Avoider-Enforcer star games

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Abstract

In this paper, we study $(1 : b)$ Avoider-Enforcer games played on the edge set of the complete graph on n vertices. For every constant $k \geq 3$ we analyse the k-star game, where Avoider tries to avoid claiming k edges incident to the same vertex. We analyse both versions of Avoider-Enforcer games – the strict and the monotone – and for each provide explicit winning strategies for both players. Consequentially, we establish bounds on the threshold biases $f^{mon}_{\mathcal{F}}$, $f^{-}_{\mathcal{F}}$ and $f^{+}_{\mathcal{F}}$, where $\mathcal{F}$ is the hypergraph of the game. We also study two more related monotone games.

1 Introduction

Let a and b be two positive integers, let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets of X. In an $(a : b)$ Avoider-Enforcer game $\mathcal{F}$, two players, called Avoider and Enforcer, alternately claim a and b previously unclaimed elements of X per move, respectively. If the number of unclaimed elements is strictly less than a (respectively b)

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before Avoider's (respectively Enforcer's) move, then he claims all these elements. The game ends when all the elements of X have been claimed by either of the players. Avoider loses the game if by the end of the game he has claimed all the elements of some $F \in \mathcal{F}$, and wins otherwise. Throughout this paper we assume that Avoider is the first player to play, although usually it makes very little difference. We refer to X as the *board* of the game, to $\mathcal{F}$ as the *target sets*, and to a and b as the *bias* of Avoider and Enforcer, respectively. Since the pair $(X, \mathcal{F})$ is a hypergraph that represents the game, we often refer to $\mathcal{F}$ as the hypergraph of the game, or as the game itself.

Avoider-Enforcer games are the misère version of the well studied Maker-Breaker games. In an $(a : b)$ Maker-Breaker game $\mathcal{F}$, the two players are now called Maker and Breaker, they claim respectively a and b elements of X per move, and Maker *wins* if and only if by the end of the game he has claimed all the elements of some $F \in \mathcal{F}$. Both Maker-Breaker and Avoider-Enforcer games are finite, perfect information games, and there is no possibility of a draw. Hence, for every given setup – $a, b, \mathcal{F}$ – one of the players has a winning strategy. We say that this player wins the game.

Maker-Breaker games are *bias monotone*: if Maker wins some game $\mathcal{F}$ with bias $(a : b)$, he also wins this game with bias $(a' : b')$, for every $a' \geq a$, $b' \leq b$. Similarly, if Breaker wins $\mathcal{F}$ with bias $(a : b)$, he also wins this game with bias $(a' : b')$, for every $a' \leq a$, $b' \geq b$. This bias monotonicity enables the definition of the *threshold bias*: for a given hypergraph $\mathcal{F}$, the threshold bias $f_{\mathcal{F}}$ is the unique integer for which Maker wins the $(1 : b)$ game $\mathcal{F}$ for every $b < f_{\mathcal{F}}$, and Breaker wins the $(1 : b)$ game $\mathcal{F}$ for every $b \geq f_{\mathcal{F}}$.

It is very natural to play both Avoider-Enforcer and Maker-Breaker games on the edge set of a given graph G. In this case, the board is $X = E(G)$, and the target sets are all the edge sets of subgraphs $H \subseteq G$ which possess some given monotone increasing graph property $\mathcal{P}$. For example: in the *connectivity* game $\mathcal{C}(G)$, the target sets are all edge sets of spanning trees of G; in the *perfect matching* game $\mathcal{M}(G)$ the target sets are all sets of $\lfloor |V(G)|/2 \rfloor$ independent edges of G; in the *Hamiltonicity* game $\mathcal{H}(G)$ the target sets are all edge sets of Hamilton cycles of G. These three games were initially studied by Chvátal and Erdős in their seminal paper [3], for $G = K_n$, the complete graph on n vertices. They proved that Breaker can win all these $(1 : b)$ games by isolating a vertex in Maker's graph, provided that $b \geq \frac{(1+\varepsilon)n}{\ln n}$ for any $\varepsilon > 0$, and showed that the threshold bias for the connectivity game is $\Theta(\frac{n}{\ln n})$.

It turns out that in fact $(1 + o(1))\frac{n}{\ln n}$ is the threshold bias for all three games, as later proved by Gebauer and Szabó [4] (the connectivity game) and by Krivelevich [7] (the perfect matching and the Hamiltonicity games). These are examples of the so called "random graph intuition": in many games played on the edges of K_n the threshold bias is of the same order of magnitude as the threshold bias for a *random* game on K_n, where in every round Maker claims one free edge at random, and then Breaker claims b free edges at random. At the end of such random game Maker's graph G_M roughly satisfies $G_M \sim G(n, \frac{1}{b})$ (the graph on n vertices where each potential edge appears in the graph independently with probability

$\frac{1}{b}$), so the threshold bias for the random game where Maker wishes to acquire some graph property $\mathcal{P}$ approximately equals the reciprocal of the threshold function for the appearance of $\mathcal{P}$ in $G(n,p)$.

Unfortunately, Avoider-Enforcer games do not have bias monotonicity in general (see e.g. [5], [6]): although intuitively each player wishes to claim as less elements as possible, it is sometimes a disadvantage to claim less elements per move, for any of the players. This fact makes Avoider-Enforcer games much harder to analyse. In particular, it prevents the possibility of defining an analogous version of the threshold bias. What can be defined is the following, which was introduced by Hefetz, Krivelevich and Szabó in [6]: let $\mathcal{F}$ be a hypergraph; the *lower threshold bias* $f^-_{\mathcal{F}}$ is the largest integer such that Enforcer wins the $(1:b)$ game $\mathcal{F}$ for every $b \le f^-_{\mathcal{F}}$; the *upper threshold bias* $f^+_{\mathcal{F}}$ is the smallest integer such that Avoider wins the $(1:b)$ game $\mathcal{F}$ for every $b > f^+_{\mathcal{F}}$. Except for some trivial cases, $f^-_{\mathcal{F}}$ and $f^+_{\mathcal{F}}$ always exist and satisfy $f^-_{\mathcal{F}} \le f^+_{\mathcal{F}}$. When $f^-_{\mathcal{F}} = f^+_{\mathcal{F}}$ we call this number $f_{\mathcal{F}}$ and refer to it as the *threshold bias* of the game $\mathcal{F}$.

In order to overcome this obstacle, Hefetz, Krivelevich, Stojaković and Szabó proposed in [5] a bias monotone version for Avoider-Enforcer games: they suggested that Avoider and Enforcer will claim **at least** a and b board elements per move, respectively. It is trivial to see that this new version is indeed bias monotone, i.e. each player can only benefit from lowering his bias. This fact allowed them to define for any given hypergraph $\mathcal{F}$ the *monotone threshold bias* $f^{mon}_{\mathcal{F}}$ as the largest non-negative integer for which Enforcer wins the $(1:b)$ game $\mathcal{F}$ under the new set of rules if and only if $b \le f^{mon}_{\mathcal{F}}$. Throughout this paper we refer to this new set of rules as the *monotone* rules, to distinguish it from the *strict* rules. Accordingly, we refer to the games played under each set of rules as monotone games and as strict games.

Interestingly, these seemingly minor adjustments in the rules can completely change the game. For example, even in such a natural game as the connectivity game, the two versions of the game are essentially different. In the strict game, Avoider wins the $(1:b)$ connectivity game played on $E(K_n)$ if and only if at the end of the game he has at most $n-2$ edges, therefore the threshold bias exists and is of linear order [6]. On the other hand, the asymptotic monotone threshold bias for this game is $\Theta(\frac{n}{\ln n})$ [5, 8].

