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Maximal Antichains of Isomorphic Subgraphs of the Rado Graph

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Abstract. If $\langle R, E\rangle$ is the Rado graph and $\mathcal{R}(R)$ the set of its copies inside R, then $\langle \mathcal{R}(R), \subset\rangle$ is a chain-complete and non-atomic partial order of the size $2^{\aleph_0}$. A family $\mathcal{A} \subset \mathcal{R}(R)$ is a maximal antichain in this partial order iff (1) $A \cap B$ does not contain a copy of R, for each different $A, B \in \mathcal{A}$ and (2) For each $S \in \mathcal{R}(R)$ there is $A \in \mathcal{A}$ such that $A \cap S$ contains a copy of R. We show that the partial order $\langle \mathcal{R}(R), \subset\rangle$ contains maximal antichains of size $2^{\aleph_0}$, $\aleph_0$ and n, for each positive integer n (thus, of all possible cardinalities, under CH). The results are compared with the corresponding known results concerning the partial order $\langle [\omega]^\omega, \subset\rangle$.

1. Introduction

The object of our study is the Rado graph (the countable random graph) introduced by Erdős and Rényi [3] and characterized as the unique (up to isomorphism) countable graph $\langle R, E\rangle$ such that the set

$$R_K^{H\cup K} = \Big\{ r \in R \setminus (H \cup K) : \forall h \in H\ (rh \in E)\ \wedge\ \forall k \in K\ (rk \notin E)\Big\}$$

is non-empty, for each pair of disjoint finite subsets H, K of R. This rich combinatorial structure and various related structures (for example the automorphism group and the endomorphism monoid of $\langle R, E\rangle$, various topologies on R etc.) were extensively explored (see [1]).

Since for each partition of the Rado graph R into two pieces at least one of them is isomorphic to R, one of the structures naturally related to the Rado graph (and providing additional information about it) is the partial order $\langle \mathcal{R}(R), \subset\rangle$, where $\mathcal{R}(R)$ is the set of all isomorphic copies of R contained in R, that is the set of all subsets A of R such that $\langle A, E \cap [A]^2\rangle$ is a countable random graph. It is easy to see that $\langle \mathcal{R}(R), \subset\rangle$ is a chain-complete and non-atomic partial order with the largest element R and of the cardinality continuum. Concerning the question of how "tall" is this partial order, we note that, by [4], the class of order types of maximal chains in the poset $\langle \mathcal{R}(R), \subset\rangle$ is exactly the class of order types of linear orders of the form $K \setminus \{\min K\}$, where K is a compact subset of the real line, $\mathbb{R}$, having the minimum non-isolated. Thus, for example, there is a maximal chain of isomorphic subgraphs of the Rado graph $\langle R, E\rangle$ order isomorphic to the interval $(0, 1]_{\mathbb{R}}$.

Our main goal is to determine how "wide" is the partial order $\langle \mathcal{R}(R), \subset\rangle$, that is to find one of its order invariants - the set of cardinalities of maximal antichains in $\langle \mathcal{R}(R), \subset\rangle$. So we will show that under the CH there are maximal antichains in $\langle \mathcal{R}(R), \subset\rangle$ of all possible cardinalities κ ($1 \leq \kappa \leq 2^{\aleph_0}$). We note that, in contrast to this result, in the poset $\langle [R]^\omega, \subset\rangle$ of all infinite subsets of R, which contains our poset as a sub-order, countable maximal antichains do

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not exist. In this paper we use the terminology of set theory, rather than lattice theory, so our antichains are what some authors call *strong antichains*; we will define all our terminology in the next section to avoid confusion.

Ultimately, the motivation for this paper comes from set theory, since the study of posets and their antichains is closely related to the study of possibilities for a forcing construction.

