UNIVERSITY OF NOVI SAD
FACULTY OF SCIENCES
DEPARTMENT OF
MATHEMATICS AND INFORMATICS

**Mirjana Mikalački**

# Positional games on graphs

- PhD thesis -

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# Izvod

Proučavamo takozvane Mejker-Brejker (Maker-Breaker) igre koje se igraju na granama kompletnog grafa sa $n$ čvorova, $K_n$, čija familija pobedničkih skupova $\mathcal{F}$ obuhvata sve skupove grana grafa $G \subseteq K_n$ koji imaju neku monotono rastuću osobinu. Dva igrača, *Mejker* (*Praviša*) i *Brejker* (*Kvariša*) se smenjuju u odabiru $a$, odnosno $b$, slobodnih grana po potezu. Interesuje nas da pronađemo granični bias $b_{\mathcal{F}}(a)$ za sve vrednosti parametra $a$, tako da za svako $b$, $b \leq b_{\mathcal{F}}(a)$, Mejker pobeđuje u igri, a za svako $b$, takvo da je $b > b_{\mathcal{F}}(a)$, Brejker pobeđuje. Posebno nas interesuju slučajevi u kojima oba parametra $a$ i $b$ mogu imati vrednost veću od 1. Naša pažnja je posvećena igri povezanosti, gde su pobednički skupovi grane svih pokrivajućih stabala grafa $K_n$, kao i igri Hamiltonove konture, gde su pobednički skupovi grane svih Hamiltonovih kontura grafa $K_n$.

Zatim posmatramo igre tipa Avojder-Enforser (Avoider-Enforcer), sa biasom $(1 : b)$, koje se takođe igraju na granama kompletnog grafa sa $n$ čvorova, $K_n$. Za svaku konstantu $k$, $k \geq 3$ analiziramo igru $k$-zvezde (zvezde sa $k$ krakova), u kojoj *Avojder* pokušva da izbegne da ima $k$ svojih grana incidentnih sa istim čvorom. Posmatramo obe verzije ove igre, striktnu i monotonu, i za svaku dajemo eksplicitnu pobedničku strategiju za oba igrača. Kao rezultat, dobijamo gornje i donje ograničenje za granične biase $f_{\mathcal{F}}^{mon}$, $f_{\mathcal{F}}^{-}$ i $f_{\mathcal{F}}^{+}$, gde $\mathcal{F}$ predstavlja hipergraf igre (familija ciljnih skupova). Takođe, posmatramo i monotonu verziju $K_{2,2}$-igre, gde Avojder želi da izbegne da graf koji čine njegove grane sadrži graf izomorfan sa $K_{2,2}$.

Konačno, želimo da pronađemo strategije za brzu pobedu Mejkera u igrama savršenog mečinga i Hamiltonove konture, koje se takođe igraju na granama kompletnog grafa $K_n$. Ovde posmatramo asimetrične igre gde je bias Mejkera 1, a bias Brejkera $b$, $b \geq 1$.

# Abstract

We study Maker-Breaker games played on the edges of the complete graph on $n$ vertices, $K_n$, whose family of winning sets $\mathcal{F}$ consists of all edge sets of subgraphs $G \subseteq K_n$ which possess a predetermined monotone increasing property. Two players, Maker and Breaker, take turns in claiming $a$, respectively $b$, unclaimed edges per move. We are interested in finding the threshold bias $b_{\mathcal{F}}(a)$ for all values of $a$, so that for every $b$, $b \leq b_{\mathcal{F}}(a)$, Maker wins the game and for all values of $b$, such that $b > b_{\mathcal{F}}(a)$, Breaker wins the game. We are particularly interested in cases where both $a$ and $b$ can be greater than 1. We focus on the *Connectivity game*, where the winning sets are the edge sets of all spanning trees of $K_n$ and on the *Hamiltonicity game*, where the winning sets are the edge sets of all Hamilton cycles on $K_n$.

Next, we consider biased $(1 : b)$ Avoider-Enforcer games, also played on the edges of $K_n$. For every constant $k \geq 3$ we analyse the $k$-star game, where Avoider tries to avoid claiming $k$ edges incident to the same vertex. We analyse both versions of Avoider-Enforcer games, the strict and the monotone, and for each provide explicit winning strategies for both players. Consequentially, we establish bounds on the threshold biases $f^{mon}_{\mathcal{F}}$, $f^{-}_{\mathcal{F}}$ and $f^{+}_{\mathcal{F}}$, where $\mathcal{F}$ is the hypergraph of the game (the family of target sets). We also study the monotone version of $K_{2,2}$-game, where Avoider wants to avoid claiming all the edges of some graph isomorphic to $K_{2,2}$ in $K_n$.

Finally, we search for the fast winning strategies for Maker in Perfect matching game and Hamiltonicity game, again played on the edge set of $K_n$. Here, we look at the biased $(1 : b)$ games, where Maker's bias is 1, and Breaker's bias is $b, b \geq 1$.

# Preface

The focus of this thesis is on positional games on graphs. This branch of combinatorics has been extensively studied over the past 15 years. So, what is it all about? Positional games are combinatorial games, meaning that they are finite, perfect information games. As such, they could be, in theory, solved by an all powerful computer. But only in theory. In reality such computers are not at our disposal, and we are faced with finding alternative ways of solving the games. Computers can be of great help when simulating random play, but can hardly be of any help in the simulation of an optimal play due to the enormously large number (exponential) of possibilities to check. So, mathematical tools are used to prove the correctness of the strategies. Many results have been obtained in this field and in particular about Maker-Breaker and Avoider-Enforcer games. Reading about these games motivated me to investigate either known games in different settings or to see what happens in some new games.

In Chapter 1 we will give the overview of the background information about positional games. A positional game is a pair $(X, \mathcal{F})$, where $X$ is a finite set, and $\mathcal{F} \subseteq 2^X$ is a family of target sets. The game is played by two players who in turns claim $a$, respectively $b$, unclaimed elements of $X$. Depending on the way of determining the winner, we distinguish different types of games. In the *strong games*, both players have the same goal, which is to claim all the elements of some $A \in \mathcal{F}$. The player who does it first wins. Game ends when all the elements of the board are claimed. If none of the player wins, the game ends in a draw. This type is the most interesting type of game for playing, yet it is the most difficult to analyse.

In *Maker-Breaker* games, the players are called Maker and Breaker, and they have different goals in the game. Maker wins, if and only if he manages to claim all the elements of some $A \in \mathcal{F}$ by the end of the game. Breaker, on the other hand, wants to prevent Maker from achieving his goal. So, either Maker or Breaker wins. No draw is possible.

In *Avoider-Enforcer* games, the two players are called Avoider and Enforcer. Avoider's goal is to avoid claiming all the elements of any $A \in \mathcal{F}$ and if he manages to maintain that until the end of the game, he wins. Enforcer, however, aims at forcing Avoider to do so. Similarly to Maker-Breaker games, there can be no draw in this type of games.

In Chapter 2 we give an overview of the main results obtained in the thesis. We also compare the obtained results to the existing results and explain their significance.

In Chapter 3 we give novel results in the doubly biased $(a : b)$ Maker-Breaker games. Our focus is on the Connectivity game $\mathcal{T}_n$ and the Hamiltonicity game $\mathcal{H}_n$ played on the edge set of the complete graph on $n$ vertices, $K_n$. The target sets in the Connectivity game are all connected spanning subgraphs of $K_n$. In the Hamiltonicity game, the target sets are the edge sets of all Hamiltonian spanning subgraphs. Both $a$ and $b$ can be greater than one and in both games for all relevant values of $a$ we find the threshold biases $b_{\mathcal{T}_n}(a)$ and $b_{\mathcal{H}_n}(a)$.

The interesting thing about the $(a : b)$ Maker-Breaker games, where $a, b \geq 1$, $(a, b) \neq (1, 1)$ (or *doubly biased*, as we call them) is that the change in $a$ can have a strong impact on the outcome of the game (see e.g. [17]).

To be able to give an explicit winning strategy of Breaker, we also study the generalized Box game. In Section 3.1, we give the sufficient condition for BoxMaker's win in the $(a : b)$ Box game.

In Section 3.2 we give an explicit winning strategy for Breaker in the $(a : b)$ Connectivity game. This gives the upper bound for threshold bias for both Connectivity and Hamiltonicity games. It is easy to see that if Breaker has a strategy to prevent Maker from creating a connected graph, this graph cannot possibly contain a Hamilton cycle.

In Section 3.3 we give an explicit winning strategy for Maker in $(a : b)$

Connectivity game, thus establishing the lower bound on the threshold bias $b_{\mathcal{T}_n}(a)$.

Finally, in Section 3.4 we give an explicit winning strategy for Maker in the $(a:b)$ Hamiltonicity game.

The results in this chapter are mainly joint work with Dan Hefetz and Miloš Stojaković [60]. Section 3.4 is joint work with Miloš Stojaković.

In Chapter 4 we study $(1:b)$ Avoider-Enforcer games played on the edge set of $K_n$, the complete graph on $n$ vertices. We are interested in games where Avoider tries to avoid claiming a copy of some fixed graph $H$. In other words, the family of target sets is $\mathcal{F} = \mathcal{K}_H \subseteq 2^{E(K_n)}$ and contains all the copies of a fixed graph $H$ in $K_n$.

In Section 4.2 we look at both strict and monotone version of the *$k$-star game*, the game where the fixed graph $H$ is a $K_{1,k}$, for constant $k$. We give explicit winning strategies for both Avoider and Enforcer in this game.

In Section 4.3 we look at one more monotone $(1:b)$ game, where the fixed graph $H$ is a $K_{2,2}$.

The results in Section 4.2 are joint work with Andrzej Grzesik, Zoltán Loránt Nagy, Alon Naor, Balázs Patkós and Fiona Skerman [47].

In Chapter 5 we are interested in finding fast winning strategies in the $(1:b)$ Maker-Breaker games played on the edge set of the complete graph on $n$ vertices, $K_n$. In this chapter, we look at two natural graph games, the Perfect matching and the Hamiltonicity game and give upper and lower bound for the duration of both games.

The results in this chapter are joint work with Asaf Ferber, Dan Hefetz and Miloš Stojaković [38].

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M.M.

# Contents

# List of Figures

# Chapter 1

# Introduction

## 1.1 Overview of positional games

Positional games are combinatorial games. They are finite perfect information games played by two players in turns, with no chance moves. As such, they are completely different from the games that belong to the classical game theory, which is a branch of economics, that deals with games that are played simultaneously and with hidden information.