In [6], Hefetz, Krivelevich and Szabó provided a general sufficient condition for Avoider's win in $(a:b)$ Avoider-Enforcer games played under both sets of rules. This criterion takes only Avoider's bias into account. In [1], Bednarska-Bzdęga introduced a new sufficient condition for Avoider's win under both sets of rules, which depends on both parameters a and b.

In [5], Hefetz et al. investigated $(1:b)$ Avoider-Enforcer games played on the edge set of K_n, where Avoider wants to avoid claiming a copy of some fixed graph H. In this case $X = E(K_n)$, and $\mathcal{F} = \mathcal{K}_H \subseteq 2^{E(K_n)}$ consists of all copies of H in K_n. These games are referred to as H–games. They conjectured that for any fixed graph H, the thresholds $f^-_{\mathcal{K}_H}$ and $f^+_{\mathcal{K}_H}$ are not of the same order, and wondered about the connection between monotone H–games and strict H^-–games, where H^- is H with one edge missing. They investigated

H–games where $H = K_3$ (a triangle) and $H = P_3 = K_3^-$ (a path on three vertices) and established the following:

$$f^{mon}_{\mathcal{K}_{P_3}} = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor - 1, \quad f^{+}_{\mathcal{K}_{P_3}} = \binom{n}{2} - 2, \quad f^{-}_{\mathcal{K}_{P_3}} = \Theta(n^{\frac{3}{2}}) \text{ and } f^{mon}_{\mathcal{K}_{K_3}} = \Theta(n^{\frac{3}{2}}).$$

This example supports their conjecture, as $f^{+}_{\mathcal{K}_{P_3}}$ and $f^{-}_{\mathcal{K}_{P_3}}$ are indeed not of the same order, while $f^{mon}_{\mathcal{K}_{K_3}}$ and $f^{-}_{\mathcal{K}_{P_3}}$ are of the same order. They also wondered about the results for H–games where $|V(H)| > 3$. Bednarska-Bzdęga established in [1] general upper and lower bounds on $f^{+}_{\mathcal{K}_H}$, $f^{-}_{\mathcal{K}_H}$ and $f^{mon}_{\mathcal{K}_H}$ for every fixed graph H, but these bounds are not tight.

Our main objective in this paper is to study monotone and strict H–games played on the edges of the complete graph K_n, where H is a k-star $\mathcal{S}_k = K_{1,k}$, for some fixed $k \geq 3$ (note that $\mathcal{S}_2 = P_3$, so the case $k = 2$ is already covered). This goal is natural, since avoiding a k-star is exactly keeping the maximal degree in Avoider's graph strictly under k. We refer to this game as the *star game*, or more specifically, for a given k, we call this game the *k-star game*. We show that for any given $k \geq 3$, $f^{-}_{\mathcal{K}_{\mathcal{S}_k}}$ and $f^{+}_{\mathcal{K}_{\mathcal{S}_k}}$ are not of the same order for any sufficiently large n.In addition, as $\mathcal{S}_k^- = \mathcal{S}_{k-1}$, an immediate consequence is that $f^{mon}_{\mathcal{K}_{\mathcal{S}_k}}$ and $f^{-}_{\mathcal{K}_{\mathcal{S}_k^-}}$ are of the same order for infinitely many values of n. We also provide explicit winning strategies for both players under both sets of rules.

In order to state our main result, we need to introduce some functions: let $r = r(n, b)$ be the integer for which $1 \leq r \leq b + 1$ and $\binom{n}{2} \equiv r \bmod (b + 1)$ hold. Note that r is the number of edges which remain for Avoider to choose from in his last move when playing the strict $(1 : b)$ game. Let

$$e^{+}_{n,k} = \max\left\{ b \leq 0.4n^{\frac{k}{k-1}} : r < \tfrac{n^{k+1}}{(2b)^{k-1}} \right\}, \text{ and}$$

$$e^{-}_{n,k} = \max\left\{ b \leq 0.4n^{\frac{k}{k-1}} : r < \tfrac{n^{k+1}}{(2b')^{k-1}} \text{ for every } 1 \leq b' \leq b \right\}.$$

The main result in our paper is the following theorem:

Theorem 1.1. *Let $k \geq 3$. In the $(1 : b)$ k-star game $\mathcal{K}_{\mathcal{S}_k}$ we have*

(i) $f^{mon}_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k}{k-1}})$;

(ii) $e^{+}_{n,k} \leq f^{+}_{\mathcal{K}_{\mathcal{S}_k}} = O(n^{\frac{k}{k-1}})$ *holds for all values of n, and* $f^{+}_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k}{k-1}})$ *holds for infinitely many values of n;*

(iii) $\Omega(n^{\frac{k+1}{k}}) = e^{-}_{n,k} \leq f^{-}_{\mathcal{K}_{\mathcal{S}_k}} = O(n^{\frac{k+1}{k}} \log n)$ *holds for all values of n, and* $f^{-}_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k+1}{k}})$ *holds for infinitely many values of n.*

Remark. We would like to show the correlation between these results and the random graph intuition. For different values of b let us compare the outcome of the Avoider-Enforcer $(1 : b)$ k-star game to the corresponding random graph $G \sim G(n, \frac{1}{b})$. All the following statements about G hold w.h.p. (i.e. with probability tending to 1 as n tends to infinity). For details the reader may refer to [2], Theorem 3.1.

- For $b = \omega(n^{\frac{k}{k-1}})$ the maximal degree in G is at most $k-2$, and Avoider wins.
- At $b = \Theta(n^{\frac{k}{k-1}})$ vertices of degree $k-1$ emerge in G. If Avoider claims the last edge in the game, the appearance of a vertex of degree $k-1$ in his graph before the last round means he loses, and this is indeed the order of magnitude of $f^{mon}_{\mathcal{S}_k}$ and $f^{+}_{\mathcal{S}_k}$, where presumably Avoider claims the last edge.
- In the range $\omega(n^{\frac{k+1}{k}}) \leq b \leq o(n^{\frac{k}{k-1}})$ the maximal degree in G is exactly $k-1$, and so the outcome of the strict game heavily depends on the number of free edges Avoider will be able to choose from in his last move, and so the winner oscillates.
- Finally, for $b \leq Cn^{\frac{k+1}{k}}$, where C is a sufficiently small constant, vertices of degree k emerge in G, and Enforcer wins.

The rest of the paper is organized as follows: in Section 2 we give some preliminaries and notation. In Section 3 we prove Theorem 1.1. In Section 4 we study two related monotone star games. Finally, in Section 5 we give concluding remarks and present some open problems.

2 Preliminaries

Throughout this paper we use the following notation:

A previously unclaimed edge is called a *free* edge, or an *available* edge. The act of claiming one free edge by one of the players is called a *step*. In the strict game, Enforcer's b (Avoider's 1) successive steps are called a *move*. In the monotone game, each move of Enforcer, respectively Avoider, consists of at least b, respectively 1, steps. A *round* in the game consists of one move of the first player (Avoider), followed by one move of the second player (Enforcer). Whenever one of the players claims an edge incident to some vertex u, we say that the player *touched* u.

Our graph-theoretic notation is standard and follows that of [9]. In particular, throughout the paper G stands for a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. By $N_G(v)$ we denote the set of neighbours of a vertex v in a graph G, i.e. $\{u \in V(G) : (u, v) \in E(G)\}$. For any subset $U \subseteq V$ we say that an edge $(u, v) \in E$ lies *inside* U if $u, v \in U$. For any subset $U \subseteq V$ we denote by $G[U]$ the *induced* graph on U, i.e. the graph with vertex set U and edge set $\{(u, v) \in E : u, v \in U\}$. By A_i and F_i we denote the graphs with vertex set V, whose edges were claimed by Avoider, respectively Enforcer, in the first i rounds. For any vertex $v \in V$, let $d_{A_i}(v)$ and $d_{F_i}(v)$ denote the degree of v in A_i, respectively F_i. We sometimes omit the sub index i when its value is clear, unknown, or unimportant. In these cases we also refer to $d_{A_i}(v)$ as the A-degree of v. The union of the two graphs A_i and F_i is called the *global graph* and is denoted by G_i.