2. Preliminaries

Our notation is mainly standard. So $\omega = \{0, 1, 2, \ldots\}$ is the set of non-negative integers and, also, the minimal infinite cardinal ($\omega = \aleph_0$). $\mathfrak{c} = 2^{\aleph_0}$ is the cardinality of the **continuum**. For a set X, by $|X|$ we denote its **cardinality** and, for a cardinal κ, $[X]^\kappa = \{A \subset X : |A| = \kappa\}$ and $[X]^{<\kappa} = \{A \subset X : |A| < \kappa\}$.

If $\langle \mathbb{P}, \leq \rangle$ is a partial order, the elements p and q of $\mathbb{P}$ are said to be **incompatible** iff there is no $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$. A subset $\mathcal{A}$ of $\mathbb{P}$ is called an **antichain** iff each two elements of $\mathcal{A}$ are incompatible and $\mathcal{A}$ is a **maximal antichain** iff it is not properly contained in any antichain of $\mathbb{P}$. By Zorn's Lemma each antichain in $\mathbb{P}$ is contained in some maximal antichain. **Chains** in $\mathbb{P}$ are its linearly ordered subsets. A partial order $\langle \mathbb{P}, \leq \rangle$ is called: **chain complete** iff each chain in $\mathbb{P}$ has a least upper bound; **non-atomic** iff below each element of $\mathbb{P}$ there are incompatible elements of $\mathbb{P}$.

We will compare our results with the corresponding known results concerning the partial order $\langle [\omega]^\omega, \subset \rangle$ (isomorphic to $\langle [R]^\omega, \subset \rangle$) where maximal antichains are called **maximal almost disjoint families** or, shortly, **mad families**. So, $\mathcal{A} \subset [\omega]^\omega$ is a maximal antichain in the poset $\langle [\omega]^\omega, \subset \rangle$ (that is, a mad family) iff

- $|A \cap B| < \omega$, for each different $A, B \in \mathcal{A}$;
- For each $S \in [\omega]^\omega$ there is $A \in \mathcal{A}$ such that $|A \cap S| = \omega$.

Let $\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \subset [\omega]^\omega$ is an infinite mad family$\}$. Then we have (see [2]).

Fact 1. *In the partial order $\langle [\omega]^\omega, \subset \rangle$*
(a) There are maximal antichains of size n, for each positive integer n;
(b) There are no maximal antichains of size $\aleph_0$, so $\aleph_0 < \mathfrak{a} \leq \mathfrak{c}$;
(c) There are maximal antichains of size $\mathfrak{c}$.

A pair $\langle R, E \rangle$ is a **graph** if R is a non-empty set and $E \subset [R]^2$ or, equivalently, $E \subset R^2$ is a symmetric and irreflexive relation. In order to simplify notation, instead of $\{r, s\} \in E$ or $\langle r, s \rangle \in E$ we will write $rs \in E$. We will use the following known facts concerning the Rado graph (see [1]).

Fact 2. *Let $\langle R, E \rangle$ be a Rado graph and F a finite subset of R. Then*
(a) $R \setminus F \in \mathcal{R}(R)$;
(b) $R^F_S = \{r \in R \setminus F : \{u \in F : ur \in E\} = S\} \in \mathcal{R}(R)$, for each $S \subset F$, and $R = F \cup \bigcup_{S \subset F} R^F_S$ is a partition of R.
(c) If $R = X_1 \cup \cdots \cup X_k$ is a partition, then $X_i \in \mathcal{R}(R)$, for some $i \leq k$.

Concerning the partial order $\langle \mathcal{R}(R), \subset \rangle$ preliminarily we have

Proposition 1. *Let $\langle R, E \rangle$ be a Rado graph. The partial order $\langle \mathcal{R}(R), \subset \rangle$ is a non-atomic, chain complete suborder of the order $\langle [R]^\omega, \subset \rangle$ (isomorphic to the order $\langle [\omega]^\omega, \subset \rangle$), has the largest element, R, contains all cofinite subsets of R and, hence, has countably many co-atoms: $R \setminus \{v\}$, $v \in R$.*