The pioneering papers in the study of positional games appeared in 1963, by Hales and Jewett [49], and in 1973, by Erdős and Selfridge [32] and with them starts the systematic study of the positional games. József Beck plays a very significant role in the study of positional games. He made the positional games a well studied field of combinatorics by publishing numerous papers about them over the last 30 years. Moreover, his book [17] covers a great deal of the subject and also poses a lot of open problems. The book itself is a good study base for the positional games because it provides a wide variety of tools and methods for analysing them.

Positional games are deterministic, so if we suppose that players are playing according to their optimal strategies, by using an all-powerful computer, we could (at least theoretically) determine what will be the outcome of a positional game: first player's win, second player's win or a draw. So, basically, the outcome is known even before the game starts. However, even today's computers are of limited help here, to exhaustively search through the whole exponentially large game tree. This brings us to conclusion that mathematical tools and algorithms are of utmost importance for studying these games.

Interestingly, and surprisingly (since the games are of perfect information), the probabilistic method can be adapted to develop very useful tools for the game analysis based on potential arguments. Beck extensively studied this in his book [17]. Actually, positional games turned out to be a powerful instrument for derandomization and algorithmization of important probabilistic tools, thus having a strong impact on theoretical computer science. Apart from probability, there is also a strong relationship between positional games and other fields of combinatorics, such as extremal theory and Ramsey theory.

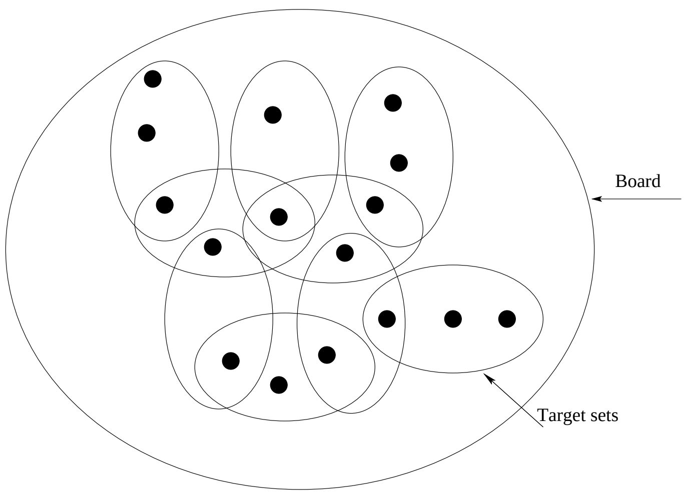


Figure 1.1: Positional game

Slika 1.1: Poziciona igra

Formally said, the positional game is a pair $(X, \mathcal{F})$, where $X$ is a finite set and $\mathcal{F} = \{A_1, A_2, \dots, A_n\} \subseteq 2^X$. We call the set $X$ a *board*, and $\mathcal{F}$—the family of *target sets* (see Figure 1.1). $(X, \mathcal{F})$ is also called the *hypergraph* of the game, whose vertices are the elements of set $X$ and whose *hyperedges* are sets $A_1, A_2, \dots, A_n$. When it is clear on which board the game $(X, \mathcal{F})$ is played, we just use $\mathcal{F}$ to denote the hypergraph of the game. There are also two more parameters, $a$ and $b$. In $(a : b)$ positional game, players claim $a$, respectively $b$, unclaimed elements of the board in each turn. When $a = b = 1$, we call such games *fair* or *unbiased*. Otherwise, they are called *biased games*.

Positional games have been extensively studied throughout last 15 years. There is a wide variety of games that belong to this field. By the way of determining the winner, we can divide them into two large classes of *strong* and *weak* games. Now, each class has its subclasses and we will speak about some of them in the following sections.

Some results related to various positional games can be found in e.g. [1, 2, 7, 8, 11–16, 32, 34, 43, 45, 51, 56, 64–68, 70]. Also, one of the fields of research are games on random graphs and some results related to them can be found in [10, 24, 29, 35, 41, 71].

## 1.2 Strong games

Strong positional games (formally defined by Beck in [17]) are popular, some of them actually played games among people. In this setting, the target sets in $\mathcal{F}$ are called the *winning sets*. In strong games both players have the same goal. They want to claim all the elements of a winning set. The player who does it *first* wins the game. If all the elements of the board are claimed and none of the players has won, then the game ends in a *draw*.

**Example 1.1.** The most famous strong positional game is certainly Tic-Tac-Toe (or Crosses and Noughts). Two players, Xena and Obelix, play the game by putting marks “X”, respectively “O”, in turns on empty squares of a $3 \times 3$ square lattice. Each of them puts one mark per turn on the board. There are nine empty squares at the beginning and 8 winning lines (winning sets) in total—three horizontal lines, three vertical lines and two diagonals. The winner is the first player who claims three marks of his own that lie on one winning line. If both players play according to their optimal strategies, it is well-known that this game ends in a draw.

**Example 1.2.** Another example of the strong game is the so-called $n^d$ game. Actually, this is the generalization of Tic-Tac-Toe game. The board of the game is $X = [n]^d$, a $d$-dimensional hypercube consisting of $n^d$ elements. Formally, the board is $X = \{\mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{Z} : 1 \leq a_j \leq n$ for each $1 \leq j \leq d\}$. The winning sets in this game are the so-called *combinatorial lines*. Each combinatorial line is an $n$-tuple $\left(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}\right)$, such that the $j$th coordinate, $1 \leq j \leq d$, is the sequence $a_j^k$, $1 \leq k \leq n$ which is either increasing sequence $(1, 2, \ldots, n)$, decreasing sequence $(n, n-1, \ldots, 1)$ or a constant sequence. The player who claims one combinatorial line first is a winner. In this setting Tic-Tac-Toe would be $3^2$ game. The $n^d$ game is pretty difficult to analyse. We will see later that for some fixed values of $n$ we can determine $d$ such that $n^d$ game is the first player's win.

### 1.2.1 Tools of the trade

*Strategy Stealing* argument is a very powerful tool in the analysis of strong games, used a lot in the literature. By Strategy Stealing argument (first published in [49], later formulated under this name in [17]), the first player can force at least a draw in any strong positional game. This argument affirms the advantageous first move of the first player. Indeed, suppose for a contradiction that the second player has a winning strategy. The winning strategy is a list of moves and counter moves that a player follows in order to win. Playing an extra move in a positional game cannot harm the player. If the second player had a winning strategy, then the first player could *steal* his strategy for this game and do the following: he could play one arbitrary move at the beginning of the game, and afterwards pretend he was the second player and play according to the strategy he has stolen. If the strategy tells him to claim the element he has already claimed, then he chooses another element of the board, arbitrarily. Thus, playing by the strategy he has stolen, till the end of the game, first player would be the winner, a contradiction. In the games where drawing position is not possible, this argument guarantees the winning position for the first player.

Unfortunately, Strategy Stealing argument tells us nothing about how the first player should play in order to win. This argument is strong by means of its extensive applicability, but inexplicit.

Another, very useful tool, is the Ramsey type argument. Before we state what that means in the game setting, let us first define what Ramsey number is. Ramsey number $R(p, q)$ is the least positive integer $n$ such that any two-coloring of edges of $K_n$, a complete graph on $n$ vertices, in red and blue gives a monochromatic red $K_p$, or a monochromatic blue $K_q$. In the game setting, this means that a game has a Ramsey property if any 2-coloring of the board in red and blue, gives a monochromatic set $A \subseteq \mathcal{F}$. Thus, if a game has a Ramsey property, by Strategy Stealing argument, the game is the first player's win.

Now, by Ramsey type argument, Hales and Jewett proved in [49] that for a given $n$, there exists $d(n)$, such that for every $d \geq d(n)$, every 2-coloring of $n^d$ gives a monochromatic winning line, so there can be no draw in this game, which combined with Strategy Stealing implies that this game is the first player's win. However, this argument only gives the existence of the parameter $d$, but it is very difficult to find the smallest such number.

For $n = 3$ and $n = 4$, it is known that $d = 3$. For the game $3^3$, Beck proved there exists an easy first player's winning strategy. Patashnik gave an explicit winning strategy of the first player in $4^3$ game, very complicated one, obtained by searching the game tree. However, the question about whether first player wins in the game $5^3$ is still an open problem.

Apart from aforementioned arguments, there is also another useful tool in the analysis of the strong games—*the pairing strategy.* Second player can guarantee a draw, if there is a paring of the elements of the board, such that there exist $\{x_1, y_1\}, \dots, \{x_k, y_k\} \subseteq X$, with $\{x_i, y_i\} \cap \{x_j, y_j\} = \emptyset$, for every $i \neq j$, and for every $A \subseteq \mathcal{F}$ there exists $i$ such that $\{x_i, y_i\} \subseteq A$. Whichever element of $\{x_i, y_i\}$ the first player claims, the second player will claim another element from the pair thus ensuring that he claimed an element of every winning set.

No other tools exist for the analysis of strong games. Recently, however, the strategies for the first player's win in some natural graph games appeared in the literature [36, 37]. Also, some other results about strong games can be found in e.g. [11, 17, 31, 49]. All in all, not so much is known about these games.

## 1.3 Maker-Breaker games

If we are interested in "what is achievable configuration, achievable, but not necessarily first", [17], i.e. if the focus is not on the competitiveness, but rather on the goal itself, then there is another concept—*Maker-Breaker games*, that appeared as first representatives of the so-called *weak games* (Beck, [17]).

Similarly to strong games, the target sets are called the *winning sets* here as well. But the rules are different. One player, called *Maker*, aims at claiming all the elements of some winning set, but not necessarily first. The other player, *Breaker*, wants to prevent Maker from achieving his goal, i.e. to claim at least one element of each winning set. Let $a$ and $b$ be positive integers. In the $(a : b)$ Maker-Breaker game $(X, \mathcal{F})$, Maker claims $a$ elements of $X$ per move, and Breaker claims $b$ elements per move. Parameters $a$ and $b$ are referred to as bias of Maker, respectively Breaker. If there are less elements of $X$ than the bias of the player who is on the move, he has to claim them all. If Maker has claimed all the elements of one winning set at any point of the game, he won the game. If all the elements of the board are claimed and Maker did not win, then Breaker won the game. It is clear that no draw is possible when playing by these rules. When Maker has a strategy to win against every strategy of Breaker, we say that the game is *Maker's win*. *Breaker's win* is defined analogously.