For the sake of simplicity and clarity of presentation, no real effort has been made here to optimize the constants appearing in our results. We also omit floor and ceiling signs whenever these are not crucial. Our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large.

As the value of $r = r(n,b)$, the number of free edges before the last round of the strict game, may be very significant in determining the identity of the winner in this game, we will need the following two number theoretic statements, due to Bednarska-Bzdęga ([1]):

Fact 2.1. *Let $r < 2$ be a rational number and $c > 0$ be an integer. Then:*

(i) There are infinitely many natural numbers n such that $q | \binom{n}{2}$ for some q with $cn^r < q < 2cn^r$;

(ii) There are infinitely many natural numbers n such that $q | \left(\binom{n}{2} - 1\right)$ for some q with $cn^r < q < 4cn^r$.

Fact 2.2. *For every $\delta \in (0,1)$ there exists an integer N_δ such that if $N \geq N_\delta$ and $N^\delta < q < \frac{\delta N}{2 \log N}$ hold, then there exists an integer k such that $q \leq k \leq 2q \log q/\delta$ and the remainder of the division of N by k is at least q.*

3 The k - star game

In this section we prove Theorem 1.1, by providing winning strategies for both players.

3.1 Enforcer's strategies – lower bounds

In this subsection we give lower bounds on the threshold biases $f^{mon}_{\mathcal{KS}_k}, f^{+}_{\mathcal{KS}_k}, f^{-}_{\mathcal{KS}_k}$. First, for $b = 0.4n^{\frac{k}{k-1}}$ we describe a strategy for Enforcer in the $(1:b)$ monotone game, then prove it is a winning strategy, thus establishing the lower bound on $f^{mon}_{\mathcal{KS}_k}$.

Enforcer's strategy – the monotone game:

Enforcer's strategy is basically to wait for a vertex with A-degree $k-1$ to appear, then to claim all free edges but one incident to that vertex. In every move before such a vertex appears, Enforcer maintains a dynamic partition $V = I \cup C$ – where I_i and C_i represent the respective sets at the end of the ith round of the game – in such a way that the following two properties hold: $d_{F_i}(v) = 0$ for every $v \in I_i$ (i.e. all Enforcer's edges are inside C_i), and $G_i[C_i]$, the global graph induced on C_i, is a clique (i.e. there are no free edges inside C_i). Explicitly, Enforcer's strategy is as follows:

At the beginning of the game, Enforcer sets $I_0 = V$ and $C_0 = \emptyset$. At any point during the game, if he cannot follow the proposed strategy he forfeits the game. If at any point during the game Avoider creates a vertex of A-degree at least k, he loses. Therefore, for simplicity, we assume that in this case Avoider immediately forfeits.

Let $i > 0$, and suppose that $i-1$ rounds of the game have been played and that Enforcer was able to ensure the existence of I_{i-1} and C_{i-1} as described above. In his ith move, Enforcer checks if there is a vertex $v \in V$ of A-degree $k-1$, and if there is, he claims all free edges in the graph but one, incident to v. Avoider in his next move will be forced to claim that last edge and increase $d_A(v)$ to k, and lose. If there is no such vertex, Enforcer enumerates the vertices $v_1^i, v_2^i, \dots, v_{|I_{i-1}|}^i$ in a non-decreasing A-degree order and determines the smallest integer n_i such that the number of free edges inside $C_{i-1} \cup \{v_1^i, \dots, v_{n_i}^i\}$ is at least b. Then he claims all these edges and sets $C_i := C_{i-1} \cup \{v_1^i, \dots, v_{n_i}^i\}$ and $I_i := I_{i-1} \setminus \{v_1^i, \dots, v_{n_i}^i\}$. This new partition clearly possesses the two required properties. Note that by the definition of n_i, as long as there is no vertex of A-degree $k-1$, Enforcer does not claim more than $b + n = (1 + o(1))b$ edges per move. In addition, Avoider can claim less than nk edges throughout the game (otherwise he would have a vertex of degree k), so the number of edges claimed by both players in each round is also at most $(1+o(1))b$.

In order to prove that the proposed strategy is indeed a winning strategy for Enforcer, we have to show that no matter how Avoider plays, if Enforcer plays according to this strategy, a vertex of A-degree $k-1$ appears and that at this point (if Avoider had not already lost) there is still a free edge incident to that vertex and the number of free edges is strictly larger than b. For the analysis, we fix some arbitrary strategy for Avoider, and divide the course of the game into stages: the game begins at stage 1; for every $1 \le j \le k-2$, stage j ends (and stage $j+1$ begins) at the end of the ith round, if $\frac{1}{|I_i|}\sum_{v \in I_i} d_{A_i}(v) < j$ and $\frac{1}{|I_{i+1}|}\sum_{v \in I_{i+1}} d_{A_{i+1}}(v) \ge j$ hold, i.e. the average A-degree in I became at least j during the $(i+1)$st round. This is well defined, as this value can only increase during the game: Avoider's moves increase this value, and Enforcer removes from I vertices of minimal A-degree so he does not decrease it. It is possible that several stages will end at the same time if the average A-degree in I increases by more than one between two consecutive rounds. In this case there will be stages of length zero. Note that if at the end of round i the game is in stage j, then every vertex in C_i was of A-degree strictly less than j when it was added to C. Indeed, Enforcer's strategy implies that otherwise every vertex in I_i is of A-degree at least j and so stage j would have ended.

The following lemma estimates the size of I at the end of stage j, for $1 \le j \le k-2$. It takes into account the only thing we know about Avoider – that he claims at least one edge per move, thus increasing the A-degree of at least one vertex in I by at least one.

Lemma 3.1. *For every $1 \leq j \leq k-2$, stage j ends, and at that moment the inequality $|I| \geq 0.9n^{1-\frac{j}{k-1}}$ holds.*

Proof. First, let us consider the size of I just before the beginning of the last round in the game. It cannot be that $|I| = \Theta(n)$ because that would imply that there are $\Theta(n^2) = \omega(b)$ free edges remaining, in contradiction to the number of edges claimed by both players in each round. Therefore, before the last round $|I| = o(n)$ and $|C| = (1-o(1))n$ and so there are at least $(1-o(1))|I|n$ free edges, and since this number must be at most $(1+o(1))b$ we get $|I| \leq 0.5n^{\frac{1}{k-1}}$. It means that I cannot be too large just before the game ends.

Now, we prove the lemma by induction on j. Suppose for contradiction that the statement of the lemma does not hold for $j = 1$. Since I gets small enough, it means that for some i, after the ith round we have $|I_i| < 0.9n^{1-\frac{1}{k-1}}$ but the inequality $\frac{1}{|I_i|}\sum_{v\in I_i} d_{A_i}(v) < 1$ holds. Together we get $\sum_{v\in I_i} d_{A_i}(v) < 0.9n^{1-\frac{1}{k-1}}$. Due to Enforcer's strategy, in each move $1 \leq l \leq i$, Avoider must claim an edge having at least one of its endpoints – say u – in I_l, increasing u's A-degree by one. As during these rounds Enforcer moves from I_l to C_{l+1} only vertices with A-degree zero (otherwise it would mean that the average size of I is at least 1, in contradiction), $u \notin C_{l+1}$. Therefore the sum $\sum_{v\in I_l} d_{A_l}(v)$ increases by at least one in each round and thus $i < 0.9n^{1-\frac{1}{k-1}}$. Each vertex in C_i is connected by Enforcer's edges to all but at most $k-1$ vertices in C_i, so the number of Enforcer's edges in C_i is more than $\frac{1}{2}(|C_i|-k)^2$. On the other hand, Enforcer claims only $(1+o(1))b$ edges per move, hence $\frac{1}{2}(|C_i|-k)^2 < i(1+o(1))b < 0.4n^2$. Thus, $|C_i| < 0.9n$ and $|I_i| = n - |C_i| > 0.1n$, a contradiction.

