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DECIDABILITY PROBLEMS FOR THE VARIETY GENERATED BY TOURNAMENTS

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Abstract

This paper deals with a representation of tournaments (complete directed graphs with loops) by algebras having one binary operation. The variety generated by such algebras is our main object of investigation. We focus on those varietal properties which involve questions of decidability, i.e. existence of different kinds of algorithms. It is shown that the considered variety has decidable equational theory and solvable word problem. However, its elementary theory is undecidable.

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1. Introduction

In the sense of graph theory (see e.g. [5] and [8]) a tournament is a complete directed graph, that is, a digraph in which every pair of different vertices is connected by exactly one directed edge. Of course, it will make no major difference if we consider tournaments to have loops (edges leading from a vertex to itself). More precisely, the concept of a tournament can be formalized as a relational structure $\mathbf{T} = \langle T, \rightarrow \rangle$, where T is a non-empty finite set and $\rightarrow$ is a reflexive binary relation on T such that for each $x, y \in T$, $x \neq y$, exactly one of the assertions $x \rightarrow y$, $y \rightarrow x$ is true.

In 1965, Z. Hedrlín observed that any tournament $\mathbf{T}$ can be transformed into a groupoid by defining the multiplication such that $xy = yx = x$ if and only if $x \rightarrow y$ for all $x, y \in T$ (note that this yields $x^2 = x$ for all $x \in T$). It is easy to see that the described construction is in fact a bijective correspondence between the class of all tournaments and the class of all finite commutative groupoids satisfying $xy \in \{x, y\}$ for all x and y. Therefore, the groupoids obtained from tournaments in this way will be also called tournaments, whenever it makes no confusion.

Since then, algebraic representations of tournaments have become the subject of many interesting investigations. Such is the paper of Müller, Nešetřil and Pelant [10] – which was the inspiration for our work – who followed Hedrlín's approach. Another algebraization of tournaments is presented and studied in [1], [2], [3], [4] and [9], just to mention some of the papers dealing with the topic. Finally, we should refer to a recent paper [6], where it was proved that the equational theory of the class of all tournaments is not finitely based, thereby solving a problem posed in [10].

In this paper we consider the variety of groupoids generated by the class of all tournaments, which will be denoted by $\mathcal{T}$ in the sequel. Our aim is to examine main algorithmic problems concerning $\mathcal{T}$. First of all, we prove that the equational theory of $\mathcal{T}$ is decidable. We proceed by locating the finitely generated free algebras of $\mathcal{T}$ and showing that $\mathcal{T}$ is locally finite in order to solve its word problem in a uniform way. In the final section, it is proved that $\mathcal{T}$ has undecidable elementary theory by making use of some methods and techniques of model theory.

2. The word problem for $\mathcal{T}$

Let Ω_n denote the set of all n-element tournaments having $\{1, 2, \ldots, n\}$ as its universe. Clearly, for each natural number n, Ω_n is a finite set and each n-element tournament is isomorphic to some member of Ω_n. We begin this section with a lemma, which shows that the tournaments in Ω_n are sufficient to decide whether an equation with n variables holds in $\mathcal{T}$ or not.

Lemma 2.1. *Let $p = p(x_1, \ldots, x_n)$ and $q = q(x_1, \ldots, x_n)$ be arbitrary groupoid terms. Then $\mathcal{T} \models p \approx q$ if and only if $\Omega_n \models p \approx q$.*

Proof. ($\Rightarrow$) is obvious. ($\Leftarrow$) Suppose $p \approx q$ does not hold in $\mathcal{T}$. Then there is a tournament $\mathbf{T}$ and $a_1, \ldots, a_n \in T$ such that $p^{\mathbf{T}}(a_1, \ldots, a_n) \neq q^{\mathbf{T}}(a_1, \ldots, a_n)$. But the chosen elements $a_1, \ldots, a_n$ form a subtournament of $\mathbf{T}$, say $\mathbf{T}_1$, and we have $\mathbf{T}_1 \not\models p \approx q$. Now it just remains to expand $\mathbf{T}_1$ to a tournament $\mathbf{T}_2$ having exactly n elements. Clearly $\mathbf{T}_2 \not\models p \approx q$ and so $\Omega_n \not\models p \approx q$. □

Corollary 2.2. *The equational theory of $\mathcal{T}$ is decidable.*

Lemma 2.3. $\mathbf{F}_{\mathcal{T}}(n) \in \mathsf{ISP}_{fin}(\Omega_n)$.

