H-distributions via Sobolev spaces

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Abstract

H-distributions associated to weakly convergent sequences in Sobolev spaces are determined. It is shown that a weakly convergent sequence (u_n) in $W^{-k,p}(\mathbf{R}^d)$ has the property that θu_n converges strongly in $W^{-k,p}(\mathbf{R}^d)$ for every $\theta \in \mathcal{S}(\mathbf{R}^d)$ if and only if all H-distributions related to this sequence are equal to zero. Results are applied on a weakly convergent sequence of solutions to a family of linear first order PDEs.

1. *Introduction*

H-measures, or *Microlocal defect measures*, of Tartar [**21**] and Gérard [**8**] obtained for weakly convergent sequences in $L^2(\mathbf{R}^d)$, and their generalization to $L^p(\mathbf{R}^d)$, $p \in (1,\infty)$, called H-distributions [**5**], are widely used to determine whether a weakly convergent sequence of solutions to certain classes of equations converges strongly. For example, by using H-measures the authors of [**3**] obtained L^1_{loc}-precompactness of solutions to diffusion-dispersion approximation for a scalar conservation law. In homogenization theory applications of these objects can be found e.g. in [**4**] and [**12**]. In [**14**], H-measures are applied to family of entropy solutions of a first order quasilinear equation and in [**18**] to ultraparabolic equation. The list of applications of these objects is far from being complete.

Our aim in this paper is to extend the concept of H-distributions to the Sobolev spaces. For the purposes of this paper, we introduce in Subsection 2.1 new tensor product - spaces of test functions and distributions. For the reader's convenience, we give full description of such spaces in the Appendix (Propositions 4.1 and 4.2).

In order to use the duality $W^{-k,p}$-$W^{k,q}$, $q = \frac{p}{p-1}$, $k \in \mathbf{N}_0$, we prove the existence result for H-distributions associated to a weakly convergent sequence in $L^p(\mathbf{R}^d)$; in Theorem 2.1 we extend the result of [**5**, Theorem 2.1] since we did not use the localization coming from the compactly supported test functions. H-distributions of Theorem 2.1 are defined on the space of rapidly decreasing functions. This leads to the improvements of results of [**5**] in the case of L^p-spaces. In Theorem 3.1 we prove the existence of H-distributions for weakly convergent sequences in Sobolev spaces. Our main theorem, Theorem 3.2, shows that if for a given weakly convergent sequence $u_n \rightharpoonup 0$ in $W^{-k,p}(\mathbf{R}^d)$ and every weakly convergent sequence $v_n \rightharpoonup 0$ in $W^{k,q}(\mathbf{R}^d)$ the corresponding H-distributions are equal to zero, then for every $\varphi \in \mathcal{S}(\mathbf{R}^d)$, (φu_n) converges strongly to zero in $W^{-k,p}(\mathbf{R}^d)$. Clearly, the converse assertion also holds. As an application, we analyze in Theorem 3.3 a weakly convergent sequence (u_n) of solutions to $\sum_{i=1}^d \partial_i\,(A_i(x)u_n) = f_n$ in $W^{-k,p}(\mathbf{R}^d), d > \frac{p}{p-1}$, and show that the supports of the corresponding H-distributions are concentrated on the characteristic set $\{(x,\xi) : \sum_{i=1}^d A_i(x)\xi_i = 0\}$, under the new condition that for every $\varphi \in \mathcal{S}(\mathbf{R}^d)$, (φf_n) strongly converges to zero in $W^{-k-1,p}(\mathbf{R}^d)$. Moreover, if all H-distributions assigned to this equation are equal to zero, then (φu_n) converges strongly to

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zero in $W^{-k,p}(\mathbf{R}^d)$. The corresponding results for $L^2(\mathbf{R}^d)$ and $L^q(\mathbf{R}^d)$ are obtained in [**21**] and [**5**], respectively. Even in mentioned cases our results with $k=0$ extend previous results since the non-locality is the essential part of our approach. Moreover, the results from the recent contributions in which the H-distributions were used cf. [**11**, **13**, **17**] can be extended to a more general situations (in the Sobolev spaces with negative coefficients) by using results from this paper.

2. *Basic definitions and assertions*

2.1. *Some spaces of distributions*

We refer to [**2**] for the Sobolev spaces $W^{k,q}(\mathbf{R}^d)$. If $k > \frac{d}{q}$, then $W^{k,q}(\mathbf{R}^d) \subset C_0(\mathbf{R}^d)$, where $C_0(\mathbf{R}^d)$ is the space of continuous functions vanishing at infinity. The dual $\big(W^{k,q}(\mathbf{R}^d)\big)' =: W^{-k,p}(\mathbf{R}^d)$ is isometrically isomorphic to the Banach space consisting of distributions $u \in \mathcal{S}'(\mathbf{R}^d)$ of the form $u = \sum_{|\alpha|\le k} \partial^\alpha u_\alpha$, where all $u_\alpha \in L^p(\mathbf{R}^d)$, normed by $\|u\| := \inf\Big\{\Big(\sum_{|\alpha|\le k}\|u_\alpha\|_p^p\Big)^{1/p} : u = \sum_{|\alpha|\le k}\partial^\alpha u_\alpha\Big\}$, cf. [**2**, Theorem 3.10, p. 50].

In order to give clear explanations concerning a new space, which will be denoted by $\mathcal{SE}(\mathbf{R}^d \times \mathbf{S}^{d-1})$, and its dual $\mathcal{SE}'(\mathbf{R}^d \times \mathbf{S}^{d-1})$, we will use some classical results, [**16**] and [**7**], of L^2 and Sobolev theory for the unit sphere $\mathbf{S}^{d-1}$ as well as of [**6**] for some results for C^k and L^2 functions on $\mathbf{S}^{d-1}$. Concerning Sobolev spaces and distributions on a manifold, we refer to [**19**] and for tensor product of test spaces, to [**22**].

We define the space of smooth functions $\mathcal{SE}(\mathbf{R}^d \times \mathbf{S}^{d-1})$ by the sequence of norms

$$p^{\infty}_{\mathbf{R}^d\times\mathbf{S}^{d-1},k}(\theta) = \sup_{(x,\xi)\in\mathbf{R}^d\times\mathbf{S}^{d-1},|\alpha+\beta|\le k} \langle x\rangle^k |(\Delta^\star_\xi)^\alpha \partial^\beta_x\, \theta(x,\xi)|, \tag{2.1}$$

where $\langle x\rangle^k = (1+|x|^2)^{k/2}$ and $\Delta^\star$ is the Laplace-Beltrami operator. The space $\mathcal{SE}(\mathbf{R}^d\times\mathbf{S}^{d-1})$ is a Fréchet space and can be identified with the completion of tensor product $\mathcal{S}(\mathbf{R}^d)\hat{\otimes}\mathcal{E}(\mathbf{S}^{d-1})$, as was shown in Proposition 4.2 in the Appendix. Complete description of this space can be found in the Appendix.

2.2. *H-distributions on L^p spaces*

A bounded function ψ, on $\mathbf{R}^d$, is called L^p-Fourier multiplier if $f \mapsto \mathcal{A}_\psi(f) := (\psi\hat{f})\check{}$ is a bounded mapping from $\mathcal{S}(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$ and can be continuously extended to a mapping from $L^p(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$. Here $\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbf{R}^d} e^{-2\pi i x\cdot\xi} f(x)\,dx$ denotes the Fourier transform on $\mathbf{R}^d$, while $\check{g}(x) = \mathcal{F}^{-1}[g](x) = \int_{\mathbf{R}^d} e^{2\pi i x\cdot\xi} g(\xi)\,d\xi$ denotes the inverse Fourier transform. The space of L^p-Fourier multipliers, denoted by $\mathcal{M}_p(\mathbf{R}^d)$, $1<p<\infty$ (cf. [**9**]), is supplied by the norm $\|\psi\|_{\mathcal{M}_p} := \|\mathcal{A}_\psi\|_{L^p\to L^p}$, where $\|\cdot\|_{L^p\to L^p}$ is the standard operator norm.