**Example 1.3.** Let us look at the Maker-Breaker version of the game Tic-Tac-Toe. This is a $(1 : 1)$ game, as both players claim just one element each move. When the rules are changed, we can easily find the winning strategy for Maker in this game (by the simple case analysis). So, the outcome of the game is different compared to its original, strong, version. The condition *not necessarily first* is enough to enable Maker's win in this game.

It is very common to play Maker-Breaker game on the hypergraph $(X, \mathcal{F})$ whose set of winning sets $\mathcal{F}$ is *monotone increasing*. In that case, it is enough for Maker to claim all the elements of a minimal winning set to win. Also, it is sufficient for Breaker to claim at least one element from each minimal winning set, to win. Therefore, we can restrict our game to minimal winning sets, and when there is no possibility of confusion to denote by $\mathcal{F}$ the hypergraph of minimal winning sets.

The game ends at latest when all the elements of the board are claimed. If some player wins before that, we can stop the game at that point, as none of the following moves of any player can have an influence on already obtained goal.

In the games studied in this thesis the board is the edge set of the complete graph on $n$ vertices, $E(K_n)$, where $n$ is sufficiently large integer. The winning sets are various well-known graph-theoretic structures like spanning trees, Hamilton cycles, perfect matching etc. These games have been extensively studied in the last couple of years and various results can be found in [9, 18, 20–23, 30, 42, 44, 46, 53, 61–63].

**Example 1.4.** Let us look at the $(1 : 1)$ Maker-Breaker Connectivity game where the board is $E(K_4)$ and the winning sets are all spanning trees on $E(K_4)$. We denote this family of winning sets by $\mathcal{T}_4$. Formally, this game can be written as $(E(K_4), \mathcal{T}_4)$. Without going into too much details, we will present Maker's winning strategy in this game. We assume that Maker starts the game. In the first two moves, Maker claims two edges incident with the same vertex. He needs to claim just one more edge out of three edges incident to the untouched vertex to complete a spanning tree. Before his third (and last) move, there is at least one free edge that suits his needs. So, he can claim it and win.

When the game is played on $E(K_2)$ Maker wins in his first move and when the board is $E(K_3)$ Maker also wins, as there are three edges from which Maker can take any two and win.

When the game is played on the larger board, $n \geq 5$, it is even easier for Maker to win the game $\mathcal{T}_n$. Lehman showed in [64] that the necessary and sufficient condition for Maker's win in this game is that the board contains two disjoint spanning trees minus an edge. Since for every $n \geq 4$, $K_n$ contains that structure, together with the aforementioned, we can conclude that Maker wins this game as the first player for all $n \geq 2$. In general it holds that he wins this game, as either the first or the second player for all $n \geq 4$. Moreover, it can be shown that Maker can win this game quite *easily*, in exactly $n-1$ moves.

Since it is very easy for Maker to win in some unbiased games, Chvátal and Erdős in [30] introduced *biased games*. They studied the $(1 : b)$ game, where $b > 1$, to find the smallest value of $b$ for which Breaker wins the game.

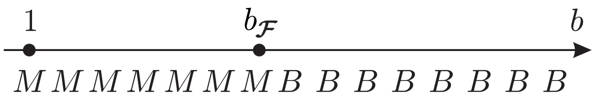


Figure 1.2: Threshold bias $b_{\mathcal{F}}$

Slika 1.2: Granični bias $b_{\mathcal{F}}$

They observed that Maker-Breaker games are bias monotone, that is, if some $(1 : b)$ Maker-Breaker game $(X, \mathcal{F})$ is a Breaker's win, then the $(1 : b+1)$ game $(X, \mathcal{F})$ is a Breaker's win as well. Since, unless $\emptyset \in \mathcal{F}$, the $(1 : |X|)$ game $(X, \mathcal{F})$ is clearly a Breaker's win, it follows that, unless $\emptyset \in \mathcal{F}$ or $\mathcal{F} = \emptyset$ (we refer to these cases as *degenerate*), there exists a unique non-negative integer $b_{\mathcal{F}}$ such that the $(1 : b)$ game $(X, \mathcal{F})$ is a Maker's win if and only if $b \leq b_{\mathcal{F}}$, for a monotone increasing $\mathcal{F}$. This value of $b_{\mathcal{F}}$ is known as the *threshold bias* of the game $(X, \mathcal{F})$ (see Figure 1.2).

When the unbiased game $(X, \mathcal{F})$ is Breaker's win, then we can search for the smallest $a_{\mathcal{F}}$, such that for every $a \geq a_{\mathcal{F}}$ the $(a : 1)$ game is Maker's win.

Also, in general, the $(a : b)$ Maker-Breaker games $(X, \mathcal{F})$ are *bias monotone*. That means that if an $(a : b)$ game is a Maker's win, then Maker wins $(a+1 : b)$ and $(a : b-1)$ games. Analogously, if $(a : b)$ game is won by

Breaker, then Breaker wins in $(a : b+1)$ and $(a-1 : b)$ games as well. This implies that claiming more elements per move cannot harm a player.

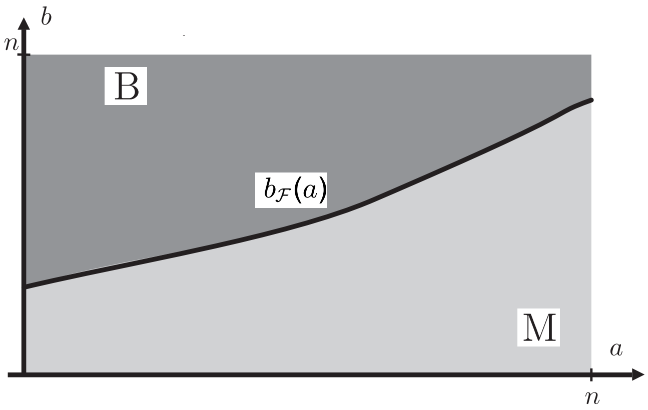


Figure 1.3: Generalized threshold bias $b_{\mathcal{F}}(a)$

Slika 1.3: Opšti granični bias $b_{\mathcal{F}}(a)$

Similarly to the $(1 : b)$ game $(X, \mathcal{F})$, one can define the *generalized threshold bias* for the $(a : b)$ game $(X, \mathcal{F})$ as well. Given a non-degenerate Maker-Breaker game $(X, \mathcal{F})$ and $a \geq 1$, let $b_{\mathcal{F}}(a)$ be the unique non-negative integer such that the $(a : b)$ game $(X, \mathcal{F})$ is Maker's win if and only if $b \leq b_{\mathcal{F}}(a)$ (see Figure 1.3). It is very difficult to determine the exact value of the threshold bias, so we often have the lower and the upper bound for it.

### 1.3.1 The tools in Maker-Breaker games

Some results on strong games have implications in weak games and vice versa. First player's win in a strong game $(X, \mathcal{F})$ immediately implies Maker's win in Maker-Breaker game $(X, \mathcal{F})$. Also, Breaker's win in Maker-Breaker game $(X, \mathcal{F})$ implies second player's drawing strategy in strong game $(X, \mathcal{F})$.

Some tools that are used in strong games can be used in weak games as well. For example, the Ramsey type argument can be applied in Maker-Breaker games for the existence of Maker's strategy, since it provides the existence of the winning strategy of the first player. Also, the pairing strategy in strong game is just the implication of the pairing strategy of Breaker in Maker-Breaker game.

However, Strategy Stealing argument cannot be used in the weak games. The reason for that is simple; different goals of players. So, there is no use in stealing the strategy of the other players when they both have different goals.

Apart from the aforementioned tools there are many others. One of the first results for Breaker's win in the $(1 : 1)$ game is Erdős-Selfridge Theorem.

**Theorem 1.5** ([32], *Erdős-Selfridge Theorem*)**.** *If*

$$\sum_{A\in\mathcal{F}} \frac{1}{2^{|A|}} < \frac{1}{2},$$

*then Breaker (as the second player) has the strategy in the* $(1 : 1)$ *game* $(X, \mathcal{F})$*. When Breaker is the first player, then he wins if* $\sum_{A\in\mathcal{F}} 2^{-|A|} < 1$.

Theorem 1.5 gives not only the criterion for Breaker's win, but also gives an explicit winning strategy for Breaker. Moreover, its power lies in its applicability to wide variety of games since it is independent of the board size and of the structure of the winning sets. It only depends on the sizes of the winning sets.

In [9], József Beck gave the generalized version of Theorem 1.5.

**Theorem 1.6** ([9], *Biased Erdős-Selfridge Theorem*)**.** *Let* $a$ *and* $b$ *be positive integers and let* $(X, \mathcal{F})$ *be a positional game. If*

$$\sum_{A\in\mathcal{F}} (1+b)^{-|A|/a} < \frac{1}{1+b},$$

*then Breaker (as the second player) has the winning strategy in the* $(a : b)$ *Maker-Breaker game* $(X, \mathcal{F})$*. If Breaker is the first player, then the condition* $\sum_{A\in\mathcal{F}}(1+b)^{-|A|/a} < 1$ *ensures Breaker's win.*

There is another criterion for determining Breaker's win which is simpler, but not as applicable as Theorem 1.6.

**Theorem 1.7** ([17], *Degree Criterion for Pairing*). *Let $(X, \mathcal{F})$ be a positional game, where $\mathcal{F}$ is an $n$-uniform hypergraph, i.e. $|A| = n$, for every $A \in \mathcal{F}$. If maximum degree is at most $n/2$, $\Delta(\mathcal{F}) \leq n/2$, then Breaker has a winning (pairing) strategy in game $(X, \mathcal{F})$.*

Next, we give a general criterion for a Maker's win in an unbiased (fair) game.

**Theorem 1.8** ( [17], *Maker's wining criterion*). *Let $(X, \mathcal{F})$ be a positional game. Let $\Delta_2(\mathcal{F})$ denote the max-pair degree of $\mathcal{F}$, i.e. $\Delta_2(\mathcal{F}) = \max\{|\{A \in \mathcal{F} : \{x, y\} \subseteq A\}| : \{x, y\} \in X\}$. If*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} > \frac{1}{8} \Delta_2(\mathcal{F})|X|,$$

*then Maker has a winning strategy in the $(1 : 1)$ game $(X, \mathcal{F})$.*

Theorem 1.8 can be generalized as follows.

