advanced Mathematics 106, Cambridge University Press, 2007

[5] Farid, F. O.; Lancaster, P. Spectral properties of diagonally dominant infinite matrices. I. Proc. Roy. Soc. Edinburgh Sect. A 111 (1989), no. 3-4, 301–314.

[6] Farid, F. O.; Lancaster, P. Spectral properties of diagonally dominant infinite matrices. II. Linear Algebra Appl. 143 (1991), 7–17.

[7] Kierzkowski, J., Smoktunowicz, A. Block normal matrices and Gershgorin-type discs. Elec- tronic Journal of Linear Algebra, 22 (2011), 1059–1069

[8] Koskela, M. A characterization of non-negative matrix operators on l^p to l^q with $\infty > p \geq q > 1$. Pacific J. Math. 75 (1978), no. 1, 165–169.

[9] Kostić, V. R. On general principles of eigenvalue localizations via diagonal dominance. Adv. Comp. Math. (2014) DOI: 10.1007/s10444-014-9349-0

[10] Maddox, I. J. Matrix maps of bounded sequences in a Banach space. Proc. Amer. Math. Soc. 63 (1977), no. 1, 82–86.

[11] Mazko, A. Matrix Equations, Spectral Problems and Stability of Dynamic Systems. Stability, Oscillations and Optimization of Systems. Cambridge Scientific Publishers, 2008.

[12] Rakbud, J., Ong, S.-C. Sequence spaces of operators on l^2. J. Korean Math. Soc. 48 (2011), no. 6, 1125–1142.

[13] Ruby L. Applications of the Mathieu equation, Am. J. Phys., Vol. 64, No. 1 (1996) 39–44

[14] Shivakumar, P. N. Diagonally dominant infinite matrices in linear equations, Util. Math. 1 (1972) 235–248.

[15] Shivakumar, P. N., Sivakumar, K. C. A review of infinite matrices and their applications. Linear Algebra Appl. 430 (2009), no. 4, 976–998.

[16] Shivakumar, P. N., Williams, J.J., Rudraiah, N. Eigenvalues for infinite matrices, Linear Algebra Appl. 96 (1987) 35–63.

[17] Shivakumar, P. N. , Xue J. On the double points of a Mathieu equation, J. Comput. Appl. Math. 107 (1999) 111125.

[18] Trefethen, L. N., Embree, M. Spectra and pseudospectra - The behavior of nonnormal matrices and operators. Princeton University Press, Princeton, NJ, 2005.

[19] Varga, R. S. Geršgorin and his circles. Volume 36 of Springer Series in Computational Mathematics. Springer-Verlag, Berlin, 2004.