If $\psi \in C^\kappa(\mathbf{R}^d\backslash\{0\})$, $\kappa = [\frac{d}{2}]+1$, is homogeneous of zero degree (i.e. $\psi(\lambda\xi) = \psi(\xi)$, $\lambda>0$), then $\psi \in L^\infty(\mathbf{R}^d)$ and

$$|\partial^\alpha_\xi \psi(\xi)| \le A|\xi|^{-|\alpha|}, \quad \xi \in \mathbf{R}^d\backslash\{0\}, \tag{2.2}$$

for every $|\alpha| \le \kappa$ (with $A = \max_{|\beta|\le\kappa}\sup_{\xi\ne0}|\xi|^\alpha|\partial^\beta\psi|$, cf. [**1**, p. 120]). Thus ψ fulfills conditions from the Mihlin theorem (cf. [**9**]): *Let ψ be a complex-valued bounded function on $\mathbf{R}^d\backslash\{0\}$ that* *satisfies (2.2) for all multi-indices* $|\alpha| \le [\frac{d}{2}] + 1$. *Then* $\psi \in \mathcal{M}_p(\mathbf{R}^d)$ *for any* $1 < p < \infty$ *and*

$$\|\psi\|_{\mathcal{M}_p} \le C_d \max\left\{p, \frac{1}{p-1}\right\}(A + \|\psi\|_\infty). \tag{2.3}$$

Moreover, if $\psi \in C^\kappa(\mathbf{S}^{d-1})$, then constant A in (2.3) can be replaced by $\|\psi\|_{C^\kappa(\mathbf{S}^{d-1})}$.

Fourier multiplier operators $\mathcal{A}_\psi$ with symbol $\psi \in C^\kappa(\mathbf{S}^{d-1})$ can be defined on $W^{-k,p}(\mathbf{R}^d)$, via duality

$$_{W^{-k,p}}\langle \mathcal{A}_\psi u, v\rangle_{W^{k,q}} := {}_{W^{-k,p}}\langle u, \mathcal{A}_{\bar\psi} v\rangle_{W^{k,q}}.$$

Since $\partial^\alpha \mathcal{A}_{\bar\psi} v = \mathcal{A}_{\bar\psi}(\partial^\alpha v)$, we know that $\mathcal{A}_{\bar\psi} v \in W^{k,q}(\mathbf{R}^d)$. If $u \in W^{-k,p}(\mathbf{R}^d)$ is of the form $u = \sum_{|\alpha|\le k} \partial^\alpha u_\alpha$, then for all $v \in W^{k,q}(\mathbf{R}^d)$,

$$\begin{aligned}
_{W^{-k,p}}\langle \mathcal{A}_\psi u, v\rangle_{W^{k,q}} &= \sum_{|\alpha|\le k} {}_{W^{-k,p}}\langle \partial^\alpha u_\alpha, \mathcal{A}_{\overline{\psi}} v\rangle_{W^{k,q}} = \\
= \sum_{|\alpha|\le k} (-1)^{|\alpha|}\, {}_{L^p}\langle u_\alpha, \mathcal{A}_{\overline{\psi}}(\partial^\alpha v)\rangle_{L^q} &= \sum_{|\alpha|\le k} (-1)^{|\alpha|}\, {}_{L^p}\langle \mathcal{A}_\psi(u_\alpha), \partial^\alpha v\rangle_{L^q}.
\end{aligned}$$

One can see that every L^p-multiplier operator $\mathcal{A}_\psi$ with symbol $\psi \in \mathcal{M}_p(\mathbf{R}^d)$ is a bounded operator from $W^{-k,p}(\mathbf{R}^d)$ to $W^{-k,p}(\mathbf{R}^d)$.

In order to prove the existence of an H-distributions of Theorem 3.1 given below, we need Tartar's First commutation lemma [**21**] and the modification of this lemma given in [**5**] .

*[**21**]: Let* $\psi \in C(\mathbf{S}^{d-1})$ *and* $b \in C_0(\mathbf{R}^d)$ *define the Fourier multiplier operator* $\mathcal{A}_\psi$ *and the operator of multiplication* B, *acting on* $u \in L^2(\mathbf{R}^d)$, *as follows:* $\mathcal{F}(\mathcal{A}_\psi u)(\xi) = \psi\left(\frac{\xi}{|\xi|}\right)\mathcal{F}(u)(\xi)$, $\xi \in \mathbf{R}^d\backslash\{0\}$, *and* $Bu(x) = b(x)u(x)$, $x \in \mathbf{R}^d$. *Then the operators* $\mathcal{A}_\psi$ *and* B *are bounded on* $L^2(\mathbf{R}^d)$, *and their commutator* $C := \mathcal{A}_\psi B - B\mathcal{A}_\psi$ *is a compact operator from* L^2 *into itself.*

*Moreover, [**5**]: If a sequence* (v_n) *is bounded in both* $L^2(\mathbf{R}^d)$ *and* $L^r(\mathbf{R}^d)$, *for some* $r \in (2,\infty]$ *and* $v_n \rightharpoonup 0$ *in the sense of distributions, then the sequence* (Cv_n) *strongly converges to zero in* $L^q(\mathbf{R}^d)$, *for any* $q \in [2,r]\backslash\{\infty\}$.

THEOREM 2.1. *If* $u_n \rightharpoonup 0$ *in* $L^p(\mathbf{R}^d)$, *and* $v_n \rightharpoonup 0$ *in* $L^q(\mathbf{R}^d)$, *then there exist subsequences* $(u_{n'})$, $(v_{n'})$ *and a distribution* $\mu(x,\xi) \in \mathcal{SE}'(\mathbf{R}^d \times \mathbf{S}^{d-1})$ *of order not more than* $\kappa = [d/2] + 1$ *in* ξ, *such that for every* $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbf{R}^d)$ *and* $\psi \in C^\kappa(\mathbf{S}^{d-1})$,

$$\begin{aligned}
\lim_{n'\to\infty}\int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})(x)\overline{(\varphi_2 v_{n'})(x)}dx &= \lim_{n'\to\infty}\int_{\mathbf{R}^d} (\varphi_1 u_{n'})(x)\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(x)}dx \\
&=: \langle \mu, \varphi_1\overline{\varphi}_2\psi\rangle,
\end{aligned} \tag{2.4}$$

where $\mathcal{A}_\psi : L^p(\mathbf{R}^d) \to L^p(\mathbf{R}^d)$ *is a Fourier multiplier operator with the symbol* $\psi \in C^\kappa(\mathbf{S}^{d-1})$.

By the order of $\mu \in \mathcal{SE}(\mathbf{R}^d \times \mathbf{S}^{d-1})$ we mean that for any $\varphi \in \mathcal{S}(\mathbf{R}^d)$, $\langle \mu(x,\xi), \varphi(x)\psi(\xi)\rangle$ can be extended on $C^\kappa(\mathbf{S}^{d-1})$ (see (2.9) and (2.10) below).

Proof. First, notice that the Fourier multiplier operator $\mathcal{A}_\psi$ with $\psi \in C^\kappa(\mathbf{S}^{d-1})$ is well defined on both $\varphi_1 u_n \in L^p(\mathbf{R}^d)$ and $\varphi_2 v_n \in L^q(\mathbf{R}^d)$, and that the adjoint operator of $\mathcal{A}_\psi$ is $\mathcal{A}_{\overline{\psi}}$. Thus, the first equality in (2.4) holds.