**Theorem 1.9** ([17], *Biased Maker's winning criterion*). *Let $a$ and $b$ be positive integers and let $(X, \mathcal{F})$ be a positional game. Let $\Delta_2(\mathcal{F})$ denote the max-pair degree of $\mathcal{F}$, i.e. $\Delta_2(\mathcal{F}) = \max\{|\{A \in \mathcal{F} : \{x, y\} \subseteq A\}| : \{x, y\} \in X\}$. If*

$$\sum_{A \in \mathcal{F}} \left(\frac{a+b}{a}\right)^{-|A|} > \frac{a^2 \cdot b^2}{(a+b)^3} \Delta_2(\mathcal{F})|X|,$$

*then Maker has a winning strategy in the $(a : b)$ Maker-Breaker game $(X, \mathcal{F})$.*

**Box game**

When dealing with disjoint winning sets, we are talking about the so-called Box game. The Box game was first introduced by Chvátal and Erdős in [30]. A hypergraph $\mathcal{H}$ is said to be of type $(k, t)$ if $|\mathcal{H}| = k$, its hyperedges

$e_1, e_2, \ldots, e_k$ are pairwise disjoint, and the sum of their sizes is $\sum_{i=1}^{k} |e_i| = t$. Moreover, the hypergraph $\mathcal{H}$ is said to be *canonical* if $||e_i| - |e_j|| \leq 1$ holds for every $1 \leq i, j \leq k$. The board of the Box game $B(k, t, a, b)$ is a canonical hypergraph of type $(k, t)$. This game is played by two players, called *BoxMaker* and *BoxBreaker*, with BoxBreaker having the first move. BoxMaker claims $a$ vertices of $\mathcal{H}$ per move, whereas BoxBreaker claims $b$ vertices of $\mathcal{H}$ per move. BoxMaker wins the Box game on $\mathcal{H}$ if he can claim all vertices of some hyperedge of $\mathcal{H}$, otherwise BoxBreaker wins this game. Hyperedges are also referred to as *boxes* and their vertices as *elements of the boxes*. One example of canonical hypergraph is shown on Figure 1.4.

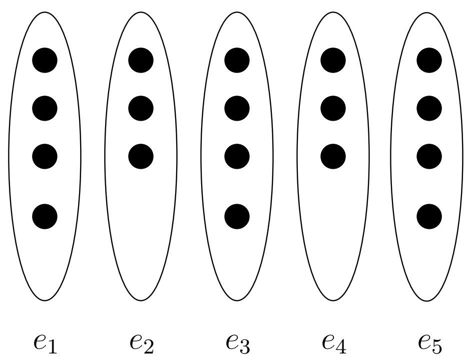


Figure 1.4: The canonical hypergraph $\mathcal{H}$ of type $(5, 18)$

Slika 1.4: Kanonički hipergraf $\mathcal{H}$ tipa $(5, 18)$

Chvátal and Erdős [30] studied the case when $a \geq 1$ and $b = 1$. In order to give a criterion for BoxMaker's win in $B(k, t, a, 1)$, the following recursive function was defined in [30]

$$f(k, a) := \begin{cases} 0, & k = 1 \\ \lfloor \frac{k(f(k-1,a)+a)}{k-1} \rfloor, & k \geq 2. \end{cases}$$

The value of $f(k,a)$ can be approximated as

$$(a-1)k\sum_{i=1}^{k}\frac{1}{i} \leq f(k,a) \leq ak\sum_{i=1}^{k}\frac{1}{i}.$$

The following theorem establishes the criterion for BoxMaker's win in $B(k,t,a,1)$. We note that the proof in [30] contained an error, which was corrected by Hamidoune and Las Vergnas in [50], but the statement formulated in [30] is correct.

**Theorem 1.10** ([30], *Box Maker's win*)**.** *Let $a$, $k$ and $t$ be positive integers. BoxMaker has a winning strategy in $B(k,t,a,1)$ if and only if $t \leq f(k,a)$.*

## 1.4 Avoider-Enforcer games

Avoider-Enforcer games are a *misére* version of Maker-Breaker games. In *misére* games the rules of play are opposite to the regular rules—namely, the rule for Maker to win in a Maker-Breaker game becomes the rule for Avoider to lose in the corresponding Avoider-Enforcer game. While Maker tends to *make* certain given property (thus its name) and Breaker aims at *breaking* the game for Maker, Avoider wants to *avoid* creating a certain property in his graph and Enforcer *forces* Avoider to create the required property. Some results about this type of games can be found in e.g. [3, 4, 6, 40, 55, 57, 65].

In an $(a:b)$ Avoider-Enforcer game $(X,\mathcal{F})$, $a,b \geq 1$, Avoider claims $a$ elements of $X$ each turn, and Enforcer claims $b$ elements of $X$ each turn. If there are less elements of the board than a bias of the player, he has to claim all the elements. In this setting, the family $\mathcal{F}$ is called the family of *losing sets*, as Avoider loses the game if he claims all the elements of any $A \in \mathcal{F}$. Otherwise, Avoider wins.

Very important thing about Maker-Breaker games is their bias monotonicity. The following simple example shows that Avoider-Enforcer games are not bias monotone in their original setting.

**Example 1.11.** Let $(X,\mathcal{F})$ be the hypergraph of game, as shown on the Figure 1.5. There are three losing sets consisting of two elements each, with

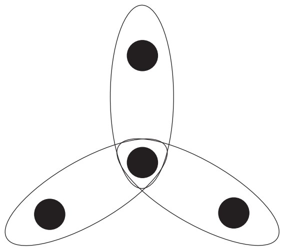

Figure 1.5: The hypergraph of the game

Slika 1.5: Hipergraf igre

one common element. Let us denote it by $v$. Suppose Avoider starts the game. If we consider a $(1 : 1)$ Avoider-Enforcer game on this hypergraph, Avoider can claim any $x \in X \setminus \{v\}$. No matter which element Enforcer claims in his first move, there is at least one element $x \neq v$ that Avoider can claim and win. Now, let us look at the $(1 : 2)$ game. Since the board is symmetric, there are two options for the first move: $v$ or any other element different from $v$. No matter how Avoider plays his first move, there are at least two elements of the board different from $v$ which Enforcer will claim in his move, and Avoider is forced to claim the remaining element that is in the same losing set as the one he claimed in his first move. So, Enforcer wins. The same situation is with $(2 : 1)$ game. Avoider has to select two elements different from $v$ in his first move, but then Enforcer will claim the remaining element that is different from $v$. Avoider is forced to claim $v$ thus losing the game. However, in the $(2 : 2)$ game, Avoider wins the game again. He chooses two elements different from $v$ in his first move. Enforcer has to claim the remaining two elements of the board, thus losing the game.

In general, Avoider-Enforcer games do not have bias monotonicity. This makes their analysis much more difficult, and in fact it is not possible to define the threshold bias in the same manner as in Maker-Breaker games. But, the following can be defined, which was introduced by Hefetz, Krivelevich and Szabó in [57]: let $(X, \mathcal{F})$ be an Avoider-Enforcer game. The *lower*


Figure 1.6: Upper and lower threshold bias

Slika 1.6: Gornji i donji granični bias

*threshold bias* is defined to be the largest integer $f^-_{\mathcal{F}}$ such that for every $b \leq f^-_{\mathcal{F}}$, the $(1 : b)$ game is an Enforcer's win. The *upper threshold bias* $f^+_{\mathcal{F}}$ is the smallest non-negative integer such that for every $b > f^+_{\mathcal{F}}$ the $(1 : b)$ game is an Avoider's win. Except in some trivial cases, $f^-_{\mathcal{F}}$ and $f^+_{\mathcal{F}}$ always exist and it always holds that $f^-_{\mathcal{F}} \leq f^+_{\mathcal{F}}$ (see Figure 1.6). When $f^-_{\mathcal{F}} = f^+_{\mathcal{F}}$ we call this number $f_{\mathcal{F}}$ and refer to it as the *threshold bias* of the game.

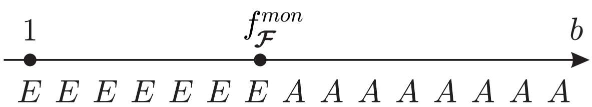


Figure 1.7: Monotone threshold bias $f^{mon}_{\mathcal{F}}$

Slika 1.7: Monotoni granični bias $f^{mon}_{\mathcal{F}}$

To overcome the non-monotonicity obstacle, Hefetz, Krivelevich, Stojaković and Szabó in [55] suggested a bias monotone version of Avoider-Enforcer games. In monotone $(a : b)$ Avoider-Enforcer game, Avoider and Enforcer claim at least $a$, respectively at least $b$, elements of the board in each move. It is easy to check that this is indeed a bias monotone game. So, if an $(a : b)$ game is an Enforcer's win, then Enforcer also wins in $(a + 1 : b)$ and $(a : b - 1)$ games. If an $(a : b)$ game is an Avoider's win, so are the $(a-1 : b)$ and $(a : b+1)$ games. The new rules enabled defining the unique *monotone threshold bias* $f^{mon}_{\mathcal{F}}$ for the game $(X, \mathcal{F})$ as the largest non-negative integer such that Enforcer wins the $(1 : b)$ Avoider-Enforcer game $(X, \mathcal{F})$ if and only if $b \leq f_{\mathcal{F}}^{mon}$ holds (see Figure 1.7).

Throughout the thesis we refer to this new set of rules as the *monotone* rules, to distinguish them from the original, *strict*, rules. Accordingly, we refer to the games played under each set of rules as monotone games and as strict games.

Interestingly, these seemingly minor adjustments in the rules can completely change the outcome of the game. For example, let us look at the $(1 : b)$ Connectivity game played on $E(K_n)$. In the strict game, the threshold bias exists and is of linear order [57]. On the other hand, the asymptotic monotone threshold bias for this game is $\Theta(\frac{n}{\ln n})$

So, which set of rules is better? The advantage of strict games is in their applicability to Maker-Breaker games (see e.g. [53]) and discrepancy games (e.g. [17,58]). However, the outcome (and thus analysis) of the games depends on how large the remainder of integer division of $|X|$ by $1 + b$ is. On the other hand, the advantage of monotone version of the games is that there exists a unique threshold bias.

### 1.4.1 The tools in Avoider-Enforcer games

There are some winning criteria established for Avoider-Enforcer games.