Proof. If $A \in \mathcal{R}(R)$ and $v \in A$ then, by Fact 2(b), the sets $\{a \in A : av \in E\}$ and $\{a \in A : av \notin E\}$ are incompatible elements of $\mathcal{R}(R)$ below A, so the poset is non-atomic. It is chain complete since the union of a chain of copies of R is a copy of R and, by Fact 2(a), it contains cofinite subsets of R. □

3. Finite and countable maximal antichains in $\langle \mathcal{R}(R), \subset \rangle$

Now, for a Rado graph $\langle R, E \rangle$ we investigate the size of maximal antichains of its copies. Clearly, $\mathcal{A} \subset \mathcal{R}(R)$ is a maximal antichain in the poset $\langle \mathcal{R}(R), \subset \rangle$ iff

- $A \cap B$ does not contain a copy of R, for each different $A, B \in \mathcal{A}$;
- For each $S \in \mathcal{R}(R)$ there is $A \in \mathcal{A}$ such that $A \cap S$ contains a copy of R.

First we show that the analogue of Fact 1(a) holds in the poset $\langle \mathcal{R}(R), \subset \rangle$.

Theorem 1. *For each integer $n \geq 2$ there is a partition of the Rado graph, $\langle R, E\rangle$, into n random subgraphs and in $\langle \mathcal{R}(R), \subset\rangle$ it is a maximal antichain of size n.*

Proof. First, using induction we show that for each $n \geq 2$ the graph R can be partitioned into n elements of $\mathcal{R}(R)$. Let $w \in R$. Then, by Fact 2(b), $R^{\{w\}}_{\{w\}}$ and $R^{\{w\}}_{\emptyset}$ are random subgraphs of R and, clearly, $R = \{w\} \cup R^{\{w\}}_{\{w\}} \cup R^{\{w\}}_{\emptyset}$ is a partition of R. According to Fact 2(a), the graph R is isomorphic to its subgraph $R_1 = R^{\{w\}}_{\{w\}} \cup R^{\{w\}}_{\emptyset}$ and, consequently, R can be partitioned into two random subgraphs.

If R is partitioned into n elements of $\mathcal{R}(R)$, $R = R_1 \cup R_2 \cup \cdots \cup R_n$, then partitioning R_n into two random subgraphs as above we obtain a partition of R into $n+1$ elements of $\mathcal{R}(R)$.

Now, let $R = R_1 \cup R_2 \cup \cdots \cup R_n$ be a partition of R, where $R_i \in \mathcal{R}(R)$, for all $i \leq n$. Clearly $\{R_1, R_2, \ldots, R_n\}$ is an antichain in the ordering $\langle \mathcal{R}(R), \subset\rangle$ and we prove its maximality. Let $S \in \mathcal{R}(R)$. Then $S = \bigcup_{i \leq n} S \cap R_i$ is a partition of S into finitely many pieces so, by Fact 2(c), at least one of them, say $S \cap R_{i_0}$, belongs to $\mathcal{R}(R)$. Hence S and R_{i_0} are compatible elements of $\mathcal{R}(R)$. Thus each element of $\mathcal{R}(R)$ is compatible with some R_i, which proves the maximality of $\{R_1, R_2, \ldots, R_n\}$. □

Now we show that, in contrast to Fact 1(b), the poset $\langle \mathcal{R}(R), \subset\rangle$ contains maximal antichains of size $\aleph_0$. For this we need the following lemma. In the sequel, if $F \in [R]^{<\omega}$, then instead of R^F_F we will write R^F.