Let $1 < p \le 2$. Consider a sequence of sesquilinear (linear in $\psi \in C^\kappa(\mathbf{S}^{d-1})$ and anti-linear in $\varphi \in \mathcal{S}(\mathbf{R}^d)$) functionals

$$\mu_n(\varphi, \psi) = \int_{\mathbf{R}^d} u_n\overline{\mathcal{A}_{\overline{\psi}}(\varphi v_n)}dx. \tag{2.5}$$

By the continuity of $\mathcal{A}_\psi$ and the boundedness of (u_n) and (v_n) in $L^p(\mathbf{R}^d)$ and $L^q(\mathbf{R}^d)$, it follows that there exists $c > 0$, such that for every $n \in \mathbf{N}$,

$$|\mu_n(\varphi,\psi)| \le \|u_n\|_{L^p} \|\mathcal{A}_{\overline{\psi}}(\varphi v_n)\|_{L^q} \le c\,\|\psi\|_{C^\kappa(\mathbf{S}^{d-1})}\|\varphi\|_{L^\infty}. \tag{2.6}$$

Fix $\varphi \in \mathcal{S}(\mathbf{R}^d)$ and denote by $(B_n\varphi)$ the sequence of functions defined on $C^\kappa(\mathbf{S}^{d-1})$ by

$$\langle B_n\varphi, \cdot\rangle = \mu_n(\varphi,\cdot). \tag{2.7}$$

For every $n \in \mathbf{N}$, the linearity of $B_n\varphi$ is clear and the continuity follows from (2.6):

$$|\langle B_n\varphi,\psi\rangle| \le c_\varphi\,\|\psi\|_{C^\kappa(\mathbf{S}^{d-1})}, \text{ where } c_\varphi = c\|\varphi\|_{L^\infty}. \tag{2.8}$$

If we fix $\psi \in C^\kappa(\mathbf{S}^{d-1})$, then (2.5) implies that the mapping $\mathcal{S}(\mathbf{R}^d) \to \mathbf{C}$, $\varphi \mapsto \langle B_n\varphi,\psi\rangle$ is anti-linear and, again by (2.6), continuous.

We continue with fixed φ and apply the Sequential Banach Alaoglu theorem to obtain weakly star convergent subsequence $(B_k\varphi)$ in $(C^\kappa(\mathbf{S}^{d-1}))'$. We denote the weak star limit of $B_k\varphi$ by $B\varphi$, i.e. for every $\psi \in C^\kappa(\mathbf{S}^{d-1})$,

$$\langle B\varphi,\psi\rangle = \lim_{k\to\infty}\langle B_k\varphi,\psi\rangle.$$

We are going to show that B can be defined on the whole $\mathcal{S}(\mathbf{R}^d)$, so that $\mathcal{S}(\mathbf{R}^d) \ni \varphi \mapsto B\varphi \in (C^\kappa(\mathbf{S}^{d-1}))'$ is linear and continuous.

By the diagonalization argument, we define B on a countable dense set $M = \{\varphi_m | m \in \mathbf{N}\} \subset \mathcal{S}(\mathbf{R}^d)$. For that purpose extract a subsequence $(B_{1,k})_k \subset (B_n)_n$ such that $(B_{1,k}\varphi_1)$ is weakly star convergent in $(C^\kappa(\mathbf{S}^{d-1}))'$ and denote the limit as $B\varphi_1$. Then extract a subsequence $(B_{2,k})_k \subset (B_{1,k})_k$ such that $(B_{2,k}\varphi_2)$ is weakly star convergent in $(C^\kappa(\mathbf{S}^{d-1}))'$ and denote the limit as $B\varphi_2$. Notice also that $B_{2,k}\varphi_1$ converges weakly star to $B\varphi_1$. Repeating this procedure (extracting subsequences for all $\varphi_m \in M$), we obtain diagonal (sub)sequence $B_{k,k} \in \mathcal{L}\left(\mathcal{S}(\mathbf{R}^d), \left(C^\kappa(\mathbf{S}^{d-1})\right)'\right)$, such that for all $\varphi_m \in M$

$$\langle B\varphi_m,\psi\rangle = \lim_{k\to\infty}\langle B_{k,k}\varphi_m,\psi\rangle, \quad \psi \in C^\kappa(\mathbf{S}^{d-1}).$$

Denote $B_{k,k} =: b_k$ and fix $\psi \in C^\kappa(\mathbf{S}^{d-1})$. By (2.7), $\varphi \mapsto \langle b_k\varphi,\psi\rangle$, is a pointwise bounded sequence in $\mathcal{S}'(\mathbf{R}^d)$ which converges on a dense set $M \subset \mathcal{S}(\mathbf{R}^d)$. By the Banach-Steinhaus theorem, see e.g. [**10**, p. 169], $\langle b_k(\cdot),\psi\rangle$ converges to $\langle B(\cdot),\psi\rangle$ on $\mathcal{S}(\mathbf{R}^d)$. In this way we show that for every $\varphi \in \mathcal{S}(\mathbf{R}^d)$ and every $\psi \in C^\kappa(\mathbf{S}^{d-1})$

$$\lim_{k\to\infty}\langle b_k\varphi,\psi\rangle = \langle B\varphi,\psi\rangle.$$

Moreover, by (2.7),

$$|\langle B\varphi,\psi\rangle| \le c\|\varphi\|_{L^\infty}\|\psi\|_{C^\kappa(\mathbf{S}^{d-1})}. \tag{2.9}$$

By [**22**, Part III, Chap. 50, Proposition 50.7, p. 524] (it is a version of the Schwartz kernel theorem) we have that there exists $\mu \in \mathcal{SE}'(\mathbf{R}^d \times \mathbf{S}^{d-1})$ defined as

$$\langle \mu(x,\xi), \varphi(x)\psi(\xi)\rangle = \lim_{k\to\infty}\langle b_k\varphi,\psi\rangle = \lim_{k\to\infty}\int u_k\overline{\mathcal{A}_{\overline{\psi}}(\varphi v_k)}dx, \tag{2.10}$$

for all $\varphi \in \mathcal{S}(\mathbf{R}^d)$, $\psi \in C^\kappa(\mathbf{S}^{d-1})$, where (u_k) is a subsequence of (u_n) and (v_k) is a subsequence of (v_n) corresponding to (b_k). Now, we will use the factorization property of $\mathcal{S}(\mathbf{R}^d)$, [**15**]: Every $\varphi \in \mathcal{S}(\mathbf{R}^d)$ can be written as $\varphi = \overline{\varphi}_1\varphi_2$, for some $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbf{R}^d)$. Then

$$\langle \mu, \varphi\psi\rangle = \lim_{k\to\infty}\int u_k\overline{\mathcal{A}_{\overline{\psi}}(\overline{\varphi}_1\varphi_2 v_k)}dx.$$

Since $\|\varphi_2 v_k\|_{L^2} \leq \|v_k\|_{L^q} \|\varphi_2\|_{L^{\frac{2q}{q-2}}}$, we can apply the commutation lemma to $\varphi_2 v_k \in L^2 \cap L^q$ and $\overline{\varphi}_1 \in \mathcal{S}(\mathbf{R}^d) \subset C_0(\mathbf{R}^d)$ to obtain that for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbf{R}^d)$ and $\psi \in C^\kappa(\mathbf{S}^{d-1})$,

$$\langle \mu, \overline{\varphi}_1 \varphi_2 \psi \rangle = \lim_{k\to\infty} \int_{\mathbf{R}^d} \varphi_1 u_k \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_k)} dx. \tag{2.11}$$

This completes the proof of Theorem 2.4 for $1 < p \leq 2$.