Hefetz, Krivelevich and Szabó in [57] gave a general criterion for Avoider's win in the $(a : b)$ game $(X, \mathcal{F})$.

**Theorem 1.12** ([57], Theorem 1.1)**.** *If*

$$\sum_{A \in \mathcal{F}} \left(1 + \frac{1}{a}\right)^{-|A|+a} < 1,$$

*then Avoider wins the biased* $(a : b)$ *game* $(X, \mathcal{F})$*, both strict and monotone, for every* $b \geq 1$*.*

This criterion depends only on Avoider's bias $a$ and does not take $b$ into account, which is not so effective when $b$ is large.

Recently, Bednarska-Bzdęga in [19] gave another criterion for Avoider's win in both monotone and strict games played on the hypergraphs with small rank. *Rank* of a hypergraph $\mathcal{F}$ is $rank(\mathcal{F}) = \max_{A \in \mathcal{F}} |A|$.

**Theorem 1.13** ([19], Theorem 1.2 (i)). *Let $(X, \mathcal{F})$ be a hypergraph of rank $r$. If*

$$\sum_{A \in \mathcal{F}} \left(1 + \frac{b}{ar}\right)^{-|A|+a} < 1,$$

*then Avoider has a winning strategy in both monotone and strict $(a : b)$ Avoider-Enforcer games $(X, \mathcal{F})$.*

## 1.5 Games on graphs

It is very natural to play both Maker-Breaker and Avoider-Enforcer games on the edge set of a given graph $G$. In this case, the board is $X = E(G)$, and the target sets are all the edge sets of subgraphs $H \subseteq G$ which possess some given monotone increasing graph property $\mathcal{P}$. For example: in the *Connectivity* game $\mathcal{T}(G)$, the target sets are all edge sets of spanning trees of $G$; in the *Perfect matching* game $\mathcal{M}(G)$ the target sets are all sets of $\lfloor |V(G)|/2 \rfloor$ independent edges of $G$; in the *Hamiltonicity* game $\mathcal{H}(G)$ the target sets are all edge sets of Hamilton cycles of $G$. When $G = K_n$, we denote these games $\mathcal{T}_n$, $\mathcal{M}_n$ and $\mathcal{H}_n$. These three games were initially studied by Chvátal and Erdős in their seminal paper [30], for $G = K_n$, the complete graph on $n$ vertices. They proved that Breaker can win all these $(1 : b)$ games by isolating a vertex in Maker's graph, provided that $b \geq \frac{(1+\varepsilon)n}{\ln n}$ for any $\varepsilon > 0$, and showed that the threshold bias for the Connectivity game is between $(1/4 - \varepsilon)n/\ln n$ and $(1 + \varepsilon)n/\ln n$. They conjectured that the upper bound is in fact asymptotically best possible. This was verified by Gebauer and Szabó [46].

Chvátal and Erdős [30] proved that the $(1 : 1)$ game $\mathcal{H}_n$ is a Maker's win for sufficiently large $n$. They conjectured that the threshold bias $b_{\mathcal{H}_n}$ tends to infinity as $n$ tends to infinity. This conjecture was verified by Bollobás and Papaioannou [27] giving the Maker's strategy to win $\mathcal{H}_n$ game against Breaker with bias $b = O\left(\frac{\ln n}{\ln \ln n}\right)$. Beck in [9] gave the explicit winning strategy for Maker's win in $(1 : b)$ for all $b < \left(\frac{\ln 2}{27} - o(1)\right) \frac{n}{\ln n}$ for sufficiently large $n$. This bound was later improved to $(\ln 2 - o(1))\frac{n}{\ln n}$ by Krivelevich and Szabó [63], and recently Krivelevich [62] showed that Maker wins in $(1:b)$ game $\mathcal{H}_n$ for all $b \leq \left(1 - \frac{30}{\ln^{1/4} n}\right) \frac{n}{\ln n}$, so the leading term of the threshold bias is $\frac{n}{\ln n}$. From this result, we obtain that the threshold bias in $(1:b)$ Perfect matching game, $\mathcal{M}_n$, is $\Theta(\frac{n}{\ln n})$.

Assume that the $(1:b)$ game $\mathcal{T}_n$ is being played, but instead of playing optimally, both players play randomly (they will thus be referred to as *RandomMaker* and *RandomBreaker*, and the resulting game will be referred to as the *Random Connectivity game*). It follows that the graph built by RandomMaker by the end of the game is a random graph $G(n, \lfloor \binom{n}{2}/(b+1) \rfloor)$. It is well known that almost surely such a graph is connected if $\lfloor \binom{n}{2}/(b+1) \rfloor \geq (1/2+\varepsilon)n \ln n$ and disconnected if $\lfloor \binom{n}{2}/(b+1) \rfloor \leq (1/2-\varepsilon)n \ln n$. Hence, almost surely RandomBreaker wins the game if $b \geq (1+\tilde{\varepsilon})n/\ln n$ but loses if $b \leq (1-\tilde{\varepsilon})n/\ln n$, just like when both players play optimally. The same holds for $(1:b)$ game $\mathcal{H}_n$ when it is played randomly. The number of edges that RandomMaker claims by the end of the game when playing $\mathcal{H}_n$ against RandomBreaker who plays with bias $b = b_{\mathcal{H}_n} = \frac{(1-\varepsilon)n}{\ln n}$ is $(1/2+\varepsilon)n \ln n$. Random graph $G(n,m)$, for $m = (1/2+\varepsilon)n \ln n$, build by RandomMaker in *Random Hamiltonicity game* is Hamiltonian [25]. This remarkable relation between positional games and random graphs, first observed in [30], has come to be known as the *probabilistic intuition* or *Erdős paradigm*. At the end of such random game Maker's graph $G_M$ satisfies $G_M \sim G(n,m)$, which is in many ways similar to $G(n,p)$, with $p = \frac{1}{b}$ (the graph on $n$ vertices where each potential edge appears in the graph independently with probability $\frac{1}{b}$), so the threshold bias for the random game where Maker wishes to acquire some graph property $\mathcal{P}$ approximately equals the reciprocal of the threshold probability for the appearance of $\mathcal{P}$ in $G(n,p)$. Much of the research in the theory of positional games has since been devoted to finding the threshold bias of certain games and investigating the probabilistic intuition. Many of these results can be found in [17].

It is also interesting to play the aforementioned games in Avoider-Enforcer setting. The results of Hefetz et al. [55] together with the results of Krivelevich and Szabó [63] show that when Connectivity game $\mathcal{T}_n$, Hamiltonicity game $\mathcal{H}_n$ and Perfect matching game $\mathcal{M}_n$ are played by monotone rules, it holds that

$$f^{mon}_{\mathcal{T}_n}, f^{mon}_{\mathcal{H}_n}, f^{mon}_{\mathcal{M}_n} = \Theta\left(\frac{n}{\ln n}\right).$$

The results in the strict versions of these games differ from the ones obtained for the monotone games. Avoider wins the strict $(1 : b)$ Connectivity game played on $E(K_n)$ if and only if at the end of the game he has at most $n-2$ edges, therefore the threshold bias exists and is of linear order [57]. In the strict $(1 : b)$ Hamiltonicity game and Perfect matching game, the lower threshold biases are $f^{-}_{\mathcal{H}_n} = (1-o(1))\frac{n}{\ln n}$ [63] and $f^{-}_{\mathcal{M}_n} = \Omega\left(\frac{n}{\ln n}\right)$ [57], respectively. In both of these games, the only upper bounds that we know are the trivial ones.

In [55], Hefetz et al. investigated $(1 : b)$ Avoider-Enforcer games played on the edge set of $K_n$, where Avoider wants to avoid claiming a copy of some fixed graph $H$. In this case $X = E(K_n)$, and $\mathcal{F} = \mathcal{K}_H \subseteq 2^{E(K_n)}$ consists of all copies of $H$ in $K_n$. This game is referred to as $H$-game. They conjectured that for any fixed graph $H$, the thresholds $f^{-}_{\mathcal{K}_H}$ and $f^{+}_{\mathcal{K}_H}$ are not of the same order, and wondered about the connection between the monotone $H$-game and the strict $H^-$-game, where $H^-$ is $H$ with one edge missing. They investigated $H$-games where $H = K_3$ (a triangle) and $H = P_3 = K_3^-$ (a path on three vertices) and established the following:

$$f^{mon}_{\mathcal{K}_{P_3}} = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor - 1,\ f^{+}_{\mathcal{K}_{P_3}} = \binom{n}{2} - 2,\ f^{-}_{\mathcal{K}_{P_3}} = \Theta(n^{\frac{3}{2}}) \text{ and } f^{mon}_{\mathcal{K}_{K_3}} = \Theta(n^{\frac{3}{2}}).$$

This example supports their conjecture, as $f^{+}_{\mathcal{K}_{P_3}}$ and $f^{-}_{\mathcal{K}_{P_3}}$ are indeed not of the same order, while $f^{mon}_{\mathcal{K}_{K_3}}$ and $f^{-}_{\mathcal{K}_{P_3}}$ are of the same order. They also wondered about the results for $H$-games where $|V(H)| > 3$. Bednarska-Bzdęga established in [19] general upper and lower bounds on $f^{+}_{\mathcal{K}_H}$, $f^{-}_{\mathcal{K}_H}$ and $f^{mon}_{\mathcal{K}_H}$ for every fixed graph $H$, but these bounds are not tight.

## 1.6 Winning fast in positional games on graphs

In strong positional games, as we already saw, either the first player wins, or the second has a drawing strategy. However, in this type of games, it is very difficult to find an explicit winning strategy, since the known tools provide criteria only for determining the existence of a winning strategy. This difficulty partly initiated the study of the weak games—Maker-Breaker and their misére version Avoider-Enforcer games.

Since draw is impossible in the weak games, the game $\mathcal{F}$ is the win for one player. In several (unbiased) standard games $\mathcal{F}$ on $E(K_n)$ it is the case that Maker or Enforcer can win quite easily. Taking that into consideration, one of the interesting questions that arise is *how fast* can Maker, respectively Enforcer, win the game $\mathcal{F}$. Parameters $\tau_M(\mathcal{F})$ and $\tau_E(\mathcal{F})$ represent the shortest *duration* of the Maker-Breaker, respectively Avoider-Enforcer, game $\mathcal{F}$ and are defined as the smallest integer $t$ such that Maker, respectively Enforcer, can win the game $\mathcal{F}$ in $t$ moves. If Breaker, respectively Avoider, wins in $\mathcal{F}$, then $t = \infty$.