Before proceeding to the second part of the proof, we make some observations. Let g be the last round of the first stage. We assume that $I_g = o(n)$, as clearly Enforcer's win under this assumption implies Enforcer's win under the assumption $I_g = \Theta(n)$. Since all of Avoider's edges, and all the free edges, have at least one endpoint in I_g, and since k is a constant, throughout the game Avoider claims only $o(n)$ edges, unless he loses. Therefore, for every $i > g$ and for every vertex $v \in I_i$ there are at least $(1-o(1))n$ free edges between v and C_i. We conclude that while there is no vertex of A-degree $k-1$, Enforcer, in each of his moves after the gth, moves at most $(1+o(1))\frac{b}{n}$ vertices from I to C.

We now proceed with our proof. Suppose for contradiction that for some $1 < j \leq k-2$, the lemma holds for stage $j-1$, but not for stage j, i.e. there exists an integer i such that $|I_i| < 0.9n^{1-\frac{j}{k-1}}$, but $\sum_{v\in I_i} d_{A_i}(v) < j|I_i|$ holds. Once again, such an i must exist since I gets small enough. Denote by m the number of rounds that have been played in stage j up to and including round i, excluding the first round of the stage. For any of those rounds l let $w(l) = \sum_{v\in I_l}(d_{A_l}(v) - (j-1))$. This is a non-negative integer after the first round of stage j. Avoider touches at least one vertex of I in each of his moves, and so he increases this sum in each move. Enforcer, however, only removes from I vertices of degree less than j (otherwise it would imply that the average size of I is at least j, a contradiction), so he does not decrease this sum in his moves. Hence, the sum increases by at least one in each round during stage j, so we get $m \le w(i)$. Note that if $w(l) \ge |I_l|$ holds, then the average A-degree in I_l is at least j, thus stage j would have ended. This yields $w(i) < |I_i| < 0.9n^{1-\frac{j}{k-1}}$. It follows that during stage j Enforcer has removed at most $(m+1)(1+o(1))\frac{b}{n} < 0.4n^{1-\frac{j-1}{k-1}}$ vertices from I, but using the induction hypothesis we get that $|I_i| \ge 0.9n^{1-\frac{j-1}{k-1}} - 0.4n^{1-\frac{j-1}{k-1}} = 0.5n^{1-\frac{j-1}{k-1}}$, a contradiction. □

By Lemma 3.1, stage $k-2$ ends, and at its end there are at least $0.9n^{\frac{1}{k-1}}$ vertices in I, and therefore at least $(1-o(1))n|I| > 0.89n^{\frac{k}{k-1}} > 2.2b$ free edges remain. If after Avoider's first move in stage $k-1$ a vertex v of A-degree $k-1$ appears, Enforcer in his next move may proceed according to his strategy and claim all free edges but one adjacent to v and win. Otherwise, he plays his standard move, leaving at least $1.1b$ free edges after his move, and at this point all vertices in I must be of A-degree exactly $k-2$. In his next move Avoider must create a vertex v with A-degree $k-1$, and then, once again, Enforcer in his next move may proceed according to his strategy and claim all free edges but one adjacent to v and win. In both cases Avoider cannot claim more than $|I| = o(b)$ edges without creating a vertex of degree k, so he can do nothing to stop Enforcer.

The above strategy cannot be used in exactly the same manner in the strict game for two reasons. First of all, as Enforcer must claim exactly b edges per move, he cannot maintain the clique C in the global graph and only claim edges inside it. More importantly, even if Avoider creates a vertex v of A-degree $k-1$, Enforcer cannot make sure (in general) that Avoider eventually will claim another edge incident to v and lose. In the remainder of this subsection, we show how to modify Enforcer's strategy to overcome these difficulties.

Enforcer's strategy – the strict game:

Note that if $b = o(n)$ then the length of the game – and therefore the number of Avoider's edges at the end of the game – is $\Theta(n^2/b) = \omega(n)$, so in this case Enforcer wins no matter how he plays. If $b = \Theta(n)$ Enforcer does the following: before the game starts he chooses an arbitrary set $U \in V$ of size $|U| = b^{\frac{k^2}{k^2+1}} < n$, and in each step he claims some arbitrary free edge with at least one endpoint outside U until he can no longer do so, i.e., until all free edges lie completely inside U. Then he pretends to start a new game on $n' := b^{\frac{k^2}{k^2+1}}$ vertices with bias $b = n'^{\frac{k^2+1}{k^2}}$ according to the strategy for the case $b = \omega(n)$. This is not exactly a new game because there may be some edges inside U already claimed by Avoider, and the "new" game may start during Enforcer's move. However, since Avoider can claim only a constant number of edges incident to each vertex, and since Enforcer makes at most b additional steps before Avoider's first move, these factors have no significant effect. They can only affect the basis of the induction in the proof of Lemma 3.2 to follow, and it is easy to see that the analysis there is still valid. The number of free edges before the last round is also affected, but for this value of b Enforcer wins regardless of that number, so it does not matter.

Until the end of this subsection we assume that $b = \omega(n)$ and $b \leq 0.4n^{\frac{k}{k-1}}$. Recall that r denotes the integer which satisfies $1 \leq r \leq b+1$ and $\binom{n}{2} \equiv r \bmod (b+1)$. This is the number of free edges before the last round of the game, i.e., the number of edges Avoider will be able to choose from before his last move. Enforcer will then claim the remaining $r-1$ edges in his last move. Enforcer strategy for the strict game is very similar to his strategy for the monotone game. If at any point during the game he cannot follow it, he immediately forfeits the game, and we assume that if Avoider has increased the maximum degree in his graph to at least k (and thus lost), he also forfeits. We say that a free edge is a *threat* if it is adjacent to a vertex of A-degree $k-1$. If at any point during the game there exist at least r threats, Enforcer switches his strategy and plays arbitrarily until the end of the game, with the only rule of not claiming a threat unless he has to. Note that if this happens, Avoider loses in his next move, so the appearance of r threats ensures Enforcer's win.

Until r threats appear, Enforcer, like in the monotone game, maintains a partition $V = I \cup C$ (where I_i and C_i represent the respective sets after the ith round) such that $F_i[I_i]$ – Enforcer's graph induced on I_i – is empty and $G_i[C_i]$ – the global graph induced on C_i – is a clique. The only difference here is that Enforcer may sometimes claim edges between C and I. Once again, initially $I_0 = V$ and $C_0 = \emptyset$. In his ith move Enforcer enumerates the vertices $v_1^i, v_2^i, \ldots, v_{|I_{i-1}|}^i$ in a non-decreasing A-degree order, with a possible tie breaking rule that will be presented shortly, and determines the largest integer n_i such that the number of free edges inside $C_{i-1} \cup \{v_1^i, \ldots, v_{n_i}^i\}$ is at most b. Then he claims all these edges and sets $C_i := C_{i-1} \cup \{v_1^i, \ldots, v_{n_i}^i\}$ and $I_i := I_{i-1} \setminus \{v_1^i, \ldots, v_{n_i}^i\}$. This new partition possesses the two required properties. As opposed to the monotone game, Enforcer must claim exactly b edges, so there are l_i more edges Enforcer must claim in order to complete his move. Enforcer chooses these edges in the following way: he picks the next $4k$ vertices of the enumeration of I_{i-1}, $v_{n_i+1}^i, v_{n_i+2}^i, \ldots, v_{n_i+4k}^i$, and for each $1 \leq h \leq 4k$ he claims arbitrarily $\lfloor \frac{l_i+h-1}{4k} \rfloor$ free edges (we call them *extra edges*) joining $v_{n_i+h}^i$ to vertices of C_i, to get a total of l_i edges. When enumerating the vertices of I_i in his $(i+1)$st move, Enforcer uses extra edges as a tie breaker: he always places vertices which received extra edges in his previous move as early as possible, that is, among all vertices with A-degree d, first come those which received extra edges in his previous move and afterwards those which did not. At this point, remaining ties are broken arbitrarily.