Proof. Suppose $\Omega_n = \{\mathbf{T}_1, \ldots, \mathbf{T}_m\}$. Consider the algebra

$$\mathbf{P}_n = \mathbf{T}_1^{n^n} \times \ldots \times \mathbf{T}_m^{n^n}.$$

We prove that $\mathbf{F}_{\mathcal{T}}(n) \in \mathsf{IS}(\mathbf{P}_n)$.

Let $\bar{X} = \{\bar{x}_1, \ldots, \bar{x}_n\}$ be the set of free generators of $\mathbf{F}_{\mathcal{T}}(n)$. Consider the mapping $\bar{\varphi} : \bar{X} \mapsto P_n$ defined for $1 \leq i \leq n$ and $1 \leq j \leq mn^n$ by

$$(\bar{\varphi}(\bar{x}_i))(j) = a^{(q+1)}_{f_r(i)},$$

where q and r are respectively the quotient and the remainder of j divided by n^n, $T_i = \{a_1^{(i)}, \ldots, a_n^{(i)}\}$ and $f_0, f_1, \ldots, f_{n^n-1}$ are all transformations of the set $\{1, 2, \ldots, n\}$.

Now $\bar{\varphi}$ extends in a unique way to a homomorphism $\varphi : \mathbf{F}_{\mathcal{T}}(n) \mapsto \mathbf{P}_n$. Thus it remains to show that φ is injective.

Suppose that $\bar{p}_1 \neq \bar{p}_2$ in $\mathbf{F}_{\mathcal{T}}(n)$. This means that $\mathcal{T} \not\models p_1 \approx p_2$. It is clear that for $i = 1, 2$ and $1 \leq j \leq mn^n$ we have

$$(\varphi(\bar{p}_i))(j) = p_i^{\mathbf{T}_{q+1}}(a^{(q+1)}_{f_r(1)}, \ldots, a^{(q+1)}_{f_r(n)}),$$

where q, r denote the same as before. By Lemma 2.1, it follows that

$$\mathbf{T}_k \not\models p_1 \approx p_2$$

for some $k \in \{1, 2, \ldots, m\}$, i.e. there are elements $a^{(k)}_{l_1}, \ldots, a^{(k)}_{l_n} \in T_k$ such that

$$p_1^{\mathbf{T}_k}(a^{(k)}_{l_1}, \ldots, a^{(k)}_{l_n}) \neq p_2^{\mathbf{T}_k}(a^{(k)}_{l_1}, \ldots, a^{(k)}_{l_n}).$$

Choose among $f_0, f_1, \ldots, f_{n^n-1}$ a mapping f_t such that $f_t(i) = l_i$ for all $1 \leq i \leq n$. Doing so, we obtain

$$p_1^{\mathbf{T}_k}(a^{(k)}_{f_t(1)}, \ldots, a^{(k)}_{f_t(n)}) \neq p_2^{\mathbf{T}_k}(a^{(k)}_{f_t(1)}, \ldots, a^{(k)}_{f_t(n)}),$$

which is nothing else than

$$(\varphi(\bar{p}_1))((k - \mathrm{sg}(t))n^n + t) \neq (\varphi(\bar{p}_2))((k - \mathrm{sg}(t))n^n + t).$$

The latter implies that $\varphi(\bar{p}_1) \neq \varphi(\bar{p}_2)$ and the lemma is proved. □

Corollary 2.4. *$\mathcal{T}$ is locally finite.*

Corollary 2.5. *There is an algorithm which for every finite groupoid* $\mathbf{G}$ *decides whether* $\mathbf{G} \in \mathcal{T}$.