Fast winning strategies for Maker-Breaker games are studied in [28, 33, 36, 37, 39, 54, 61, 69] and some results about fast strategies in Avoider-Enforcer games can be found in [3,4,6,52]. Apart from being interesting on their own, fast strategies can be very useful. Namely, the (almost) perfectly fast Maker can help first player to win the strong game. For example, we know that Maker can win in the unbiased Connectivity game, $\mathcal{T}_n$, played on the edge set of complete graph on $n$ vertices (or even any graph $G$ that contains two disjoint spanning trees with $n$ vertices), in $n-1$ moves, so $\tau_M(\mathcal{T}_n) = n-1$, as Maker needs $n-1$ edges for a spanning tree, and Lehman in [64] established the upper bound by giving a strategy that does not involve creating cycles. The same strategy works for the first player in the strong game, as the second player obviously cannot create a spanning tree in less than $n-1$ moves.

In the *Minimum degree* $k$ game, played on the edge set of $K_n$, the winning sets are edge sets of all graphs on $n$ vertices with minimum degree at least $k$, for $k \geq 1$. Hefetz et al. showed in [54] that for the $(1:1)$ Perfect matching game $\mathcal{M}_n$, $\tau_M(\mathcal{M}_n) = \lfloor \frac{n}{2} \rfloor$, when $n$ is odd and $\tau_M(\mathcal{M}_n) = \frac{n}{2}+1$, when $n$ is even. From this result, it can be easily obtained that Maker wins the Minimum degree 1 game in $\lfloor \frac{n}{2} \rfloor + 1$ moves, as in case when $n$ is odd Maker needs one more edge. In [61], Heftez and Stich proved that for the unbiased Hamiltonicity game, $\mathcal{H}_n$, $\tau_M(\mathcal{H}_n) = n + 1$. In [44], Gebauer proved that in the $(1:1)$ *$q$-clique game*, $\mathcal{K}_n^q$, where the winning sets are all copies of $K_q$ on $K_n$, $\tau_M(\mathcal{K}_n^q) = 2^{\frac{2q}{3}} poly(q)$. Note here that duration does not depend on $n$, but only on the number of vertices in the clique, $q$. Also, Ferber and Hefetz [37] recently showed that in the $(1:1)$ $k$-connectivity game on $E(K_n)$, $k \geq 1$, where the winning sets are edge sets of all graphs on $n$ vertices with minimum degree at least $k$, Maker can win in $\lfloor kn/2 \rfloor + 1$ moves, which is tight since the minimum number of edges in such a graph is $\lceil kn/2 \rceil$. This implies that Maker can win the Minimum degree $k$ game, for $k \geq 1$, in at most $\lfloor kn/2 \rfloor + 1$ moves as well.

Using the fact that explicit fast winning strategies for Maker are known in the Perfect matching game, the Hamiltonicity game and the $k$-connectivity game, Ferber and Hefetz gave in [36, 37] fast strategies for first player in the aforementioned strong games.

## 1.7 Preliminaries

Our graph-theoretic notation is standard and follows that of [72]. In particular, we use the following. For a graph $G$, $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A, B)$ denote the set of edges of $G$ with one endpoint in $A$ and one endpoint in $B$, and let $e_G(A, B) = |E_G(A, B)|$. For any subset $S \subseteq V$ we say that an edge $(u, v) \in E$ lies *inside* $S$ if $u, v \in S$. For any subset $S \subseteq V$ we denote by $G[S]$ the *induced* graph on $S$, i.e. the graph with vertex set $S$ and edge set $\{(u, v) \in E : u, v \in S\}$. For disjoint sets $S, T \subseteq V(G)$, let $N_G(S, T) = \{u \in T : \exists v \in S, uv \in E(G)\}$ denote the set of neighbours of the vertices of $S$ in $T$. We let $d_G(w) = |N_G(w)|$ denote the degree of $w$ in $G$. The minimum and maximum degrees of a graph $G$ are denoted by $\delta(G)$ and $\Delta(G)$ respectively. For a set $T \subseteq V(G)$ and a vertex $w \in V(G) \setminus T$ we abbreviate $N_G(\{w\}, T)$ to $N_G(w, T)$, and let $d_G(w,T) = |N_G(w,T)|$ denote the degree of $w$ into $T$. For a set $S \subseteq V(G)$ and a vertex $w \in V(G)$ we abbreviate $N_G(S, V(G) \setminus S)$ to $N_G(S)$ and $N_G(w, V(G) \setminus \{w\})$ to $N_G(w)$. The *closed* neighbourhood of a vertex $v \in V(G)$ is defined as $N_G[v] := N_G(v) \cup \{v\}$. Often, when there is no risk of confusion, we omit the subscript $G$ from the notation above. In a positional game, a previously unclaimed edge is called a *free* edge or an *available* edge. For any fixed graph $H$, the extremal number, $ex(n,H)$, is the maximum possible number of edges in a graph on $n$ vertices which does not contain a copy of $H$.

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize some of the constants obtained in our proofs. We also omit floor and ceiling signs whenever these are not crucial. Most of our results are asymptotic in nature and whenever necessary we assume that $n$ is sufficiently large. Throughout the thesis, ln stands for the natural logarithm.

For every positive integer $j$, we denote the *jth harmonic number* by $H_j$, that is, $H_j = \sum_{i=1}^{j} 1/i$, for every $j \geq 1$. We will make use of the following known fact,

$$\ln j + 1/2 < H_j < \ln j + 2/3, \text{ for sufficiently large } j. \tag{1.7.1}$$

# Chapter 2

# Main results

In this chapter we state the main results of the thesis. They are divided into three sections.

## 2.1 Doubly biased Maker-Breaker games

In Chapter 3, we study Maker-Breaker games played with bias $(a : b)$ on $E(K_n)$. Although $(a : b)$ games, where $a > 1$, were studied less than the case $a = 1$, they are not without merit. Indeed, the small change of going from $a = 1$ to $a = 2$ has a considerable impact on the outcome and the course of play of certain positional games (see [17]). Moreover, it was shown in [5] that the *acceleration* of the so-called *diameter*-2 game partly restores the probabilistic intuition. Namely, it was observed that, while $G(n, 1/2)$ has diameter 2 almost surely, the $(1 : 1)$ diameter-2 game (that is, the board is $E(K_n)$ and the winning sets are all spanning subgraphs of $K_n$ with diameter at most 2) is Breaker's win. On the other hand, it was proved in [5] that the seemingly very similar $(2 : 2)$ game is Maker's win. Further examples of $(a : b)$ games, where $a > 1$, can be found in [5, 17, 44].

Coming back to the Random Connectivity game, its outcome depends on the number of edges RandomMaker has at the end of the game rather than on the actual values of $a$ and $b$. Hence, if $a = a(n)$ and $b = b(n)$ are positive integers satisfying $b \leq a(1-\varepsilon)n/\ln n$, for some constant $\varepsilon > 0$, and $b$ is not too large (clearly if for example $b \geq \binom{n}{2}$, then RandomBreaker wins regardless of the value of $a$), then almost surely RandomMaker wins the game. Similarly, if $a$ is not too large and $b \geq a(1+\varepsilon)n/\ln n$, then almost surely RandomBreaker wins the game. Clearly the outcome of the random game and of the regular game could vary greatly for large values of $a$ and $b$. For example, while Breaker wins the $(\binom{n}{2} : n)$ game $\mathcal{T}_n$ in one move, the corresponding random game is almost surely RandomMaker's win. We prove that for all "reasonable" values of $a$ and $b$, the probabilistic intuition is maintained.

We give the generalized threshold bias in the $(a : b)$ Connectivity game, $\mathcal{T}_n$, and in the $(a : b)$ Hamiltonicity game, $\mathcal{H}_n$. Next theorem gives the bounds on the threshold bias, $b_{\mathcal{T}_n}$, in $(a : b)$ Maker-Breaker Connectivity game.

**Theorem 2.1.** *(i) If $a = o(\ln n)$, then*
$\frac{an}{\ln n} - (1+o(1))\frac{an(\ln\ln n + a)}{\ln^2 n} < b_{\mathcal{T}_n}(a) < \frac{an}{\ln n} - (1-o(1))\frac{an\ln a}{\ln^2 n}$.

*(ii) If $a = c\ln n$ for some $0 < c \leq 1$, then*
$(1-o(1))\frac{cn}{c+1} < b_{\mathcal{T}_n}(a) < \min\left\{cn, (1+o(1))\frac{2n}{3}\right\}$.

*(iii) If $a = c\ln n$ for some $c > 1$, then*
$(1-o(1))\frac{cn}{c+1} < b_{\mathcal{T}_n}(a) < (1+o(1))\frac{2cn}{2c+1}$.

*(iv) If $a = \omega(\ln n)$ and $a = o\left(\sqrt{\frac{n}{\ln n}}\right)$, then*
$n - \frac{n\ln n}{a} < b_{\mathcal{T}_n}(a) < n - (1-o(1))\frac{n\ln(n/a)}{2a}$.

*(v) If $a = \Omega\left(\sqrt{\frac{n}{\ln n}}\right)$ and $a = o(n)$, then*
$n - (1+o(1))\frac{2n\ln(n/a)}{a} < b_{\mathcal{T}_n}(a) < n - (1-o(1))\frac{n\ln(n/a)}{2a}$.

*(vi) If $a = cn$ for $0 < c < \frac{1}{2e}$, then*
$n - \frac{2\ln(1/c)+4}{c} < b_{\mathcal{T}_n}(a) < n - 2 - \frac{1-2c}{2c}\left(\ln(\frac{1}{2c}) - 1\right) + o(1)$.

*(vii) If $a = cn$ for $\frac{1}{2e} \leq c < \frac{1}{2}$, then*
$n - \frac{2\ln(1/c)+4}{c} < b_{\mathcal{T}_n}(a) < n - 2$.