Now we prove that Enforcer is able to play according to the proposed strategy without ever having to forfeit the game, and that at some point r threats will appear. Let us see first that Enforcer can claim the extra edges in each move as described above. Let v be a vertex that was picked by Enforcer to receive extra edges in his ith move. If Avoider does not touch v in his $(i+1)$st move, v will be among the first $4k$ vertices in Enforcer's $(i+1)$st move enumeration. Indeed, every vertex that was placed after the first $4k$ vertices of I_i in the ith enumeration had an A-degree at least as large as that of v. Since v was not touched by Avoider in his last move it still holds, and in case they have equal A-degree the tie breaker puts v earlier. Enforcer will then add v to C_{i+1} since his bias is large enough. So v remains in I only if Avoider touched it immediately after Enforcer. Thus, any vertex $v \in V$ can receive extra edges at most k times, as otherwise Avoider must have already claimed an $\mathcal{S}_k$.

Suppose now that we are in the $(i+1)$st move of Enforcer for some $i \geq 0$, just before he adds l_{i+1} extra edges to the $4k$ selected vertices from I_{i+1}. Note that $l_{i+1} < |C_{i+1}|$ by the choice of n_i, and clearly $|C_j| \leq |C_{i+1}|$ holds for any $j \leq i$. Thus, if this is the mth time that a vertex v is picked to receive extra edges in Enforcer's graph, the total number of extra edges incident to v is at most $m\lceil\frac{|C_{i+1}|}{4k}\rceil \leq k\lceil\frac{|C_{i+1}|}{4k}\rceil = (1/4+o(1))|C_{i+1}|$, as $m \leq k$ and $|C_i| = \omega(1)$ for all $i > 0$.

In addition, $d_A(v) < k$ always holds, so for every i there are at least $(3/4-o(1))|C_i|$ free edges between any vertex $v \in I_i$ and C_i, so Enforcer is able to follow the above strategy. We now need the following strict analog of Lemma 3.1:

Lemma 3.2. *For every $1 \leq j \leq k-2$, stage j ends, and at that moment the inequality $|I| \geq 0.9n\left(\frac{n}{2b}\right)^j$ holds.*

Proof. The proof – by induction on j – is identical to the one of Lemma 3.1, except for some slightly different calculations as follows. Just before the beginning of the last round, the size of I cannot be linear in n because that would imply $\Theta(n^2)$ free edges to be claimed in the last round. Therefore, before the last round $|C| = (1-o(1))n$ and there are at least $(3/4-o(1))|I|n \leq b+1$ free edges, which implies $|I| < 1.5\frac{b}{n} < 0.9n^{\frac{1}{k-1}} \leq 0.9n\left(\frac{n}{2b}\right)^{k-2}$, so I cannot be too large just before the game ends.

For $j=1$ we assume for contradiction that for some i both inequalities $|I_i| < 0.45n^2/b$ and $\sum_{v\in I_i} d_{A_i}(v) < |I_i|$ hold, and since the sum $\sum_{v\in I} d_A(v)$ increases by at least one in every round we get $i < |I_i|$. By counting Enforcer's edges in C_i and the number of edges he has claimed in the first i rounds we get the inequality $\frac{1}{2}(|C_i|-k)^2 < ib < 0.45n^2$. Thus, $|C_i| < 0.95n$ and $|I_i| = n - |C_i| > n/20$, a contradiction.

We observe that the length of the game, and therefore the number of Avoider's edges in his final graph is $\Theta(n^2/b) = o(n)$. So if g denotes the last round of the first stage we get $|I_g| = o(n)$. Since for every $i > g$ there are at least $(3/4-o(1))|C_i| \geq 2n/3$ free edges between any vertex $v \in I_i$ and C_i, Enforcer, in each move after the gth, moves at most $\frac{3b}{2n}$ vertices from I to C.

To complete the proof we assume for contradiction that for some $1 < j \leq k-2$ the lemma holds for stage $j-1$ but not for stage j, i.e. there exists an integer i such that $|I_i| < 0.9n\left(\frac{n}{2b}\right)^j$, but $\sum_{v\in I_i}(d_{A_i}(v)) < j|I_i|$, or $\sum_{v\in I_i}(d_{A_i}(v)-(j-1)) < |I_i|$. This sum is non-negative after the first round of the stage and increases by at least one in every round, so there were at most $|I_i|$ rounds in this stage. Since the lemma holds for $j-1$ it follows that $|I_i| \geq 0.9n\left(\frac{n}{2b}\right)^{j-1} - |I_i|\frac{3b}{2n} \geq \left(0.9n\left(\frac{n}{2b}\right)^{j-1}\right)\left(1-\frac{3}{4}\right)$, a contradiction. □

Let us consider the beginning of stage $k-1$. Denote by g the first round of this stage. As already shown, there are at least $(3/4-o(1))|C| = (3/4-o(1))n$ free edges between any vertex $v \in I$ and C. Thus every vertex in I of degree $k-1$ creates at least $(3/4-o(1))n$ unique threats. So for any $\varepsilon > 0$, if we denote $r' := r/\left((3/4-\varepsilon)n\right)$, then if there are – at any point before the last round – r' vertices of degree $k-1$, there are at least r threats and Enforcer wins. The following proposition shows that if the game lasts more than $g+r'$ rounds than at least r' vertices of degree $k-1$ will appear in I before the last round.

Proposition 3.3. *After Avoider's $(g+l)$th move either Avoider's graph contains an $\mathcal{S}_k$ or there are at least l vertices in I of A-degree $k-1$.*

Proof. Let $v_1, v_2, \ldots, v_t$ denote the vertices of I after the gth round with A-degree at most $k-3$ and let us write $m = \sum_{i=1}^{t}(k-2-d_{A_g}(v_i))$. If Avoider has not yet created an $\mathcal{S}_k$, all vertices have A-degree at most $k-1$, thus after the gth round there are at least m vertices in I with A-degree $k-1$, as the average A-degree in I_g is at least $k-2$. Since all edges claimed by Avoider have at least one endpoint in I, in every move after the gth but at most m he creates a new vertex of A-degree $k-1$. □

In his gth move Enforcer removes at most $(4/3+o(1))\frac{b}{n}$ vertices from I. A simple calculation yields

$$\frac{2b}{n} = \left(\frac{2b}{n}\right)^{k-1}\left(\frac{n}{2b}\right)^{k-2} \leq \left(0.8n^{\frac{1}{k-1}}\right)^{k-1}\left(\frac{n}{2b}\right)^{k-2} \leq 0.64n\left(\frac{n}{2b}\right)^{k-2},$$

and by using Lemma 3.2 we get $|I_g| \geq |I_{g-1}| - (4/3+o(1))\frac{b}{n} \geq 0.47n\left(\frac{n}{2b}\right)^{k-2}$, and therefore the number of free edges after the gth round is at least $|I_g|(3/4-o(1))n \geq 0.35\frac{n^k}{(2b)^{k-2}}$. Thus, if the inequality $r'(b+1) < 0.35\frac{n^k}{(2b)^{k-2}}$ holds, the game will last more than $g+r'$ rounds and Enforcer will win. This inequality may be relaxed to: $r < \frac{1}{2}\frac{n^{k+1}}{(2b)^{k-1}}$. Note that since $r \leq b+1$, this inequality holds if $b < \frac{1}{2}\frac{n^{k+1}}{(2b)^{k-1}} \iff (2b)^k < n^{k+1} \iff b < \frac{1}{2}n^{\frac{k+1}{k}}$ (it suffices to use b instead of $b+1$ since the constant $\frac{1}{2}$ used in the inequality is not tight). So if $b < \frac{1}{2}n^{\frac{k+1}{k}}$ Enforcer wins regardless of r, which shows that for all n, $e^-_{n,k} = \Omega(n^{\frac{k+1}{k}})$. From all that is said above , it is clear that $e^+_{n,k} \leq f^+_{\mathcal{K}_{\mathcal{S}_k}}$ and $e^-_{n,k} \leq f^-_{\mathcal{K}_{\mathcal{S}_k}}$.