In the case when $p > 2$, we define

$$\mu_n(\varphi, \psi) := \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi u_n) \overline{v}_n dx.$$

Then, in the same way as above, but now with the change of the roles of (u_n) and (v_n), we use factorization $\varphi = \varphi_1 \overline{\varphi}_2$, then the commutation lemma on $\varphi_1 u_n \in L^2(\mathbf{R}^d)$ and apply the preceding proof. □

Remark 1. The formulation of the previous theorem can be slightly changed in the case when $p \in (1, 2)$. Then, instead of $v_n \rightharpoonup 0$ in L^q we can assume that $v_n \rightharpoonup 0$ in L^r for some $r \geq q$ and obtain the same result as in Theorem 2.1. In that case for every $\varphi \in \mathcal{S}(\mathbf{R}^d)$, $\varphi v_n \in L^q \cap L^2$ and the same proof can be applied. The same idea but with compactly supported φ was used in [**5**].

3. *H-distribution and Sobolev spaces*

The next theorem determines H-distributions associated to sequences in Sobolev space.

Theorem 3.1. *If a sequence $u_n \rightharpoonup 0$ weakly in $W^{-k,p}(\mathbf{R}^d)$ and $v_n \rightharpoonup 0$ weakly in $W^{k,q}(\mathbf{R}^d)$, then there exist subsequences $(u_{n'}), (v_{n'})$ and a distribution $\mu \in \mathcal{SE}'(\mathbf{R}^d \times \mathbf{S}^{d-1})$ such that for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbf{R}^d)$ and every $\psi \in C^\kappa(\mathbf{S}^{d-1})$,*

$$\lim_{n'\to\infty} \langle \mathcal{A}_\psi(\varphi_1 u_{n'}) \,,\, \varphi_2 v_{n'} \rangle = \lim_{n'\to\infty} \langle \varphi_1 u_{n'} \,,\, \mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'}) \rangle = \langle \mu, \varphi_1 \bar{\varphi}_2 \psi \rangle. \tag{3.1}$$

Proof. Since $u_n \rightharpoonup 0$ in $W^{-k,p}(\mathbf{R}^d)$, there exist a subsequence $u_{n'} \rightharpoonup 0$ such that $u_{n'} = \sum_{|\alpha|\leq k} \partial^\alpha g_{\alpha,n'}$, where for every $|\alpha| \leq k$, $(g_{\alpha,n'})$ is a sequence of L^p-functions such that $g_{\alpha,n'} \rightharpoonup 0$ in $L^p(\mathbf{R}^d)$. Indeed, since a weakly convergent sequence forms a bounded set in $W^{-k,p}(\mathbf{R}^d)$, using the same proof of the representation theorem for elements of $W^{-k,p}(\mathbf{R}^d)$, one can obtain the existence of bounded sets $\{F_{\alpha,n}, n \in \mathbf{N}\}$, $|\alpha| \leq k$, such that $u_n = \sum_{|\alpha|\leq k} \partial^\alpha F_{\alpha,n}$. Now, since $\{F_{\alpha,n}, n \in \mathbf{N}\}$ are bounded in $L^p(\mathbf{R}^d)$, these sets are weakly precompact and every $\{F_{\alpha,n}, n \in \mathbf{N}\}$ has a weakly convergent subsequence. By the diagonalization method one can find a subsequence such that $F_{\alpha,n'} \rightharpoonup f_\alpha \in L^p(\mathbf{R}^d)$, $n' \to \infty$, $|\alpha| \leq k$, in $L^p(\mathbf{R}^d)$. Since $\sum_{|\alpha|\leq k} \partial^\alpha F_{\alpha,n'} \rightharpoonup 0$, it follows that $\sum_{|\alpha|\leq k} \partial^\alpha f_\alpha = 0$. Thus we obtain required subsequence $u_{n'} = \sum_{|\alpha|\leq k} \partial^\alpha (F_{\alpha,n'} - f_\alpha)$. In the sequel we will not relabel subsequences, so we will use u_n instead of $u_{n'}$.

Since

$$\partial_x^\alpha \Big[\mathcal{A}_\psi(u)\Big] = \mathcal{A}_{\psi_\alpha}(u) = \mathcal{A}_\psi(\partial^\alpha u), \text{ for } \psi_\alpha(\xi) = (2\pi i)^{|\alpha|} \xi^\alpha \psi(\xi),$$

we have that

$$\mathcal{A}_\psi\left(\varphi_1\, \partial^\alpha F_{\alpha,n}\right) = (-1)^{|\alpha|} \sum_{0\le\beta\le\alpha} (-1)^{|\beta|} \binom{\alpha}{\beta} \partial^\beta \left[\mathcal{A}_\psi\left(F_{\alpha,n}\, \partial^{\alpha-\beta}\varphi_1\right)\right],$$

and so

$$\begin{aligned}\langle \mathcal{A}_\psi(\varphi_1 u_n)\,,\, \varphi_2 v_n\rangle &= \textstyle\sum_{|\alpha|\le k}(-1)^{|\alpha|}\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\left\langle \mathcal{A}_\psi\left(F_{\alpha,n}\,\partial^{\alpha-\beta}\varphi_1\right)\,,\,\partial^\beta[\varphi_2 v_n]\right\rangle\\ &= \sum_{|\alpha|\le k}(-1)^{|\alpha|}\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\sum_{0\le\gamma\le\beta}\binom{\beta}{\gamma}\left\langle \mathcal{A}_\psi\left(F_{\alpha,n}\,\partial^{\alpha-\beta}\varphi_1\right)\,,\,\partial^{\beta-\gamma}\varphi_2\,\partial^\gamma v_n\right\rangle.\end{aligned} \tag{3.2}$$

For the moment, we fix α and apply Theorem 2.1 to $F_{\alpha,n}\rightharpoonup 0$ in $L^p(\mathbf{R}^d)$ and $v_n\rightharpoonup 0$ in $L^q(\mathbf{R}^d)$, thus obtaining subsequences $(F_{\alpha,n_0})_{n_0}$, $(v_{\alpha,n_0})_{n_0}$ and an H-distribution $\mu_{\alpha,0}\in\mathcal{SE}'(\mathbf{R}^d\times\mathbf{S}^{d-1})$, such that

$$\langle\,\mu_{\alpha,0}\,,\,\varphi_1\overline{\varphi}_2\psi\rangle := \lim_{n_0\to\infty}\langle \mathcal{A}_\psi\left(\varphi_1 F_{\alpha,n_0}\right),\,\varphi_2\, v_{\alpha,n_0}\rangle\,.$$

Then, applying Theorem 2.1 to $F_{\alpha,n_0}\rightharpoonup 0$ in $L^p(\mathbf{R}^d)$, and $\partial^{(1,0,\dots,0)}v_{\alpha,n_0}\rightharpoonup 0$ in $L^q(\mathbf{R}^d)$, we obtain subsequences $(F_{\alpha,n_{(1,0,\dots,0)}})_{n_{(1,0,\dots,0)}}$, $(v_{\alpha,n_{(1,0,\dots,0)}})_{n_{(1,0,\dots,0)}}$ and an H-distribution $\mu_{\alpha,(1,0,\dots,0)}\in\mathcal{SE}'(\mathbf{R}^d\times\mathbf{S}^{d-1})$. Thus, we obtain finitely many H-distributions $\mu_{\alpha,\gamma}$, $0\le\gamma\le\alpha$, such that

$$\langle\,\mu_{\alpha,\gamma}\,,\,\varphi_1\overline{\varphi}_2\psi\rangle := \lim_{n_\gamma\to\infty}\left\langle \mathcal{A}_\psi\left(\varphi_1 F_{\alpha,n_\gamma}\right),\,\varphi_2\,\partial^\gamma v_{\alpha,n_\gamma}\right\rangle.$$