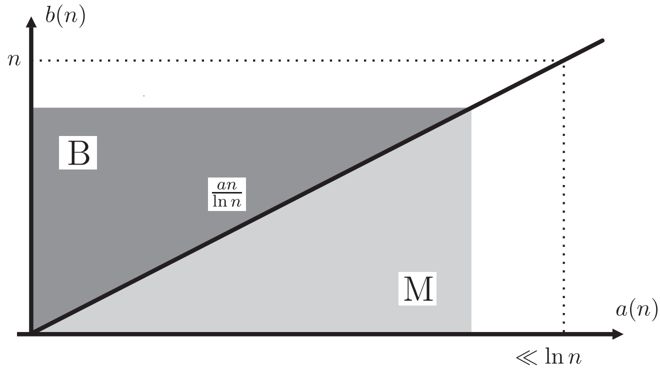


Figure 2.1: Leading term of the threshold biases $b_{\mathcal{T}_n}(a)$ and $b_{\mathcal{H}_n}(a)$ for $a = o(\ln n)$

Slika 2.1: Vodeći član graničnih biasa $b_{\mathcal{T}_n}(a)$ i $b_{\mathcal{H}_n}(a)$ kada je $a = o(\ln n)$

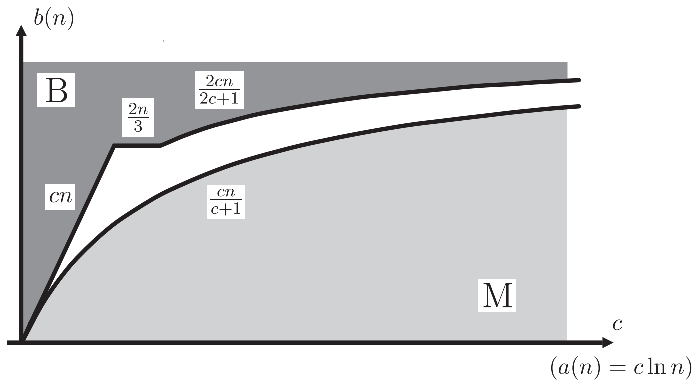


Figure 2.2: Bounds on the threshold biases $b_{\mathcal{T}_n}(a)$ and $b_{\mathcal{H}_n}(a)$ for $a = c \ln n$, where $c$ is a positive real number

Slika 2.2: Granice za granične biase $b_{\mathcal{T}_n}(a)$ i $b_{\mathcal{H}_n}(a)$ kada je $a = c \ln n$, $c > 0$

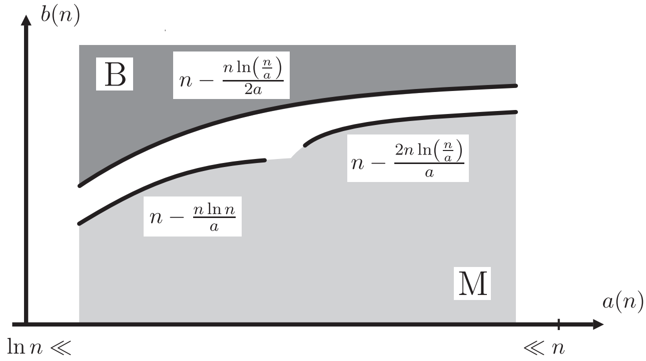


Figure 2.3: Bounds on the threshold bias $b_{\mathcal{T}_n}(a)$ for $a = \omega(\ln n)$ and $a = o(n)$

Slika 2.3: Granice za granični bias $b_{\mathcal{T}_n}(a)$ kada je $a = \omega(\ln n)$ i $a = o(n)$

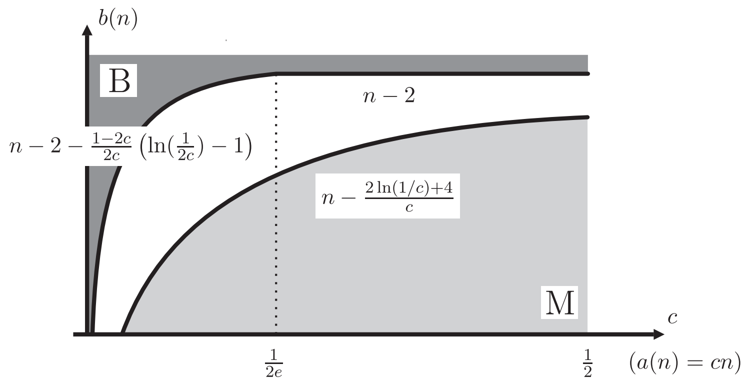


Figure 2.4: Bounds on the threshold bias $b_{\mathcal{T}_n}(a)$ for $a = cn$, where $0 < c < 1/2$ is a real number

Slika 2.4: Granice za granični bias $b_{\mathcal{T}_n}(a)$ kada je $a = cn$, $0 < c < 1/2$

Theorem 2.1 gives fairly tight bounds for the threshold bias on the whole range of the bias $a$. In particular, for $a = o(\ln n)$, the leading term of the threshold bias $b_{\mathcal{T}_n}(a)$ is determined exactly; this is depicted in Figure 2.1. Then, if $a = c \ln n$, where $c$ is a positive real number, $(ii)$ and $(iii)$ imply that the threshold biases are linear in $n$, and the upper bound we obtain is a constant factor away from the lower bound, as shown in Figure 2.2. If $a = \omega(\ln n)$ and $a = o(n)$, it follows by $(iv)$ and $(v)$ that the leading term of the threshold bias is $n$, and moreover, we obtain upper and lower bounds for the second order term which are a constant factor away from each other, see Figure 2.3. Finally, for $a = cn$, where $0 < c < 1/2$ is a real number, $(vi)$ and $(vii)$ imply that the threshold bias is just an additive constant away from $n$ as shown in Figure 2.4. For larger values of $a$ we have the trivial upper bound of $b_{\mathcal{T}_n}(a) < n - 1$ and the same lower bound as in $(vii)$ by monotonicity.

For the threshold bias $b_{\mathcal{H}_n}$, we obtain the following bounds.

**Theorem 2.2.** $(i)$ *If $a = o(\ln n)$, then*
$$\frac{an}{\ln n} - (1 - o(1))\frac{an(30\ln^{3/4} n + a)}{\ln^2 n} < b_{\mathcal{H}_n}(a) < \frac{an}{\ln n} - (1 - o(1))\frac{an \ln a}{\ln^2 n}.$$

$(ii)$ *If $a = c \ln n$ for some $0 < c \le 1$, then*
$$\frac{cn}{c+1}\left(1 - \frac{30}{\ln^{1/4} n}\right) < b_{\mathcal{H}_n}(a) < \min\left\{cn, (1 + o(1))\frac{2n}{3}\right\}.$$

$(iii)$ *If $a = c \ln n$ for some $c > 1$, then*
$$\frac{cn}{c+1}\left(1 - \frac{30}{\ln^{1/4} n}\right) < b_{\mathcal{H}_n}(a) < (1 + o(1))\frac{2cn}{2c+1}.$$

$(iv)$ *If $a = \omega(\ln n)$ and $a = o(\ln^{5/4} n)$, then*
$$n - (1 + o(1))\frac{n \ln n}{a} < b_{\mathcal{H}_n}(a) < n - (1 - o(1))\frac{n \ln(n/a)}{2a}.$$

$(v)$ *If $a = c \ln^{5/4} n$, $c > 0$, then*
$$n - (1 - o(1))\frac{(30c+1)n}{c \ln^{1/4} n} < b_{\mathcal{H}_n}(a) < n - (1 - o(1))\frac{n}{2c \ln^{1/4} n}.$$

$(vi)$ *If $a = \omega(\ln^{5/4} n)$ and $a = o(n)$, then*
$$n - \frac{30n}{\ln^{1/4} n} - (1 + o(1))\frac{n \ln n}{a} < b_{\mathcal{H}_n}(a) < n - (1 - o(1))\frac{n \ln(n/a)}{2a}.$$

$(vii)$ *If $a = cn$ for $0 < c < \frac{1}{2e}$, then*
$$n - \frac{30n}{\ln^{1/4} n} - \frac{\ln n}{c} < b_{\mathcal{H}_n}(a) < n - 2 - \frac{1-2c}{2c}\left(\ln\left(\frac{1}{2c}\right) - 1\right) + o(1).$$

*(viii) If* $a = cn$ *for* $\frac{1}{2e} \le c < 1$*, then*
$n - \frac{30n}{\ln^{1/4} n} - \frac{\ln n}{c} < b_{\mathcal{H}_n}(a) < n - 2.$

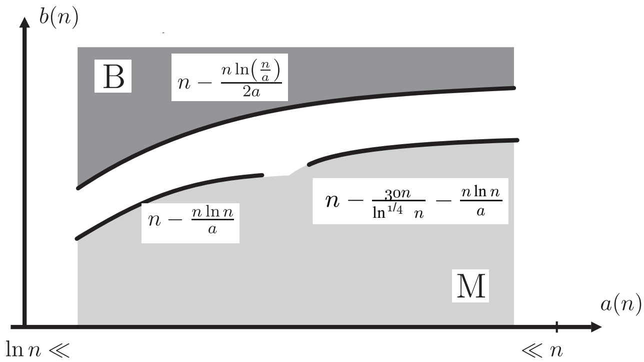


Figure 2.5: Bounds on the threshold bias $b_{\mathcal{H}_n}(a)$ for $a = \omega(\ln n)$ and $a = o(n)$

Slika 2.5: Granice za granični bias $b_{\mathcal{H}_n}(a)$ kada je $a = \omega(\ln n)$ i $a = o(n)$

Theorem 2.2 also gives fairly tight bounds for the threshold bias $b_{\mathcal{H}_n}(a)$ on the whole range of the bias $a$. In particular, for $a = o(\ln n)$, the leading term of the threshold bias $b_{\mathcal{H}_n}(a)$ is determined exactly and is equal to the leading term of the threshold bias $b_{\mathcal{T}_n}(a)$; this is depicted in Figure 2.1. Then, if $a = c \ln n$, where $c$ is a positive real number, $(ii)$ and $(iii)$ imply that the threshold biases are linear in $n$, and the upper bound we obtain is a constant factor away from the lower bound (just like in case of $b_{\mathcal{T}_n}(a)$), as shown in Figure 2.2. If $a = \omega(\ln n)$ and $a = o(\ln^{5/4} n)$, it follows by Theorem 2.2 $(iv)$ that the leading term of the threshold bias is $n$ as well and upper and lower bounds for the second order term are a constant factor away from each other, see Figure 2.5. When $a = c \ln^{5/4} n$, and $c > 0$ is a constant, by $(v)$ we obtain the same first order term $n$ and second order term differ by a constant factor. However, when $a = \omega(\ln^{5/4} n)$ and $a = o(n)$, first order term of the threshold bias is $n$, but second order term of the