Lemma 1. *If $\langle R, E\rangle$ is the Rado graph and $S, T \in [R]^{<\omega}$, where $T \not\subset R^S \cup S$, then $R^S \setminus R^{S \cup T}$ is a random graph.*

Proof. Let $w \in T \setminus (R^S \cup S)$. Then there is $r \in R \setminus (S \cup \{w\})$ such that $rw \notin E$ and $rs \in E$, for all $s \in S$. So $r \in R^S$ and $r \notin R^{S\cup T}$, which implies $R^S \setminus R^{S\cup T} \neq \emptyset$. Let $H, K \in [R^S \setminus R^{S\cup T}]^{<\omega}$ be disjoint sets. Then $H' = H \cup S$ and $K' = K \cup \{w\}$ are disjoint finite sets, so there is $v \in R \setminus (H' \cup K')$ such that (i) $\forall r \in H \cup S (vr \in E)$ and (ii) $\forall r \in K \cup \{w\}(vr \notin E)$. By (i) we have $v \in R^S$ and, by (ii), $vw \notin E$, so $v \notin R^{S\cup T}$, hence $v \in R^S \setminus R^{S\cup T}$. Also $vr \in E$ for all $r \in H$ and $vr \notin E$ for all $r \in K$. So, $R^S \setminus R^{S\cup T}$ is a random subgraph of R. □

Theorem 2. *If $\langle R, E\rangle$ is the Rado graph, then there exists an 1-1 enumeration $R = \{a^k_n : k, n < \omega\}$ such that*

(a) Each column $A_n = \{a^k_n : k < \omega\}$ of the matrix $[a^k_n : \langle k, n\rangle \in \omega \times \omega]$ is a random graph. Also, for each $n \in \omega$, $B_n = \bigcup_{m \geq n} A_m$ is a random graph.

(b) $\mathcal{A} = \{A_n : n \in \omega\}$ is a maximal antichain in the partial order $\langle \mathcal{R}(R), \subset\rangle$.

Proof. (a) Let us fix an element w of R and for each infinite subset B of R let us fix a bijection $a_B : \omega \to B$. Let the sets $B_n \subset R$, $n \in \omega$, be defined recursively by

$B_0 = R \setminus \{w\}$,

$B_1 = R^{\{w\}}$ and, for $n \geq 2$,

$$B_n = \begin{cases} R^{\{w\} \cup \{a_{B_i \setminus B_{i+1}}(k)\, :\, i+k \leq n-2\}} & \text{if } \forall i \leq n-2\ |B_i \setminus B_{i+1}| = \omega, \\ \emptyset & \text{otherwise.} \end{cases}$$

Claim 1. For each $n \in \omega$ we have $\varphi(n)$, where $\varphi(n)$ is the conjunction of the following conditions:

$\varphi_1(n) \equiv B_n \supset B_{n+1}$;

$\varphi_2(n) \equiv B_n \setminus B_{n+1} \in \mathcal{R}(R)$.

Proof of Claim 1. We prove the claim by induction. Clearly $R \setminus \{w\} \supset R^{\{w\}}$, that is $B_0 \supset B_1$, thus $\varphi_1(0)$ holds. According to Fact 2(b) we have $B_0 \setminus B_1 = (R \setminus \{w\}) \setminus R^{\{w\}} = R^{\{w\}}_{\emptyset} \in \mathcal{R}(R)$ and $\varphi_2(0)$ is proved.

Let $m > 0$ and suppose $\varphi(i)$, for each $i < m$. Then for each $i < m$ we have $B_i \setminus B_{i+1} \in \mathcal{R}(R)$, which implies $|B_i \setminus B_{i+1}| = \omega$ so, according to the definition, $B_m = R^{\{w\} \cup \{a_{B_i \setminus B_{i+1}}(k)\, :\, i+k \leq m-2\}}$ and $B_{m+1} = R^{\{w\} \cup \{a_{B_i \setminus B_{i+1}}(k)\, :\, i+k \leq m-1\}}$, which implies $B_m \supset B_{m+1}$ and $\varphi_1(m)$ is proved.