Finally, applying Fact 2.1 **(ii)** with $r = \frac{k}{k-1}$ and $c = 1$, we obtain infinitely many integers n such that there exists an integer q with $n^{\frac{k}{k-1}} < q < 4n^{\frac{k}{k-1}}$ and $q \mid \left(\binom{n}{2}-1\right)$. For each such n, q, by setting $b := \lfloor q/10 \rfloor - 1$ we get that $0.09n^{\frac{k}{k-1}} < b < 0.4n^{\frac{k}{k-1}}$. Denote $s := \left(\binom{n}{2}-1\right)/q$ and $\alpha := q \mod 10$. Both players claim together $b+1 = \lfloor q/10 \rfloor = (q-\alpha)/10$ edges in each round, so after $10s$ rounds $qs - \alpha s = \binom{n}{2} - 1 - \alpha s$ edges will be claimed, so there will be $1 + \alpha s$ free edges left on the board. Note that $s = \Theta(n^2/q) = o(n)$, and since $b = \omega(n)$ the next round will be the last round in the game, so $r = 1 + \alpha s$. Regardless of the value of α we get that $r = o(n)$. However, $\frac{n^{k+1}}{(2b)^{k-1}} = \Theta(n)$, so the inequality $r < \frac{n^{k+1}}{(2b)^{k-1}}$ holds and Enforcer wins. This shows that for these values of n we have $e^+_{n,k} = \Omega(n^{\frac{k}{k-1}})$.

3.2 Avoider's strategy – upper bounds

In this subsection we establish upper bounds on the threshold biases $f^{mon}_{\mathcal{K}_{\mathcal{S}_k}}, f^{+}_{\mathcal{K}_{\mathcal{S}_k}}, f^{-}_{\mathcal{K}_{\mathcal{S}_k}}$. As shown in Section 3.1, in the monotone game Avoider is doomed at the moment he creates a vertex of A-degree $k-1$ which still has at least one free edge incident to it. This is also the case in the strict game provided $\binom{n}{2} \equiv 1 \mod (b+1)$. To prevent this situation we provide Avoider with a simple strategy which keeps the maximum degree of his graph low. In this strategy Avoider claims exactly one edge in each move, thus this is a valid strategy for both the monotone and the strict game.

Avoider's strategy: Let E_i be the set of free edges before Avoider's ith move, then for any edge $(u, v) \in E_i$ let $d_{max}(u, v) = \max\{d_{A_{i-1}}(u), d_{A_{i-1}}(v)\}$. Avoider claims an arbitrary edge from the set of free edges with minimum d_{max}-value.

To obtain the upper bound of Theorem 1.1 **(i)** and **(ii)** it is enough to show that if Avoider plays according to this strategy, he wins independently of Enforcer's strategy provided Enforcer's bias is at least $2n^{\frac{k}{k-1}}$. For the analysis, we fix some arbitrary strategy for Enforcer, and divide the course of the game into stages according to the degrees in Avoider's graph, similarly to the previous subsection. This time, stage j starts with Avoider's move in which he creates the first vertex with A-degree j and ends right after Enforcer's last move for which Avoider's graph still has maximum degree j.

Proposition 3.4. *At the end of stage j, the vertices of A-degree at most $j-1$ form a clique in the global graph.*

Proof. According to his strategy, Avoider would otherwise claim an edge without creating a vertex of A-degree $j+1$. □

Lemma 3.5. *For any positive integer j, at the end of stage j (if it ends), the number of vertices of A-degree j is at most $(2^{j-1} + o(1))\frac{n^{j+1}}{b^j}$ and the number of free edges is at most $(2^{j-1} + o(1))\frac{n^{j+2}}{b^j}$.*

Proof. We prove the lemma by induction on j. As in every round the total number of edges taken by Avoider and Enforcer is at least $b+1$, the whole game, and thus stage 1, cannot last longer than $\lceil \binom{n}{2}/(b+1) \rceil$ rounds. As in each move during stage 1 Avoider creates two vertices of degree 1, the number of such vertices at the end of the stage is at most $\lceil n(n-1)/(b+1)\rceil = (1+o(1))\frac{n^2}{b}$. By Proposition 3.4, none of the edges between two vertices of A-degree 0 is still available, therefore the number of free edges is at most $(1+o(1))\frac{n^2}{b} \cdot n = (1+o(1))\frac{n^3}{b}$. This proves the statement of the lemma for the case $j = 1$.

Assume now that the statement of the lemma holds for j and thus the number of free edges at the end of stage j is at most $(2^{j-1} + o(1))\frac{n^{j+2}}{b^j}$. Therefore stage $j+1$ cannot last longer than $(2^{j-1} + o(1))\frac{n^{j+2}}{b^j(b+1)} = (2^{j-1} + o(1))\frac{n^{j+2}}{b^{j+1}}$. In each move Avoider claims exactly one edge, therefore in one move at most two more vertices of A-degree $j+1$ can appear and thus the number of such vertices by the end of stage $j+1$ cannot exceed $(2^j + o(1))\frac{n^{j+2}}{b^{j+1}}$. Thus, by Proposition 3.4, the number of free edges at the end of stage $j+1$ is at most $(2^j + o(1))\frac{n^{j+2}}{b^{j+1}} \cdot n = (2^j + o(1))\frac{n^{j+3}}{b^{j+1}}$. □

Lemma 3.5 easily implies the upper bounds of Theorem 1.1 on $f^{mon}_{\mathcal{KS}_k}$ and $f^{+}_{\mathcal{KS}_k}$. Indeed, if Enforcer's bias is at least $2n^{\frac{k}{k-1}}$, then if Avoider uses the above strategy, then either stage $k-2$ does not end during the run of the game and thus not even a vertex of A-degree $k-1$ is created, or at the end of stage $k-2$ the number of free edges is strictly less than $b+1$. In this case, no matter which edge Avoider claims in his first move in stage $k-1$, Enforcer will be forced to claim all remaining free edges in his move right after that, and Avoider wins.

To see that infinitely many times $f^{-}_{\mathcal{KS}_k} = O(n^{\frac{k+1}{k}})$, suppose that the number of vertices n, and Enforcer's bias b, satisfy $2n^{\frac{k+1}{k}} < b < 4n^{\frac{k+1}{k}}$ and $\binom{n}{2} \equiv 0 \mod (b+1)$. According to part (i) of Fact 2.1, with $r = \frac{k+1}{k}$ and $c = 2$, there exist infinitely many integers n with such b. Then, by applying Lemma 3.5 we get that regardless of Enforcer's strategy, at the end of stage $k-2$ the number of vertices of A-degree $k-2$ is at most $\frac{1}{2}n^{\frac{2}{k}}$ and the number of free edges is at most $\frac{1}{2}n^{\frac{k+2}{k}}$. Therefore there remain at most $\frac{1}{4}n^{\frac{1}{k}}$ rounds in the game. In each move Avoider creates at most 2 vertices of A-degree $k-1$, thus before Avoider's final move, there are at most $\frac{1}{2}n^{\frac{1}{k}}$ of them, creating at most $\frac{1}{2}n^{\frac{k+1}{k}}$ threats (free edges incident to vertices of A-degree $k-1$). But as $\binom{n}{2} \equiv 0 \mod (b+1)$ holds, Avoider in his last move has the possibility to choose from $b+1 \geq 2n^{\frac{k+1}{k}}$ free edges, so he can choose a free edge which is not a threat and win.