Proof. Let $|G| = n$. We have that $\mathbf{G} \in \mathcal{T}$ if and only if $\mathbf{G} \in \mathsf{H}(\mathbf{F}_{\mathcal{T}}(n))$. By Lemma 2.3, $\mathbf{F}_{\mathcal{T}}(n) \in \mathsf{IS}(\mathbf{P}_n)$, so $\mathsf{H}(\mathbf{F}_{\mathcal{T}}(n))$ is contained in $\mathsf{HS}(\mathbf{P}_n)$. Therefore, if $\mathbf{G} \in \mathcal{T}$ then $\mathbf{G} \in \mathsf{HS}(\mathbf{P}_n)$. The converse of the latter implication is trivial, thus we have that $\mathbf{G} \in \mathcal{T}$ if and only if $\mathbf{G} \in \mathsf{HS}(\mathbf{P}_n)$. Now the corollary follows immediately, because for each n one can effectively construct $\mathbf{P}_n$, find all its subalgebras, then all quotients of these subalgebras and finally check whether $\mathbf{G}$ is isomorphic to some of the obtained quotients. □

As a consequence of Lemma 2.3, we can also strenghten a result from Müller, Nešetřil and Pelant, [10].

Corollary 2.6. *$\mathcal{T}$ is not finitely generated.*

Proof. If $\mathcal{T}$ was generated by finitely many finite groupoids, then by Lemma 2.3 each of those groupoids would have been generated by finitely many tournaments, so that a finite set of tournaments would generate $\mathcal{T}$, which is shown to be impossible in the paper cited above, Section 5.1. □

The main result of this section is the following one.

Theorem 2.7. *$\mathcal{T}$ has uniformly solvable word problem.*

Proof. Let (G, R) be a presentation in $\mathcal{T}$. Our goal is to prove that one can effectively construct $\mathbf{P}_{\mathcal{T}}(G, R)$.

First of all, as $\mathcal{T}$ is locally finite, $\mathbf{P}_{\mathcal{T}}(G, R)$ must be a finite groupoid. Let $|G| = n$. By Lemma 2.3, $\mathbf{P}_{\mathcal{T}}(G, R) \in \mathsf{HS}(\mathbf{P}_n)$. As before, one can construct all quotients of all subalgebras of $\mathbf{P}_n$. Note that only a finite number of algebras is obtained in such a way. For each of these finite algebras, there is an obvious way to check whether they are n-generated. If so, check whether the relations from R are satisfied for some set of n generators of the considered algebra. Therefore, we picked all homomorphic images of $\mathbf{P}_{\mathcal{T}}(G,R)$. It is clear that the image having most elements must be isomorphic to $\mathbf{P}_{\mathcal{T}}(G,R)$.

The theorem now follows immediately. □

3. Undecidability of the elementary theory of $\mathcal{T}$

In this section we are going to prove that $Th(\mathcal{T})$ is undecidable. To do that, we shall apply techniques from McKenzie and Valeriote [7]. But first we need to introduce the necessary notions and terminology which will be used below.

Definition 3.1. Let $\Sigma_0 = \{r_1,\dots,r_k; f_1,\dots,f_l\}$ be a finite first-order language, where r_i, $1 \le i \le k$ are predicates, f_j, $1 \le j \le l$ are operation symbols (constants are considered as 0-ary operation symbols) and let Σ be an arbitrary first-order language. An *interpretation scheme for* Σ_0 *in* Σ is, for some fixed integers $m \ge 1$, $n \ge 0$, a $(k+l+2)$-tuple