The last one $\mu_{\alpha,\alpha}$ is obtained together with subsequences $(F_{\alpha,n_\alpha})_{n_\alpha}$, $(v_{\alpha,n_\alpha})_{n_\alpha}$ which we are going to use to define H-distribution μ^α in the following way: For $\varphi_1,\varphi_2\in\mathcal{S}(\mathbf{R}^d)$, $\psi\in C^\kappa(\mathbf{S}^{d-1})$,

$$\langle\,\mu^\alpha\,,\,\varphi_1\overline{\varphi}_2\psi\rangle := (-1)^{|\alpha|}\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\sum_{0\le\gamma\le\beta}\binom{\beta}{\gamma}\left\langle\mu_{\alpha,\gamma}\,,\,\partial^{\alpha-\beta}\varphi_1\,\partial^{\beta-\gamma}\bar{\varphi}_2\,\psi\right\rangle.$$

The sum on the right hand side is finite and all H-distributions $\mu_{\alpha,\gamma}$ can be defined via $(F_{\alpha,n_\alpha})_{n_\alpha}$ which is subsequence of F_{α,n_γ} and $(v_{\alpha,n_\alpha})_{n_\alpha}$ which is subsequence of $(v_{\alpha,n_\gamma})_{n_\gamma}$, so the H-distribution μ^α is well-defined.

Let us emphasize that we have obtained μ^α for a fixed α. Now if we take first $\alpha=0$ with previous procedure we can obtain H-distribution μ^0 defined via $(F_{0,n_0})_{n_0}$ and $(v_{n_0})_{n_0}$. Then, starting with $(F_{e_1,n_0})_{n_0}$ and $(v_{e_1,n_0})_{n_0}$ we obtain (by the same procedure) H-distribution μ^{e_1} defined via $(F_{e_1,n_{e_1}})_{n_{e_1}}$ and $(v_{e_1,n_{e_1}})_{n_{e_1}}$. Here $e_1=(1,0,\dots,0)$. Then we proceed with $e_2=(0,1,0,\dots,0)$ to obtain H-distribution μ^{e_2} and so on with all $|\alpha|\le k$.

At the end we obtain H-distribution μ defined by

$$\langle\,\mu\,,\,\varphi_1\overline{\varphi}_2\psi\rangle := \sum_{|\alpha|\le k}(-1)^{|\alpha|}\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\sum_{0\le\gamma\le\beta}\binom{\beta}{\gamma}\left\langle\mu_{\alpha,\gamma}\,,\,\partial^{\alpha-\beta}\varphi_1\,\partial^{\beta-\gamma}\bar{\varphi}_2\,\psi\right\rangle.$$

and subsequences $\left(\sum_{|\alpha|\le k}\partial^\alpha F_{\alpha,n_{(0,\dots,0,k)}}\right)_{n_{(0,\dots,0,k)}}$ and $\left(v_{n'}\equiv v_{(0,\dots,0,k),n_{(0,\dots,0,k)}}\right)_{n_{(0,\dots,0,k)}}$. □

Distribution μ obtained in Theorem 3.1 is called *H-distribution* corresponding to the (sub)sequence $(u_{n'},v_{n'})$.

Assume that the distributions μ determined by Theorem 3.1 are equal to zero. Then the local strong convergence in $W^{-k,p}(\mathbf{R}^d)$ easily follows. We will prove a more delicate assertion in the next theorem.

THEOREM 3.2. *Let $u_n \rightharpoonup 0$ in $W^{-k,p}(\mathbf{R}^d)$. If for every sequence $v_n \rightharpoonup 0$ in $W^{k,q}(\mathbf{R}^d)$ the corresponding H-distribution is zero, then for every $\theta \in \mathcal{S}(\mathbf{R}^d)$, $\theta u_n \to 0$ strongly in $W^{-k,p}(\mathbf{R}^d)$, $n \to \infty$.*

Proof. For the strong convergence we need to prove that for every $\theta \in \mathcal{S}(\mathbf{R}^d)$

$$\sup\{\langle \theta u_n, \phi\rangle \,:\, \phi \in B\} \longrightarrow 0,\ n \to \infty,\ \text{for every bounded } B \subseteq W^{k,q}(\mathbf{R}^d).$$

If it would not be true, then there would exist $\theta \in \mathcal{S}(\mathbf{R}^d)$, a bounded set B_0 in $W^{k,q}(\mathbf{R}^d)$, an $\varepsilon_0 > 0$ and a subsequence $(\theta u_k) \subset (\theta u_n)$, such that

$$\sup\{|\langle \theta u_k, \phi\rangle| \,:\, \phi \in B_0\} \geq \varepsilon_0,\ \text{for every } k \in \mathbf{N}.$$

Choose $\phi_k \in B_0$ such that $|\langle \theta u_k, \phi_k\rangle| > \varepsilon_0/2$. Since $\phi_k \in B_0$ and B_0 is bounded in $W^{k,q}(\mathbf{R}^d)$, (ϕ_k) is weakly precompact in $W^{k,q}(\mathbf{R}^d)$, i.e. up to a subsequence, $\phi_k \rightharpoonup \phi_0$ in $W^{k,q}(\mathbf{R}^d)$. Moreover, since ϕ_0 is fixed, $\langle u_k, \phi_0\rangle \to 0$ and

$$|\langle \theta u_k, \phi_k - \phi_0\rangle| > \frac{\varepsilon_0}{4}, \quad k > k_0. \tag{3.3}$$

Applying Theorem 3.1 on $u_k \rightharpoonup 0$ and $\phi_k - \phi_0 \rightharpoonup 0$, we obtain that for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbf{R}^d)$

$$\lim_{k\to\infty} {}_{W^{-k,p}}\langle \mathcal{A}_\psi(\varphi_1 u_k), \varphi_2(\phi_k - \phi_0)\rangle_{W^{k,q}} = 0. \tag{3.4}$$

With $\psi \equiv 1$ on $\mathbf{S}^{d-1}$, (3.4) implies

$$\lim_{k\to\infty} \langle \varphi_1 u_k\,,\, \varphi_2(\phi_k - \phi_0)\rangle = 0.$$

Again, we use the factorization property of $\mathcal{S}(\mathbf{R}^d)$. So if $\theta \in \mathcal{S}(\mathbf{R}^d)$, then $\theta = \phi_1\bar{\phi}_2$, for some $\phi_1, \phi_2 \in \mathcal{S}(\mathbf{R}^d)$, and we have that $\lim_{k\to\infty} \langle \phi_1 u_k\,,\, \phi_2(\phi_k - \phi_0)\rangle = 0$, i.e. $\lim_{k\to\infty} \langle \theta u_k\,,\, (\phi_k - \phi_0)\rangle = 0$. This contradicts (3.3) and completes the proof. □

3.1. *Localization property*

Recall [**20**, p. 117], the Riesz potential of order s, $Re(s) > 0$ is the operator $I_s = (-\Delta)^{-\frac{s}{2}}$., see also [**9**]. Consideration of the Fourier transform and convolution theorem reveals that I_α, for $0 < \alpha < d$, is a Fourier multiplier, i.e. $\mathcal{F}[I_\alpha[f]](\xi) := (2\pi|\xi|)^{-|\alpha|}\mathcal{F}[f](\xi)$. We will use the potential I_1 with the following properties:

$$\|I_1(f)\|_{L^{\frac{qd}{d-q}}} \leq C\|f\|_{L^q},\ \text{for } f \in L^q(\mathbf{R}^d), \quad 1 < q < d; \tag{3.5}$$

$$\partial_j I_1(f) = -R_j(f), \quad f \in L^q(\mathbf{R}^d),\ \text{where } R_j := \mathcal{A}_{\xi_j/\imath|\xi|}. \tag{3.6}$$

Moreover, $R_j : L^q \to L^q$ is continuous.