Finally, to see the general upper bound on $f^{-}_{\mathcal{KS}_k}$, by applying Fact 2.2 with $\delta = \frac{k+1}{2k}$, $N = \binom{n}{2}$ and $q = 2n^{\frac{k+1}{k}}$, we obtain that for any sufficiently large n there exists an integer b with $2n^{\frac{k+1}{k}} \leq b \leq 8n^{\frac{k+1}{k}} \log n$ such that the remainder $r = r(n,b)$ (the number of free edges before the last round) is at least $2n^{\frac{k+1}{k}}$. A computation identical to the one above yields the general statement about the upper bound on $f^{-}_{\mathcal{KS}_k}$.

4 Monotone double star games

In this section we analyse two more classes of graphs H, for which $f^{mon}_H = f^{-}_{H^-}$ holds.

Let the *double star* $\mathcal{S}_{k,k}$ be a graph on $2k$ vertices $u, u_1, \ldots, u_{k-1}, v, v_1, \ldots, v_{k-1}$ such that the edge set of $\mathcal{S}_{k,k}$ is $\{(uv)\} \cup \{(uu_i) : 1 \le i \le k-1\} \cup \{(vv_i) : 1 \le i \le k-1\}$ (see Figure 1) and let $\mathcal{K}_{\mathcal{S}_{k,k}}$ be the hypergraph of the game.

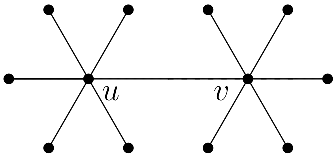

Figure 1: $\mathcal{S}_{6,6}$ on vertices (u, v)

Let the *path double star* $\mathcal{PS}_{k,k}$ be a graph on $2k+1$ vertices $w, u, u_1, \ldots, u_{k-1}, v, v_1, \ldots, v_{k-1}$ such that $E(\mathcal{PS}_{k,k}) = \{\{(u, u_i), 1 \le i \le k-1\} \cup \{(v, v_i), 1 \le i \le k-1\} \cup (v, w) \cup (u, w)\}$, as shown in Figure 2, and let $\mathcal{K}_{\mathcal{PS}_{k,k}}$ be the hypergraph of the game.

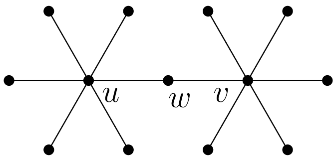

Figure 2: $\mathcal{PS}_{6,6}$ on vertices (u, v, w)

Theorem 4.1. *Let $k \ge 3$. In the $(1 : b)$ double star $\mathcal{S}_{k,k}$ and path double star $\mathcal{PS}_{k,k}$ games, we have*

(i) $f^{mon}_{\mathcal{K}_{\mathcal{S}_{k,k}}} = \Theta(n^{\frac{k}{k-1}})$,

(ii) $f^{mon}_{\mathcal{K}_{\mathcal{PS}_{k,k}}} = \Theta(n^{\frac{k+1}{k}})$.

In the previous section we saw that for $H = \mathcal{S}_k$, the equality $f^-_{H^-} = \Theta(f^{mon}_H)$ holds. Now we show that the same holds for the two classes of graphs considered in this section.

For $\widehat{H} = \mathcal{S}_{k,k}$, let $\widehat{H}^- = \mathcal{S}_{k,k} - (u, v)$, where u and v are the centres of the connected stars. Note that $\widehat{H}^-$ is a disjoint union of two $(k-1)$-stars. From the analysis of Enforcer's strategy in the star game and Theorem 4.1 we get $f^-_{\widehat{H}^-} = \Theta(n^{\frac{k}{k-1}}) = \Theta(f^{mon}_{\widehat{H}})$. For $\tilde{H} = \mathcal{PS}_{k,k}$, let $\tilde{H}^- = \mathcal{PS}_{k,k} - (u, w)$, where w is the common neighbour of the two stars and u is the centre of one of them. Note that $\tilde{H}^-$ is a disjoint union of a k-star and a $(k-1)$-star. Once again, from the analysis of Enforcer's strategy in the star game and Theorem 4.1 we get $f^-_{\tilde{H}^-} = \Theta(n^{\frac{k+1}{k}}) = \Theta(f^{mon}_{\tilde{H}})$.

We apply the results and strategies from the $\mathcal{S}_k$ game in these two games to obtain the results in Theorem 4.1. Avoider simply uses the exact same strategy to ensure his win, while Enforcer uses some more involved strategy. The full proofs of Enforcer's strategies are very similar to the ones in the $\mathcal{S}_k$ game, and involve some tedious calculations. Therefore, we only describe Enforcer's strategies in details without the explicit proofs.

Avoider's strategies for the monotone $\mathcal{S}_{k,k}$ and $\mathcal{PS}_{k,k}$ games:

For $\mathcal{S}_{k,k}$, Avoider can simply avoid a vertex of degree k when playing against a bias $b \geq 2n^{\frac{k}{k-1}}$. For $\mathcal{PS}_{k,k}$, recall the proof of the k-star game. Until there are at most $b+1$ free edges left (for this large enough b), the maximum degree in Avoider's graph is at most $k-2$. Therefore, playing against bias $b \geq 2n^{\frac{k+1}{k}}$, Avoider can make sure that no vertex in his graph reaches degree $k-1$ until his last move. It is evident that he cannot lose by claiming just one more edge.

Enforcer's strategy in the monotone $\mathcal{S}_{k,k}$ game:

Let $b < cn^{\frac{k+1}{k}}$, where $c = c(k) = \frac{1}{4k^3}$. We present a winning strategy for Enforcer in the monotone $(1:b)$ $\mathcal{S}_{k,k}$ game. At any point during the game, if an $\mathcal{S}_{k,k}$ minus some edge e appears, and such an e is still free, Enforcer claims all free edges but e (provided that there are at least $b+1$ free edges) and wins. If an $\mathcal{S}_{k,k}$ appears Enforcer has already won. At any point, if Enforcer cannot follow his strategy he forfeits the game. As long as none of this happens, let us fix some arbitrary strategy for Avoider, and divide the course of the game into two phases:

Phase I: Until a vertex of A-degree at least $k-1$ appears, Enforcer follows the strategy he used in the $\mathcal{S}_k$ game. When such a vertex appears, Enforcer performs a switching move: Let v be a vertex of maximal A-degree (hence $d_A(v) \geq k-1$). Enforcer chooses arbitrarily $k-1$ A-neighbours of v. Denote by $v_1, v_2, \ldots, v_{k-1}$ those neighbours and let $N := \bigcup_{i=1}^{k-1}\{v_i\} \cup N_A(v_i) \setminus \{v\}$. For every vertex in N Enforcer claims all free edges adjacent to it. Finally, Enforcer moves v from I to C and claims all free edges in $E(\{v\}, C)$. This is the end of Phase I and Enforcer proceeds to Phase II.