$$\Psi = \langle Un, Eq, R_1,\dots,R_k, F_1,\dots,F_l\rangle$$

of formulæ of type Σ such that

$$\begin{aligned} Un &= Un(\mathbf{x},\mathbf{y}),\\ Eq &= Eq(\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}),\\ R_i &= R_i(\mathbf{x}_1,\dots,\mathbf{x}_{p_i},\mathbf{y}),\\ F_j &= F_j(\mathbf{x}_1,\dots,\mathbf{x}_{q_j},\mathbf{x}_{q_j+1},\mathbf{y}),\end{aligned}$$

where $\mathbf{x},\mathbf{x}_1,\mathbf{x}_2,\dots$ denote sequences of m variables, while $\mathbf{y}$ is a sequence of n variables. The integers p_i and q_j denote, respectively, the arities of the symbols r_i, $1\le i\le k$ and f_j, $1 \le j \le l$. All variables involved above are different and all free variables of the formulæ just listed are among the variables appearing in the brackets following the names of the corresponding formulæ.

Definition 3.2. Let $\mathbf{A}$ be a model of type Σ and $\mathbf{w}\in A^n$. We say that the pair $\langle \mathbf{A},\mathbf{w}\rangle$ *accepts the scheme* Ψ if there exists a model

$$\mathbf{A}_0 = \langle A_0, r_1^{\mathbf{A}_0},\dots,r_k^{\mathbf{A}_0}, f_1^{\mathbf{A}_0},\dots,f_l^{\mathbf{A}_0}\rangle$$

of the type Σ_0 and a mapping φ from $Un^{\langle \mathbf{A},\mathbf{w}\rangle} = \{\mathbf{v} \in A^m \mid \mathbf{A} \models Un(\mathbf{v},\mathbf{w})\}$ onto A_0 such that for all $\mathbf{v}_1, \mathbf{v}_2, \ldots \in Un^{\langle \mathbf{A},\mathbf{w}\rangle}$ the following three conditions are satisfied:

(1) $\varphi(\mathbf{v}_1) = \varphi(\mathbf{v}_2)$ if and only if $\mathbf{A} \models Eq(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w})$.

(2) If $1 \leq i \leq k$ and p_i is the arity of r_i, then

$$\langle \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_{p_i})\rangle \in r_i^{\mathbf{A}_0} \text{ if and only if } \mathbf{A} \models R_i(\mathbf{v}_1, \ldots, \mathbf{v}_{p_i}, \mathbf{w}).$$

(3) If $1 \leq j \leq l$ and q_j is the arity of f_j, then

$$f_j^{\mathbf{A}_0}(\varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_{q_j})) = \varphi(\mathbf{v}_{q_j+1}) \text{ if and only if}$$
$$\mathbf{A} \models F_j(\mathbf{v}_1, \ldots, \mathbf{v}_{q_j+1}, \mathbf{w}).$$

It is not difficult to see that if the pair $\langle \mathbf{A}, \mathbf{w}\rangle$ accepts the scheme Ψ, then $\mathbf{A}_0$ is uniquely determined up to an isomorphism. Note that

$$Eq^{\langle \mathbf{A},\mathbf{w}\rangle} = \{\langle \mathbf{v}_1, \mathbf{v}_2\rangle \in A^m \times A^m \mid \mathbf{A} \models Eq(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w})\}$$

is an equivalence relation on A^m. Therefore, we are going to use the notation $\mathbf{A}(\Psi, \mathbf{w})$ for the model $\mathbf{A}_0$ as above, having the set of equivalence classes of the restriction of $Eq^{\langle \mathbf{A},\mathbf{w}\rangle}$ to $Un^{\langle \mathbf{A},\mathbf{w}\rangle}$ as its universe.

We are prepared to give the definition of interpretability of theories.