Consider now a sequence $u_n \rightharpoonup 0$ in $W^{-k,p}(\mathbf{R}^d)$ satisfying the following sequence of equations:

$$\sum_{i=1}^{d} \partial_i \left(A_i(x)u_n(x)\right) = f_n(x), \tag{3.7}$$

where $A_i \in \mathcal{S}(\mathbf{R}^d)$ and f_n is a sequence of temperate distributions such that

$$\varphi f_n \to 0 \text{ in } W^{-k-1,p}(\mathbf{R}^d),\ \text{for every } \varphi \in \mathcal{S}(\mathbf{R}^d). \tag{3.8}$$

THEOREM 3.3. *Let $1 < q < d$. If $u_n \rightharpoonup 0$ in $W^{-k,p}(\mathbf{R}^d)$ satisfies (3.7), (3.8), then for any sequence $v_n \rightharpoonup 0$ in $W^{k,q}(\mathbf{R}^d)$ the corresponding H-distribution μ satisfies*

$$\sum_{j=1}^{d} A_j(x)\xi_j\mu(x,\xi) = 0 \quad \text{in } \mathcal{SE}'(\mathbf{R}^d \times \mathbf{S}^{d-1}). \tag{3.9}$$

Moreover, if (3.9) implies $\mu(x,\xi)=0$, we have the strong convergence $\theta u_n \longrightarrow 0$, in $W^{-k,p}(\mathbf{R}^d)$, for every $\theta \in \mathcal{S}(\mathbf{R}^d)$.

Proof. Let $v_n \rightharpoonup 0$ in $W^{k,q}(\mathbf{R}^d)$, $\varphi_1 \in \mathcal{S}(\mathbf{R}^d)$, $\varphi_2 \in \mathcal{S}(\mathbf{R}^d)$ and let $\psi \in C^\kappa(\mathbf{S}^{d-1})$. We have to prove (3.9), i.e. (multiplying (3.9) by $(i|\xi|)^{-1}$, $|\xi|\neq 0$) that, up to a subsequence,

$$0=\sum_{j=1}^{d}\left\langle \mu, A_j\varphi_1\varphi_2\frac{\xi_j}{i|\xi|}\psi\right\rangle = \lim_{n\to\infty}\sum_{j=1}^{d}\left\langle u_nA_j\varphi_1\,,\,\mathcal{A}_{\bar{\Psi}_j}(\varphi_2 v_n)\right\rangle, \tag{3.10}$$

where $\Psi_j=\dfrac{\xi_j}{i|\xi|}\psi\Big(\dfrac{\xi}{|\xi|}\Big)$. Moreover, $\mathcal{A}_{\bar{\Psi}_j}=-R_j\circ\mathcal{A}_{\bar{\psi}}=\partial_j I_1\circ\mathcal{A}_{\bar{\psi}}$, see (3.6). Thus (3.10) is equivalent to

$$\lim_{n\to\infty}\left\langle \sum_{j=1}^{d}\partial_j(u_nA_j)\,,\,\bar{\varphi}_1 I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))\right\rangle+\sum_{i=1}^{d}\lim_{n\to\infty}\left\langle u_nA_j\,,\,\partial_j(\bar{\varphi}_1)I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))\right\rangle=0. \tag{3.11}$$

Since $\mathcal{A}_{\bar{\psi}}(\varphi_2v_n)\in W^{k,q}(\mathbf{R}^d)$ it follows from (3.5) that

$$\partial^\alpha I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))=I_1(\mathcal{A}_{\bar{\psi}}(\partial^\alpha(\varphi_2v_n)))\in L^{\frac{qd}{d-q}}(\mathbf{R}^d),\ \text{for all } 0\le|\alpha|\le k. \tag{3.12}$$

Now, since $q<\dfrac{qd}{d-q}$, we have that for all $\varphi\in\mathcal{S}(\mathbf{R}^d)$,

$$\|\varphi\, I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))\|_{L^q}\le\|I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))\|_{L^{\frac{qd}{d-q}}}\|\varphi\|_{L^d}. \tag{3.13}$$

From (3.12) and (3.13) we see that

$$\partial^\alpha[\varphi\, I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))]\in L^q(\mathbf{R}^d),\ \text{for all } 0\le|\alpha|\le k.$$

Now,

$$\partial^{\alpha+e_j}[I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))]=-R_j(\mathcal{A}_{\bar{\psi}}(\partial^\alpha(\varphi_2v_n)))\in L^q(\mathbf{R}^d),$$

which gives us that for all $\varphi\in\mathcal{S}(\mathbf{R}^d)$,

$$\varphi\, I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))\in W^{k+1,q}(\mathbf{R}^d),$$

and moreover

$$\varphi\, I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))\rightharpoonup 0 \text{ in } W^{k+1,q}(\mathbf{R}^d). \tag{3.14}$$

Take $\bar{\varphi}_1=\bar{\varphi}_{11}\bar{\varphi}_{12}$ all in $\mathcal{S}(\mathbf{R}^d)$. From (3.8) and (3.14) we conclude that

$$\left\langle\varphi_{11}f_n,\bar{\varphi}_{12}\, I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))\right\rangle\to 0.$$

From here and (3.7) we conclude that the first term in (3.11) converges to zero.

Now we analyze the second term in (3.11). We will prove that $\partial_j(\bar{\varphi}_1)I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))$ converges strongly to zero in $W^{k,q}(\mathbf{R}^d)$. For that purpose we write $\partial_j\bar{\varphi}_1=\bar{\varphi}_{13}\bar{\varphi}_{14}$, all in $\mathcal{S}(\mathbf{R}^d)$, and denote by L_m the open ball centered at the origin with radius $m\in\mathbf{N}$. By Rellich lemma $W^{k+1,q}(L_m)$ is compactly embedded in $W^{k,q}(L_m)$. Since $\bar{\varphi}_{14}I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))$ weakly converges to zero in $W^{k+1,q}(\mathbf{R}^d)$, by the diagonalization procedure we can extract a subsequence (not relabeled) such that for all $m\in\mathbf{N}$

$$\bar{\varphi}_{14}I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))\longrightarrow 0 \text{ in } W^{k,q}(L_m). \tag{3.15}$$

Take smooth cutoff functions χ_m such that $\chi_m(x)=1$ for $x\in L_m$ and $\chi_m(x)=0$ for $x\in\mathbf{R}^d\backslash L_{m+1}$ and write $\bar{\varphi}_{13}=\chi_m\bar{\varphi}_{13}+(1-\chi_m)\bar{\varphi}_{13}$. We have that

$$\|\bar{\varphi}_{13}\bar{\varphi}_{14}I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))\|_{W^{k,q}}\le\sup_{|\alpha|\le k,|x|>m}|\partial^\alpha\bar{\varphi}_{13}|\,\|\bar{\varphi}_{14}I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))\|_{W^{k,q}} \tag{3.16}$$

$$+\|\chi_m\bar{\varphi}_{13}\bar{\varphi}_{14}I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2v_n))\|_{W^{k,q}}. \tag{3.17}$$