According to Lemma 1 and since $B_m \setminus B_{m+1} = R^S \setminus R^{S\cup T}$, where

$$S = \{w\} \cup \{a_{B_i \setminus B_{i+1}}(k) : i + k \leq m-2\} \text{ and}$$

$$T = \{a_{B_i \setminus B_{i+1}}(k) : i + k = m-1\},$$

for a proof of $\varphi_2(m)$ it is sufficient to show that $T \not\subset R^S \cup S$. Clearly we have $a_{B_0\setminus B_1}(m-1) \in T$ and

$$a_{B_0\setminus B_1}(m-1) \in B_0 \setminus B_1, \tag{1}$$

which implies $a_{B_0\setminus B_1}(m-1) \notin \{w\}$. Suppose that $a_{B_0\setminus B_1}(m-1) = a_{B_i\setminus B_{i+1}}(k)$ for some i and k satisfying $i+k \leq m-2$. Then $i=0$, since $i>0$ would imply $a_{B_i\setminus B_{i+1}}(k) \in B_i \subset B_1$, which is impossible by (1). Now, since $a_{B_0\setminus B_1}$ is a bijection, $a_{B_0\setminus B_1}(m-1) = a_{B_0\setminus B_1}(k)$ implies $k = m-1$, but $k \leq m-2$, a contradiction. Thus $a_{B_0\setminus B_1}(m-1) \notin S$.

According to the induction hypothesis we have $B_m \subset B_1$, which, together with (1) implies $a_{B_0\setminus B_1}(m-1) \notin B_m = R^S$. So $a_{B_0\setminus B_1}(m-1) \in T \setminus (R^S \cup S)$ and $\varphi_2(m)$ is true. Claim 1 is proved.

For convenience, let the element $a_{B_i\setminus B_{i+1}}(k)$ be denoted by a_i^k. Then, according to Claim 1, for each $n \in \omega$ we have

$$B_n = R^{\{w\}\cup\{a_i^k : i+k\leq n-2\}} \in \mathcal{R}(R), \tag{2}$$

$$A_n =_{def} B_n \setminus B_{n+1} = \{a_n^k : k < \omega\} \in \mathcal{R}(R). \tag{3}$$

Claim 2. $\bigcap_{n\in\omega} B_n = \emptyset$.

Proof of Claim 2. Suppose that there exists $u \in \bigcap_{n\in\omega} B_n$. Let v be an element of R satisfying

$$uv \notin E \text{ and } a_0^0 v \notin E. \tag{4}$$

Since $u \in B_1 = R^{\{w\}}$ we have $uw \in E$ and, by (4), $v \neq w$, which implies $v \in B_0$. Since $v \in B_2$ would imply $a_0^0 v \in E$, which contradicts (4), we have $v \notin B_2$. So, $n_0 = \min\{n \in \omega : v \notin B_n\} \in \{1,2\}$ and $v \in B_{n_0-1} \setminus B_{n_0}$ which implies that $v = a_{n_0-1}^{k_0}$, for some $k_0 \in \omega$. But, since $u \in B_{n_0+1+k_0} = R^{\{w\}\cup\{a_i^k : i+k\leq n_0-1+k_0\}}$ we have $uv = ua_{n_0-1}^{k_0} \in E$. A contradiction to (4). Claim 2 is proved.

By Claim 2, $R = \{w\} \cup \bigcup_{n\in\omega} A_n$ is a partition of R. By the uniqueness of the Rado graph and Fact 2(a), the graphs R and $R \setminus \{w\}$ are isomorphic so we can identify R and $\bigcup_{n\in\omega} A_n$ and (a) is proved.

(b) Since the sets A_n, $n \in \omega$, are disjoint elements of $\mathcal{R}(R)$, $\mathcal{A} = \{A_n : n \in \omega\}$ is an antichain in the ordering $\langle \mathcal{R}(R), \subset\rangle$. Suppose that $\mathcal{A}$ is not a maximal antichain. Then some $S \in \mathcal{R}(R)$ is incompatible with each A_n, that is

$$\forall n \in \omega \;\; \neg\exists C \in \mathcal{R}(R) \;\; C \subset S \cap A_n. \tag{5}$$

Let $i_0 = \min\{i \in \omega : S \cap A_i \neq \emptyset\}$ and let $k_0 = \min\{k \in \omega : a_{i_0}^k \in S\}$. Then $a_{i_0}^{k_0} \in S \cap A_{i_0}$ and we prove that

$$C = (\bigcup_{i=i_0}^{i_0+k_0+1} S \cap A_i) \setminus \{a_{i_0}^{k_0}\} \notin \mathcal{R}(R). \tag{6}$$

Suppose $C \in \mathcal{R}(R)$. Then, by Fact 2(c), $S \cap A_i \setminus \{a_{i_0}^{k_0}\} \in \mathcal{R}(R)$, for some $i \in [i_0, i_0+k_0+1]$, which contradicts (5).