Definition 3.3. Let Σ_0 and Σ denote the same as above and let $\mathcal{K}_0$ and $\mathcal{K}$ be classes of models of type Σ_0 and Σ, respectively. The class $\mathcal{K}_0$ is *interpretable in* $\mathcal{K}$ if there exists an interpretation scheme Ψ such that for all $\mathbf{B} \in \mathcal{K}_0$ there exist $\mathbf{A} \in \mathcal{K}$ and $\mathbf{w} \in A^n$ such that $\mathbf{B} \cong \mathbf{A}(\Psi, \mathbf{w})$. A theory Γ_0 of type Σ_0 is *interpretable in the theory* Γ of type Σ if there exists a class $\mathcal{K}_0$ of models of type Σ_0 such that $Th(\mathcal{K}_0) = \Gamma$ and $\mathcal{K}_0$ is interpretable in $Mod_\Sigma(\Gamma)$.

This whole aparatus was built up in the previous paragraphs to enable us to formulate the following theorem from [7].

Theorem 3.4. (McKenzie and Valeriote, [7]) *Let Γ_0 be a theory on a finite language which is intepretable in the theory Γ. Then:*

(1) *if* Γ_0 *is finitely axiomatizable and decidable, then* Γ *is decidable,*

(2) *if* Γ_0 *is hereditarily undecidable, so is* Γ.

Now we can turn to the demonstration of our main result.

Theorem 3.5. *The variety* $\mathcal{T}$ *has a hereditarily undecidable elementary theory.*

Proof. Denote the class of *finite graphs* (finite relational structures with an irreflexive, symmetric binary relation) by $\mathcal{FG}$. It is well known that $Th(\mathcal{FG})$ is a hereditarily undecidable theory. Hence, the theorem will be proved if we show that $\mathcal{FG}$ is interpretable in $\mathcal{T}$, because the latter implies that $\Gamma_0 = Th(\mathcal{FG})$ is interpretable in $Th(\mathcal{T})$ ($\mathcal{T}$ is a variety, so $\mathcal{T} = Mod(Th(\mathcal{T}))$) and then Theorem 3.4 gives the desired result.

First we define an intepretation scheme $\Psi = \langle Un, Eq, R\rangle$ for the language with one binary predicate r in the language with one binary operation symbol $*$ for $m = n = 1$. Let $Un(x, y)$ be the formula

$$x * y \approx x \wedge \neg(x \approx y),$$

$Eq(x_1, x_2, y)$ be the formula

$$x_1 \approx x_2,$$

and finally, let $R(x_1, x_2, y)$ denote

$$\begin{aligned} &\neg(x_1 \approx x_2) \ \wedge \ (\exists z)(\exists t)(\neg(z \approx y) \ \wedge \ \neg(z \approx t) \ \wedge \\ &\wedge \ y * z \approx y \ \wedge \ y * t \approx y \ \wedge \ x_1 * z \approx x_1 \ \wedge \ x_2 * t \approx x_2 \ \wedge \\ &((x_1 * x_2 \approx x_1 \ \wedge \ z * t \approx z) \ \vee \ (x_1 * x_2 \approx x_2 \ \wedge \ z * t \approx t))). \end{aligned}$$

Now let $\mathbf{G} = (G, \rho)$ be a finite graph, $G = \{a_1, a_2, \ldots, a_n\}$. Define a $(2n+1)$-element tournament $\mathbf{T} = \langle T, \cdot\rangle$, $T = \{b_1, \ldots, b_n, c_1, \ldots, c_n, d\}$, in the following way:

(1) $b_i d = b_i$ for all $1 \leq i \leq n$,

(2) $c_i d = d$ for all $1 \leq i \leq n$,

(3) $c_i c_j = c_i$ for all $1 \leq i < j \leq n$,

(4) $b_i c_i = b_i$ for all $1 \leq i \leq n$,

(5) $b_i c_j = c_j$ for all $1 \leq i \neq j \leq n$,

(6) $$b_i b_j = \begin{cases} b_i, & \text{if } i < j,\ \langle a_i, a_j \rangle \in \rho \text{ or } i > j,\ \langle a_i, a_j \rangle \notin \rho \\ b_j, & \text{otherwise} \end{cases}$$

(7) $x^2 = x$ for all $x \in T$,

(8) $xy = yx$ for all $x, y \in T$.