Let $\varepsilon > 0$. The sequence $\bar\varphi_{14} I_1(\mathcal{A}_{\bar\psi}(\varphi_2 v_n))$ is bounded in $W^{k,q}(\mathbf{R}^d)$, i.e. there is $M > 0$ such that $\|\bar\varphi_{14} I_1(\mathcal{A}_{\bar\psi}(\varphi_2 v_n))\|_{W^{k,q}} \leq M$. Since $\bar\varphi_{13} \in \mathcal{S}(\mathbf{R}^d)$, there exists $m_0 \in \mathbf{N}$ such that for all $m \geq m_0$

$$\sup_{|\alpha|\leq k, |x|>m} |\partial^\alpha \bar\varphi_{13}| < \frac{\varepsilon}{2M}.$$

Next, from (3.15) we have that (3.17) goes to zero as $n \to \infty$. So, for given ε, there exists $n_0 \in \mathbf{N}$ such that (3.17) is less than $\varepsilon/2$ for all $n \geq n_0$. Thus the left hand side in (3.16) is less than ε for $n > n_0$, i.e. $\partial_j(\bar\varphi_1) I_1(\mathcal{A}_{\bar\psi}(\varphi_2 v_n))$ converges strongly in $W^{k,q}(\mathbf{R}^d)$ and (3.11) holds, which completes the proof of (3.9).

If coefficients A_j are such that $\sum_{j=1}^d A_j(x)\xi_j \neq 0$, $\xi \in \mathbf{S}^{d-1}$, then Theorem 3.2 implies the strong convergence $\theta u_n \longrightarrow 0$, for every $\theta \in \mathcal{S}(\mathbf{R}^d)$.

□

4. *Appendix*

We give here the full description of spaces $\mathcal{SE}$ and $\mathcal{SE}'$ introduced in Subsection 2.1. We recall from [**6**, Section 3.8.] the basic properties of Sobolev spaces on the unit sphere with respect to the surface measure dS^{d-1}. In the sequel we assume that $d > 2$. Let $\Omega_l = \{x \in \mathbf{R}^d; |x| \in [1-l, 1+l]\}$, $0 < l < 1$, and $k \in \mathbf{N}_0$. Then $\phi \in C^k(\mathbf{S}^{d-1})$ if for some and hence for all $0 < l < 1$, $\phi^* \in C^k(\Omega_l)$, where $\phi^*(x) = \phi(x/|x|)$. Moreover, [**6**, p. 9], $C^k(\mathbf{S}^{d-1})$ is equipped with the norm

$$p_{\mathbf{S}^{d-1},k}(\phi) = |\phi|_{C^k(\mathbf{S}^{d-1})} = \sup_{|\alpha|\leq k, x\in\Omega_l} |\partial^\alpha \phi^*(x)|, \tag{4.1}$$

and this norm does not depend on $l \in (0,1)$. Then $C^\infty(\mathbf{S}^{d-1}) = \bigcap_{k\in\mathbf{N}_0} C^k(\mathbf{S}^{d-1})$. The completion of $C^\infty(\mathbf{S}^{d-1})$ with respect to the norm

$$\|v\|_{H^s(\mathbf{S}^{d-1})} = \Big\| \Big(-\Delta^\star + \Big(\frac{d-2}{2}\Big)^2 \Big)^{s/2} v \Big\|_{L^2(\mathbf{S}^{d-1})},$$

where $\Delta^\star$ is the Laplace-Beltrami operator, is the Sobolev space $H^s(\mathbf{S}^{d-1})$, $s \in \mathbf{N}_0$. The case $d = 2$, when $\mathbf{S}^1$ is given by $x = \cos\theta$, $y = \sin\theta$, $\theta \in [0, 2\pi)$, is the simple one which we do not consider. Note, in this case one can take $-\Delta^\star + 1$ instead of $-\Delta^\star + ((d-2)/2)^2$.

Denote by $\{Y_{n,j},\ 1 \leq j \leq N_{n,d}, n \in \mathbf{N}_0\}$ the orthonormal basis of $L^2(\mathbf{S}^{d-1})$ (cf. [**6**, p. 121] or [**19**, Proposition 10.2, p. 92]), where $N_{n,d} \sim O(n^{d-2})$, [**6**, p. 16], is the dimension of the set of independent spherical harmonics $Y_{n,j}$ of order n. Then we have

$$\|v\|_{H^s(\mathbf{S}^{d-1})} = \sqrt{\sum_{n=0}^{\infty} \sum_{j=1}^{N_{n,d}} \Big(n + \frac{d-2}{2}\Big)^{2s} \|v_{n,j}\|^2}, \tag{4.2}$$

where $v_{n,j} = \int_{\mathbf{S}^{d-1}} v \overline{Y}_{n,j}\, dS^{d-1}$.

The space $C^\infty(\mathbf{S}^{d-1})$, supplied by the sequence of norms (4.1), $k \in \mathbf{N}_0$, is denoted by $\mathcal{E}(\mathbf{S}^{d-1})$. By the Sobolev lemma for compact manifolds [**7**, Theorems 2.20, 2.21 (see also Theorem 2.10)], explicitly written in [**19**, Theorem 7.6, p. 61], we have that

$$\mathcal{E}(\mathbf{S}^{d-1}) = \bigcap_{s\in\mathbf{N}_0} H^s(\mathbf{S}^{d-1}). \tag{4.3}$$

This is a Fréchet space. Since all elements of $C^\infty(\mathbf{S}^{d-1})$ are compactly supported, we also have that $\mathcal{E}(\mathbf{S}^{d-1}) = \mathcal{D}(\mathbf{S}^{d-1})$.

By [**16**, Theorem II.10, p. 52], if we have orthonormal bases $(\psi_n)_{n\in\mathbf{N}}$ and $(\tilde{\psi}_m)_{m\in\mathbf{N}}$ for $L^2(\mathbf{R}^d, dx)$ with Lebesgue measure dx and $L^2(\mathbf{S}^{d-1}, dS^{d-1})$ with surface measure dS^{d-1} respectively, then $\psi_n(t_1)\tilde{\psi}_m(t_2)$, $(t_1,t_2)\in\mathbf{R}^d\times\mathbf{S}^{d-1}$, is an orthonormal basis for $L^2(\mathbf{R}^d\times\mathbf{S}^{d-1})$.

By [**16**, Appendix to V.3, p. 141] (where the case $d=1$ is treated), one has that the product of one-dimensional harmonic oscillators $N_x = N_1...N_d$, $N_i = x_i^2 - (d/dx_i)^2, i = 1, ..., d$, and the Hermite basis $h_n(x) = h_{n_1}(x_1)...h_{n_d}(x_d), n\in\mathbf{N}_0^d$, of $L^2(\mathbf{R}^d)$ satisfy

$$N_x^k h_n = (2n_1+1)^k...(2n_d+1)^k h_n,\ n\in\mathbf{N}_0^d,\ k\in\mathbf{N}_0.$$

Moreover, $\mathcal{S}(\mathbf{R}^d)$ is determined by the sequence of norms

$$|||\phi|||_k = ||N^k\phi||_2 = \sum_{n\in\mathbf{N}_0^d}(2n_1+1)^{2k}...(2n_d+1)^{2k}|a_n|^2,\quad k\in\mathbf{N}, \tag{4.4}$$

where $\phi = \sum_{n\in\mathbf{N}^d} a_n h_n \in \mathcal{S}(\mathbf{R}^d)$. This sequence of norms is equivalent to the usual one for $\mathcal{S}(\mathbf{R}^d)$.

Now, we define the space of smooth functions $\mathcal{SE}(\mathbf{R}^d\times\mathbf{S}^{d-1})$ by the sequence of norms (2.1). By the quoted Sobolev lemma for compact manifolds [**7**] and (4.3), we have the next proposition.