By (6) there are disjoint finite subsets $H, K \subset C$ such that $R_H^{H\cup K} \cap C = \emptyset$ and, moreover, $R_H^{H\cup K\cup\{a_{i_0}^{k_0}\}} \cap C = \emptyset$. Since $S \in \mathcal{R}(R)$ there exists $v \in R_H^{H\cup K\cup\{a_{i_0}^{k_0}\}} \cap S$. Since $v \notin C$ and $v \neq a_{i_0}^{k_0}$ we have $v \in B_{i_0+k_0+2} \cap S \subset B_{i_0+k_0+2} = R^{\{w\}\cup\{a_i^k : i+k\leq i_0+k_0\}}$ which implies $va_{i_0}^{k_0} \in E$. But $v \in R_H^{H\cup K\cup\{a_{i_0}^{k_0}\}}$ implies $va_{i_0}^{k_0} \notin E$. A contradiction. $\square$

4. Uncountable maximal antichains in $\langle \mathcal{R}(R), \subset\rangle$

In this section we show that the poset $\langle \mathcal{R}(R), \subset\rangle$ contains maximal antichains of size $\mathfrak{c}$ (so the analogue of Fact 1(c) is true).

For reader's convenience we list some definitions and facts from set theory. The sets V_n, $n \in \omega$, are defined recursively by: $V_0 = \emptyset$ and $V_{n+1} = P(V_n)$. The union $V_\omega = \bigcup_{n\in\omega} V_n$ is the collection of **hereditarily finite sets** (the **combinatorial universe**) and, for $n \in \omega$, the set $\mathrm{Lev}_n = V_{n+1} \setminus V_n$ is the **n-th level** of V_ω. The **rank** of a set $x \in V_\omega$ is defined by $\mathrm{rank}(x) = \min\{n \in \omega : x \in V_{n+1}\}$. So $V_n = \{x \in V_\omega : \mathrm{rank}(x) < n\}$ and it is easy to check that $\mathrm{Lev}_n = \{x \in V_\omega : \mathrm{rank}(x) = n\}$ and $\mathrm{rank}(x) = \sup\{\mathrm{rank}(y)+1 : y \in x\}$. The **transitive closure** of a set x is the set $\mathrm{trcl}(x) = \bigcup_{n\in\omega} \cup^n x$, where $\cup^0 x = x$, and $\cup^{n+1} x = \bigcup \cup^n x$.

Fact 3. *(a) V_ω is a countable transitive set (i.e. $x \in V_\omega$ implies $x \subset V_\omega$).*

(b) The structure $\langle V_\omega, \in\rangle$ satisfies all the axioms of set theory ZFC except the Axiom of Infinity (Inf). In particular, if $x, y \in V_\omega$, then $\{x\}, x \cup y \in V_\omega$ etc.

(c) $V_\omega \cap \mathrm{Ord} = \omega$.

(d) $x \in V_\omega$ iff x is a finite subset of V_ω.

(e) $V_\omega = \{x : |\mathrm{trcl}(x)| < \omega\}$.

In the sequel by ε we will denote the binary relation on the class of all sets defined by: $x\varepsilon y$ if and only if $x \in y$ or $y \in x$. Also, instead of $\langle x, y\rangle \in \varepsilon$ we will write $xy \in \varepsilon$ and, if H is a set, instead of $\langle H, \varepsilon \cap H^2\rangle$ we will write $\langle H, \varepsilon\rangle$, whenever confusion is impossible.