Phase II: Enforcer continues to follow his strategy from the $\mathcal{S}_k$ game with the slight modification that now he maintains a dynamic partition $V \setminus N = I \cup C$, where I and C are being updated as before. Note that due to the switching move, no vertex in I can have one of $v_1, \ldots, v_{k-1}$ as a neighbour in Avoider's graph, nor can it be one of them. By Enforcer's strategy, when another vertex $u \in I$ of A-degree at least $k-1$ appears, the edge (u, v) is either free or has already been claimed by Avoider. In the former case Enforcer claims all free edges but (u, v). In the latter case $d_A(v) \geq k$ and Enforcer claims all free edges but one adjacent to u. In both cases, Avoider is forced to complete an $\mathcal{S}_{k,k}$ in his next move, thus losing the game.

Enforcer's strategy in the monotone $\mathcal{PS}_{k,k}$ game:
Enforcer's winning strategy in the $\mathcal{S}_{k+1}$ game can be modified to obtain a winning strategy for the $\mathcal{PS}_{k,k}$ game. Let $b < cn^{\frac{k+1}{k}}$, where $c = c(k) = \frac{1}{32(k+1)^4}$.

Before we describe the strategy of Enforcer, let us introduce some terminology. We use the notation from the proof of monotone k-star game, namely sets I and C. We also define sets *centres*, X, and *neighbours*, N. The sets X and N determine a modification of the A-degree which we call A^*-degree that will influence which vertices Enforcer adds to C. Note that $N_G(v)$ still denotes the set of neighbours of vertex v in graph G.
The A^*-degree of vertex v is:

$$d^*_A(v) := \big|N_A(v)\backslash(N \cup X)\big|.$$

The sets X and N are enlarged in the *updating move.*
The *updating move*: Let v be the vertex with the largest A^*-degree in I, s.t. $d^*_A(v) \geq k$. Suppose $|X| = h - 1$, for some integer $h \geq 1$. Enforcer labels v by c_h and then chooses k vertices of the lowest A^*-degree from $N_A(v)\backslash(N \cup X)$ and labels them $v^1_h, \dots, v^k_h$. He then *updates* X and N by setting $X := X \cup \{c_h\}$, $N := N \cup \{v^1_h, v^2_h, \dots, v^k_h\}$ and $I := I\backslash\{c_h\}$.

Enforcer follows the strategy he uses in $\mathcal{S}_{k+1}$ game with some slight modifications until a $\mathcal{PS}_{k,k}$ appears in Avoider's graph. Namely, he manages the dynamic partition $V = I \cup C \cup X$ such that the following holds: $d_F(v) = 0$ for every $v \in I$, the global graph induced on the vertices of C, $G[C]$, is a complete graph and $d_A(x) + d_F(x) = n - 1$ for every $x \in X$.

Initially, $X = \emptyset$, $N = \emptyset$, $C = \emptyset$ and $I = V$.

Before each his move Enforcer does the following (in the given order):

(i) If a $\mathcal{PS}_{k,k}$ is a subgraph of A, Avoider has lost the game and Enforcer can claim all the remaining edges to finish the game.

(ii) If a $\mathcal{PS}_{k,k}$ minus some edge e is a subgraph of A, with e unclaimed, then Enforcer takes all the remaining edges except for e (provided there are at least $b+1$ free edges). Avoider is forced to claim the remaining edge in his next move, thus losing the game.

(iii) While there exists a vertex $x \in I \cup N$ such that $d^*_A(x) \geq k$, Enforcer performs the updating move and claims all unclaimed edges incident with x. Note that after each updating move, Enforcer rechecks the A^*-degrees of all the vertices in $I \cup N$, as A^*-degree of a vertex can change after the updating move.

(iv) With any remaining edges Enforcer plays according to the strategy in the $\mathcal{S}_{k+1}$ game, but taking into consideration A^*-degree. Namely, he repeatedly chooses a vertex $v \in I$ of minimum A^*-degree, connecting it to all the vertices in C and afterwards removing v from I and adding it to C.

If at any point of the game, Enforcer is unable to follow the proposed strategy he immediately forfeits the game. Note that if two vertices, u, v in Avoider's graph simultaneously reach A^*-degree k by Avoider's edge (u, v), then Enforcer chooses one of them arbitrarily to add it to X, say u, meaning that the edge (u, v) no longer contributes to $d_A^*(v)$ and so the A^*-degree of v is back to $k-1$. Thus Enforcer adds at most one vertex to X for any Avoider's edge. Before any *updating* move is played the A^*-degree is equal to the A-degree.

The definition of d_A^* ensures that each time a vertex $v \in I \cup N$ of A^*-degree at least k appears in Avoider's graph, there are at least k neighbours of v different from all vertices from $X \cup N$. So, at latest when $|X| = 2$ and another vertex in $I \cup N$ of A^*-degree at least k appears, Avoider's graph will contain a $\mathcal{PS}_{k,k}$ or a $\mathcal{PS}_{k,k}$ minus some unclaimed edge e.

5 Concluding remarks and open problems

In the Introduction, we already mentioned that Bednarska-Bzdęga obtained bounds on the different threshold biases for arbitrary fixed graph G. Her bounds depend on the following three parameters of G:

$$m(G) = \max_{F \subseteq G: v(F) \geq 1} \frac{e(F)}{v(F)}; \qquad m'(G) = \max_{F \subseteq G: v(F) \geq 1} \frac{e(F)-1}{v(F)};$$

$$m''(G) = \max_{F \subseteq G: v(F) \geq 3} \frac{e(F)+1}{v(F)-2}.$$

She proved that for an arbitrary graph G, a positive real ε and a large enough integer n the inequalities $f^{mon}_{\mathcal{K}_G}, f^+_{\mathcal{K}_G} = O(n^{1/m'(G)})$ and $\Omega(n^{1/m''(G)-\varepsilon}) = f^-_{\mathcal{K}_G} = O(n^{1/m(G)} \log n)$ hold. Our results show that these bounds are far from being tight for the star game, except for the upper bound on $f^-_{\mathcal{K}_G}$, which equals ours. However, at least the upper bound on $f^{mon}_{\mathcal{K}_G}$ cannot be improved in general, since it is tight for the case $G = K_3$, as observed by Bednarska-Bzdęga herself in her paper.

In this paper we show that for any sufficiently large n and for every $k \geq 3$, the threshold biases $f^-_{\mathcal{K}_{\mathcal{S}_k}}$ and $f^+_{\mathcal{K}_{\mathcal{S}_k}}$ are not of the same order. However, for each of them we gave general upper and lower bounds for every n, and the exact order of magnitude only for infinitely many values of n. Moreover, the general lower bound of $f^+_{\mathcal{K}_{\mathcal{S}_k}}$ is only implicit. Recall that in order to obtain the general upper bound of $f^-_{\mathcal{K}_{\mathcal{S}_k}}$ we used Fact 2.2, which for every k shows the existence of a bias b of the appropriate order of magnitude such that the corresponding remainder $r = r(n, b)$ is close to b. In order to obtain an explicit lower bound of $f^+_{\mathcal{K}_{\mathcal{S}_k}}$ we would need an analogous number theoretic statement that ensures the existence of a bias b of the right order, such that the remainder $r(n, b)$ is small enough. Nevertheless, we believe that in fact $f^+_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k}{k-1}})$ and $f^-_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k+1}{k}})$ hold for any sufficiently large n and not only infinitely often. Moreover, if indeed these equalities hold for every n, another question arises: are there exact constants that can be used in these equalities instead of just the order of magnitude?

Note that our results for the k-star game only hold for a constant k. Not only that some parts of our proofs rely on the fact that k is a constant, we also get that for any $k = \omega(1)$, all threshold biases, f^{mon}, f^- and f^+, equal $\Theta(n)$. This is unlikely to be the correct threshold bias for **every** such k. It will be interesting to analyse this game for the non constant case.

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