PROPOSITION 4.1. *The family of norms (2.1) is equivalent to any of the following two families of norms:*

$$p^2_{\mathbf{R}^d\times\mathbf{S}^{d-1},k}(\theta) = \left(\int_{\mathbf{R}^d\times\mathbf{S}^{d-1}} |N_x^k(\Delta_\xi^\star)^\alpha\partial_x^\beta\,\theta(x,\xi)|^2\,dxd\xi\right)^{\frac{1}{2}}, \tag{4.5}$$

$$p_{\mathbf{R}^d\times\mathbf{S}^{d-1},k}(\theta) = \sup_{(x,\xi)\in\mathbf{R}^d\times\Omega_l,|\alpha+\beta|\le k}\langle x\rangle^k|\partial_\xi^\alpha\partial_x^\beta\theta^*(x,\xi)|, \tag{4.6}$$

where $\theta^*(x,\xi) = \theta(x,\xi/|\xi|)$, $\langle x\rangle^k = (1+|x|^2)^{k/2}$ *and the derivatives with respect to* ξ *are defined as above, with fixed* x.

In particular, $\mathcal{SE}(\mathbf{R}^d\times\mathbf{S}^{d-1})$ *is a Fréchet space.*

Note that $\mathcal{SE}(\mathbf{R}^d\times\mathbf{S}^{d-1})$ induces the π-topology on $\mathcal{S}(\mathbf{R}^d)\otimes\mathcal{E}(\mathbf{S}^{d-1})$, see [**22**, Chap. 43] for the $\pi-$topology. Since $\mathcal{S}(\mathbf{R}^d)$ is nuclear, the completion $\mathcal{S}(\mathbf{R}^d)\hat{\otimes}\mathcal{E}(\mathbf{S}^{d-1})$ is the same for the π and the ε topologies, cf. [**22**, Part III, Chap. 50, Theorem 50.1, p. 511].

PROPOSITION 4.2.

$$\mathcal{S}(\mathbf{R}^d)\hat{\otimes}\mathcal{E}(\mathbf{S}^{d-1}) = \mathcal{SE}(\mathbf{R}^d\times\mathbf{S}^{d-1}). \tag{4.7}$$

Proof. Clearly the embedding $\mathcal{S}(\mathbf{R}^d)\hat{\otimes}\mathcal{E}(\mathbf{S}^{d-1})\to\mathcal{SE}(\mathbf{R}^d\times\mathbf{S}^{d-1})$ is continuous. Thus for the proof of (4.7), it is enough to prove that the left space is dense in the right one. As for general manifolds, we have that $h_m(x)\times Y_{n,j}(\xi)$, $1\le j\le N_{n,d}$, $n,m\in\mathbf{N}$, is an orthonormal basis for $L^2(\mathbf{R}^d\times\mathbf{S}^{d-1})$. Now, by (4.3) – (4.5), it follows that

$$\theta(x,\xi) = \sum_{n=0}^\infty\sum_{j=1}^{N_{n,d}}\sum_{m\in\mathbf{N}_0^d} a_{n,j,m}h_m(x)Y_{n,j}(\xi)\in\mathcal{SE}(\mathbf{R}^d\times\mathbf{S}^{d-1}) \tag{4.8}$$

if and only if for every $r > 0$,

$$\sum_{n=0}^{\infty} \sum_{j=1}^{N_{n,d}} \sum_{m \in \mathbf{N}_0^d} |a_{n,j,m}|^2 (1 + n^2 + |m|^2)^r < \infty \tag{4.9}$$

(cf. [**23**, Chapter 9]). Now taking finite sums of the right-hand side of (4.8), we obtain that $\mathcal{S}(\mathbf{R}^d)\hat{\otimes}\mathcal{E}(\mathbf{S}^{d-1})$ is dense in $\mathcal{SE}(\mathbf{R}^d \times \mathbf{S}^{d-1})$. This completes the proof. □

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References

1. Abels, H. *Pseudodifferential and singular integral operators. An introduction with applications.* De Gruyter, Berlin, 2012

2. Adams, R. A. *Sobolev spaces. Pure and Applied Mathematics*, Vol. 65. Academic Press, New York-London, 1975.

3. Aleksić, J.; Mitrović, D.; Pilipović, S. *Hyperbolic conservation laws with vanishing nonlinear diffusion and linear dispersion in heterogeneous media.* J. Evol. Equ. 9 (2009), no. 4, 809–828.

4. Antonić N., Lazar M., *Parabolic variant of H-measures in homogenisation of a model problem based on Navier-Stokes equation*, Nonlinear Analysis—Real World Appl., 11 (2010), 4500–4512.

5. Antonić, N.; Mitrović, D. *H-distributions: an extension of H-measures to an $L^p - L^q$ setting.* Abstr. Appl. Anal. 2011, Art. ID 901084, 12 pp.

6. Atkinson, K.; Han, W. *Spherical harmonics and approximations on the unit sphere: an introduction.* Lecture Notes in Mathematics, 2044. Springer, Heidelberg, 2012.

7. Aubin, Thierry *Some nonlinear problems in Riemannian geometry.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

8. Gérard, P. *Microlocal defect measures.* Comm. Partial Differential Equations 16 (1991), no. 11, 1761–1794.

9. Grafakos, L. *Classical and Modern Fourier Analysis.* Pearson Education, Inc., 2004.

10. Köthe, G. *Topological vector spaces. I.* Springer-Verlag, 1969.

11. Lazar, M., Mitrovic, D., *On an extension of a bilinear functional on $L^p(R^d)xE$ to a Bochner space with an application on velocity averaging*, C. R. Acad. Sci. Paris Ser. I Math. 351 (2013), 261–264.

12. Mielke A., *Macroscopic behavior of microscopic oscillations in harmonic lattices via Wigner-Husimi transforms.* Arch. Rational Mech. Anal. 181(2006), 401–448.

13. Misur M., Mitrovic,D., *On a generalization of compensated compactness in the $L^p - L^q$ setting*, Journal of Functional Analysis, 268 (2015) 1904–1927.

14. E. Yu. Panov, *Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux.* Arch. Ration. Mech. Anal. 195 (2010), no. 2, 643–673.

15. Pilipović, S. *On the convergence in $\mathcal{S}'$.* Proc. Amer. Math. Soc. 111 (1991), no. 4, 949–954.

16. Reed, M., Simon, B., *Methods of modern mathematical physics. I. Functional analysis.* Second edition. Academic Press, New York, 1980.

17. Rindler, F. *Directional oscillations, concentrations, and compensated compactness via microlocal compactness forms*, Arch. Ration. Mech. Anal. 215 (2015), 1–63.

18. Sazhenkov S. A., *The genuinely nonlinear Graetz-Nusselt ultraparabolic equation*, (Russian. Russian summary) Sibirsk. Mat. Zh. 47 (2006), no. 2, 431–454; translation in Siberian Math. J. 47 (2006), no. 2, 355–375.

19. Shubin Mikhail, Pseudodifferential Operators and Spectral Theory, Springer-Verlag, Berlin Heidelberg New York, 2001.

20. Stein, E. M. *Singular Integrals and Differential Properties of Functions.* Princeton, 1970.

21. Tartar, L. *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations.* Proc. Roy. Soc. Edinburgh Sect. A 115 (1990), no. 3-4, 193–230.

22. Tréves, F. *Topological vector spaces, distributions and kernels.* Academic Press, New York-London 1967.

23. Zemanian, A. H. *Generalized integral transformations.* Pure and Applied Mathematics, Vol. XVIII. Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1968.

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