For this tournament we have

$$Un^{\langle \mathbf{T}, d \rangle} = \{x \in T |\ \mathbf{T} \models Un(x,d)\} = \{x \in T \setminus \{d\} |\ xd = x\},$$

which obviously equals to $\{b_1, b_2, \ldots, b_n\}$. Since $Eq^{\langle \mathbf{T}, d \rangle}$ is just the diagonal relation on T, it follows that the universe of $\mathbf{T}(\Psi, d)$ is $\{\{b_1\}, \{b_2\}, \ldots, \{b_n\}\}$ and that $\langle \{b_i\}, \{b_j\} \rangle \in r^{\mathbf{T}(\Psi, d)}$ if and only if $\mathbf{T} \models R(b_i, b_j, d)$. But the formula R interprets in $\mathbf{T}$ for the given valuation as follows: $b_i \neq b_j$ and there exist $z, t \in T \setminus \{d\}$ such that $dz = d$, $dt = d$, $b_i z = b_i$, $b_j t = b_j$ and either $b_i b_j = b_i$, $zt = z$ or $b_i b_j = b_j$, $zt = t$. Clearly, $b_i \neq b_j$ is equivalent to $i \neq j$, while by (1) and (2), $dz = d$ and $dt = d$ fulfils for $z, t \neq d$ if and only if z and t are one of the c_m's, say, $z = c_k$ and $t = c_l$. So, $zt = z$ means that $k \leq l$, while $zt = t$ means that $k \geq l$. Summing up, $\mathbf{T} \models R(b_i, b_j, d)$ if and only if $i \neq j$ and one of the two cases, either $b_i b_j = b_i$, $b_i c_k = b_i$ and $b_j c_l = b_j$ for some $k \leq l$ or $b_i b_j = b_j$, $b_i c_k = b_i$ and $b_j c_l = b_j$ for some $k \geq l$ is true. By definition of the multiplication in $\mathbf{T}$, $b_p c_q = b_p$ holds if and only if $p = q$. Applying this observation repeatedly above, we conclude that it must be either $b_i b_j = b_i$ and $i < j$ or $b_i b_j = b_j$ and $i > j$, which is by (6) precisely the same as $\langle a_i, a_j \rangle \in \rho$.

What we just proved is that the mapping $\xi : G \mapsto \{\{b_1\}, \ldots, \{b_n\}\}$ defined by $\xi(a_i) = \{b_i\}$ for all $1 \leq i \leq n$ is an isomorphism of $\mathbf{G}$ and $\mathbf{T}(\Psi, d)$. Thus, $\mathcal{FG}$ is interpretable in $\mathcal{T}$. □

References

[1] Erdős, P., Fried, E., Hajnal A. and Milner E. C., Some remarks on simple tournaments, Algebra Universalis 2 (1972), 238–245.

[2] Erdős, P., Hajnal A. and Milner, E. C., Simple one–point extensions of tournaments, Mathematika 19 (1972), 57–62.

[3] Fried, E. and Grätzer, G., A nonassociative extension of the class of distributive lattices, Pacific J. Math. 49 (1973), 59–78.

[4] Grätzer, G., Kisielewicz, A. and Wolk, B., An equational basis in four variables for the three-element tournament, Colloq. Math. 63 (1992), 41–44.

[5] Harary, F., Graph Theory, Addison–Wesley, Reading, 1969.

[6] Ježek J., Marković, P., Maróti M. and McKenzie, R., Equations of tournaments are not finitely based, Discrete Math., to appear.

[7] McKenzie, R. and Valeriote, M., The Structure of Locally Finite Decidable Varieties, Progress in Mathematics, Birkhäuser, Boston, 1989.

[8] Moon, J. W., Topics on Tournaments, Holt, Rinehart and Winston, New York, 1968.

[9] Moon, J. W., Embedding tournaments in simple tournaments, Discrete Math. 2 (1972), 389–395.

[10] Müller, V., Nešetřil J. and Pelant, J., Either tournaments or algebras?, Discrete Math. 11 (1975), 37–66.

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