Fact 4. *The structure $\langle V_\omega, \varepsilon\rangle$ is a Rado graph.*

Proof. Let H and K be disjoint finite subsets of V_ω. Then $H, K \in V_\omega$ and $n = \mathrm{rank}(K) < \omega$. Since $V_\omega \models$ ZFC - Inf and $H, n \in V_\omega$, we have $v = H \cup \{n\} \in V_\omega$. Now, for each $h \in H$ we have $h \in v$, thus $hv \in \varepsilon$. On the other hand, for $k \in K$ there holds $k \notin H$ (since $H \cap K = \emptyset$) and $k \neq n$ (since $\mathrm{rank}(k) < n = \mathrm{rank}(n)$) so $k \notin v$. Since $\mathrm{rank}(v) \geq n + 1$, we have $v \notin k$, thus $kv \notin \varepsilon$, for all $k \in K$. □

Lemma 2. *If A is an infinite subset of ω, then $S_A = \bigcup_{n\in A} \mathrm{Lev}_n$ is a random subgraph of the graph $\langle V_\omega, \varepsilon\rangle$.*

Proof. Let $H, K \in [S_A]^{<\omega}$ be disjoint sets. The set $\{\mathrm{rank}(x) : x \in H \cup K\}$ is a finite subset of ω, hence $m = \max\{\mathrm{rank}(x) : x \in H \cup K\} + 2 < \omega$. Clearly $n = \min(A \setminus (m + 1)) \in A$ and $n > m$. Let $v = H \cup (n \setminus m)$.

We prove $v \in S_A$. Since $H, n \setminus m \in [V_\omega]^{<\omega}$, we have $H, n \setminus m \in V_\omega$ and $H \cup (n \setminus m) \in V_\omega$ (because $V_\omega \models$ ZFC - Inf) so $v \in V_\omega$. Moreover $\mathrm{rank}(v) = \sup\{\mathrm{rank}(x) + 1 : x \in H \cup (n \setminus m)\} = n \in A$, thus $v \in \mathrm{Lev}_n \subset S_A$.

For each $h \in H$ we have $h \in v$, so $vh \in \varepsilon$ for all $h \in H$.

Let $k \in K$. Then $k \notin H$ (since $H \cap K = \emptyset$) and $k \notin n \setminus m$ (because $\mathrm{rank}(k) < m$ and for $l \in n \setminus m$ we have $\mathrm{rank}(l) = l \geq m$) thus $k \notin v$. On the other hand, $\mathrm{rank}(v) = n > m > \mathrm{rank}(k)$ implies $v \notin k$ so $vk \notin \varepsilon$, for all $k \in K$. □

Theorem 3. *Let $\mathcal{A}$ be an almost disjoint family in ω. Then*

(a) $\mathcal{A}_{V_\omega} = \{S_A : A \in \mathcal{A}\}$ is an almost disjoint family on V_ω consisting of random subgraphs of $\langle V_\omega, \varepsilon\rangle$.

(b) In $\langle \mathcal{R}(R), \subset\rangle$ there exists a maximal antichain of size $\mathfrak{c}$.

Proof. (a) By Lemma 2, for each $A \in \mathcal{A}$ the set $S_A = \bigcup_{n\in A} \mathrm{Lev}_n \in \mathcal{R}(V_\omega)$. If $A, B \in \mathcal{A}$ and $A \neq B$, then $|A \cap B| < \aleph_0$ so $S_A \cap S_B = \bigcup_{n\in A\cap B} \mathrm{Lev}_n$ is a finite set, since the sets Lev_n are finite.

(b) By Fact 1(c), there are mad families on ω of size $\mathfrak{c}$. If $\mathcal{A}$ is one, then, by (a), $\mathcal{A}_{V_\omega}$ is an antichain in $\langle \mathcal{R}(\mathcal{A}_{V_\omega}), \subset\rangle$ of size $\mathfrak{c}$ and, by Zorn's Lemma, it is contained in a maximal antichain of the same size, because $|[V_\omega]^\omega| = \mathfrak{c}$. □

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