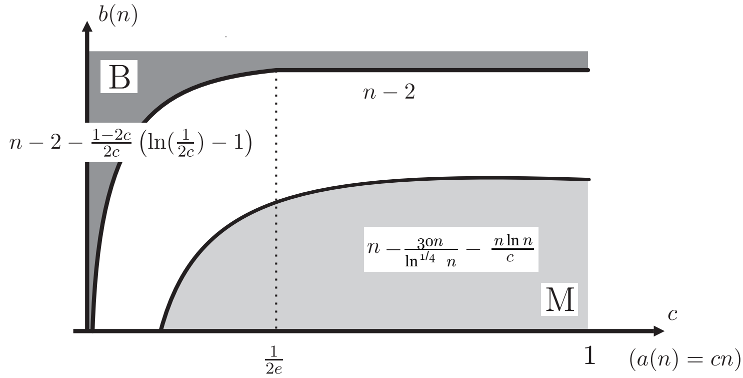


Figure 2.6: Bounds on the threshold bias $b_{\mathcal{H}_n}(a)$ for $a = cn$, where $0 < c < 1$ is a real number

Slika 2.6: Granice za granični bias $b_{\mathcal{H}_n}(a)$ kada je $a = cn$, $0 < c < 1$

lower bound is fixed to be $\frac{30n}{\ln^{1/4} n}$. So, there is a gap from then on in the lower and upper threshold bias. When $a = cn$, where $0 < c < 1$ is a real number, $(vii)$ and $(viii)$ from Theorem 2.2 imply that the second order term in the lower bound for the threshold bias $b_{\mathcal{H}_n}$ is $\frac{30n}{\ln^{1/4} n}$ away from $n$ and the upper bound is just an additive constant away from $n$, as shown in Figure 2.6.

## 2.2 Avoider-Enforcer star games

In Chapter 4, we are interested in monotone and strict $H$-games played on the edges of the complete graph $K_n$, where $H$ is a $k$-star $\mathcal{S}_k = K_{1,k}$, for some fixed $k \geq 3$ (note that $\mathcal{S}_2 = P_3$, so the case $k = 2$ is already covered in [55]). Note also that avoiding a $k$-star is exactly keeping the maximal degree in Avoider's graph strictly under $k$. We refer to this game as the *star game*, or more specifically, for a given $k$, we call this game the *$k$-star game*. We show that for any given $k \geq 3$, $f^-_{\mathcal{K}_{\mathcal{S}_k}}$ and $f^+_{\mathcal{K}_{\mathcal{S}_k}}$ are not of the same order for any sufficiently large $n$. In addition, as $\mathcal{S}_k^- = \mathcal{S}_{k-1}$, an immediate consequence is that $f^{mon}_{\mathcal{K}_{\mathcal{S}_k}}$ and $f^-_{\mathcal{K}_{\mathcal{S}_k^-}}$ are of the same order for infinitely many values of $n$.

**Theorem 2.3.** *Let $k \geq 3$. In the $(1:b)$ $k$-star game $\mathcal{K}_{\mathcal{S}_k}$ we have*

*(i)* $f^{mon}_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k}{k-1}})$;

*(ii)* $f^{+}_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k}{k-1}})$ *holds for infinitely many values of $n$;*

*(iii)* $f^{-}_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k+1}{k}})$ *holds for infinitely many values of $n$.*

We also study one related monotone $H$-game played on $E(K_n)$, where $H$ is a $K_{2,2}$. Note here that $K_{2,2} = C_4$ and also that it can be seen as two stars $K_{1,2}$ that are "glued" together via their leaves. The hypergraph of the game is denoted by $\mathcal{K}_{K_{2,2}}$.

**Theorem 2.4.** *In the monotone $K_{2,2}$-game, it holds that*

$$\frac{1}{4}n^{\frac{4}{3}} < f^{mon}_{\mathcal{K}_{K_{2,2}}} < n^{\frac{4}{3}}.$$

In Chapter 1 (Section 1.5), we already mentioned that Bednarska-Bzdęga obtained bounds on the different threshold biases in $H$-game for arbitrary fixed graph $H$. Her bounds depend on the following three parameters of $H$:

$$m(H) = \max_{F \subseteq H : v(F) \geq 1} \frac{e(F)}{v(F)}; \qquad m'(H) = \max_{F \subseteq H : v(F) \geq 1} \frac{e(F)-1}{v(F)};$$

$$m''(H) = \max_{F \subseteq H : v(F) \geq 3} \frac{e(F)+1}{v(F)-2}.$$

She proved that for an arbitrary graph $H$, a positive real $\varepsilon$ and a large enough integer $n$ the inequalities $f^{mon}_{\mathcal{K}_H}, f^{+}_{\mathcal{K}_H} = O(n^{1/m'(H)})$ and $\Omega(n^{1/m''(H)-\varepsilon}) = f^{-}_{\mathcal{K}_H} = O(n^{1/m(H)} \ln n)$ hold. Our results show that these bounds are far from being tight for the star game. For the $K_{2,2}$-game, obtained results match her bounds. However, at least the upper bound on $f^{mon}_{\mathcal{K}_H}$ cannot be improved in general, since it is tight for the case $H = K_3$, as observed by Bednarska-Bzdęga in [19].

## 2.3 Fast biased Maker-Breaker games

In Chapter 5, we study biased $(1 : b)$ Perfect matching game and Hamiltonicity game played on $E(K_n)$. Motivated by the results obtained for the unbiased games, we want to see what happens in the biased $(1 : b)$ games, when $b > 1$. The shortest duration of the biased game can be defined analogously as in unbiased games, so let $\tau_M(\mathcal{F}, b)$ denote the least integer $t$ such that Maker wins $(1 : b)$ Maker-Breaker game $(X, \mathcal{F})$ in $t$ moves. When the game $\mathcal{F}$ is Breaker's win, then $\tau_M(\mathcal{F}, b) = \infty$.

Not much is known about fast winning strategies in these games. Strategy of Gebauer and Szabó [46] for $(1 : b)$ Connectivity game is the fastest possible since it gives $\tau_M(\mathcal{T}_n, b) = n - 1$. In [29], the authors have found fast winning strategies for the Perfect matching, Hamiltonicity and $k$-connectivity games played on the random graph $G(n, p)$.

Our main results in this chapter are the following.

**Theorem 2.5.** *In the $(1 : b)$ Maker-Breaker Perfect matching game played on $E(K_n)$, $\mathcal{M}_n$, it holds that*

$$\frac{n}{2} + \frac{b}{4} \leq \tau_M(\mathcal{M}_n, b) \leq \frac{n}{2} + O(b \ln b)$$

*for all $b \leq \frac{\delta n}{100 \ln n}$, where $\delta > 0$ is a small constant.*

**Theorem 2.6.** *In the $(1 : b)$ Maker-Breaker Hamiltonicity game played on $E(K_n)$, $\mathcal{H}_n$, it holds that*

$$n + \frac{b}{2} \leq \tau_M(\mathcal{H}_n, b) \leq n + O(b^2 \ln^5 n)$$

*for all $b \leq \delta \sqrt{\frac{n}{\ln^5 n}}$, where $\delta > 0$ is a small constant.*

For given $b$, we can see that these bounds are asymptotically optimal. In the Perfect matching game, the second order term is $o(n)$ for values of $b$ that we consider.

# Chapter 3

# Doubly biased Maker-Breaker games

In this chapter, our attention is dedicated to the biased $(a : b)$ Maker-Breaker *Connectivity* game $\mathcal{T}_n$ and *Hamiltonicity* game $\mathcal{H}_n$ on $E(K_n)$. Theorem 2.1 and Theorem 2.2 are direct consequences of the results we will prove in this chapter. In particular, all lower bounds on the threshold bias $b_{\mathcal{T}_n}(a)$ in Theorem 2.1 are obtained via Theorem 3.6. All lower bounds on the threshold bias $b_{\mathcal{H}_n}(a)$ in Theorem 2.2 are obtained via Theorem 3.10. The upper bounds on threshold biases in both theorems are obtained as follows. For Theorem 2.1 (*i*) and Theorem 2.2 (*i*) we use Theorem 3.3, for Theorem 2.1 (*iii*), (*iv*), (*v*) and (*vi*) and Theorem 2.2 (*iii*), (*iv*), (*v*), (*vi*) and (*vii*) we use Theorem 3.4. To find an upper bound in Theorem 2.1 (*ii*) we use Theorem 3.3 and Theorem 2.1 (*iii*) together with bias monotonicity of Maker-Breaker games and in Theorem 2.2 (*ii*) we also use Theorem 3.3 and Theorem 2.2 (*iii*) together with bias monotonicity of Maker-Breaker games. Finally, for Theorem 2.1 (*vii*) and Theorem 2.2 (*viii*) we use Theorem 3.5.

At any point during the games $\mathcal{T}_n$ and $\mathcal{H}_n$, we denote by $M$ (respectively $B$) the graph which is spanned by the edges Maker (respectively Breaker) has claimed thus far. In all the games in this chapter, we suppose that

Breaker starts the game.

## 3.1 $(a : b)$ Box game

In order to present Breaker's winning strategy for the $(a : b)$ Maker-Breaker Connectivity game, we first look at the so-called Box game. Here, however, we suppose that BoxMaker starts the game as it is more suitable for our needs.

In [50], Hamidoune and Las Vergnas have provided a necessary and sufficient condition for BoxMaker's win in the Box game $B(k, t, a, b)$ for positive integers $k, a, b$ and $t = kb + 1$. This result can be extended to all positive integers $k, t, a$ and $b$. Unfortunately, this condition can rarely be used in practise. A more applicable criterion for BoxMaker's win was also provided in [50], but it turns out to be not so tight for certain values of $a$ and $b$. Hence, in this section, we derive a sufficient condition for BoxMaker's win in the Box game $B(k, t, a, b)$ which is better suited to our needs. In particular, it enables us to get a smaller additive constant in part $(vi)$ of Theorem 2.1. Furthermore, it improves the low order terms of the bound obtained in Theorem 3.4. We will apply this new criterion whenever we use the Box game.

Given positive integers $a$ and $b$, we define the following function,

$$f(k; a, b) := \begin{cases} (k-1)(a+1) & , \text{ if } 1 \leq k \leq b \\ ka & , \text{ if } b < k \leq 2b \\ \left\lfloor \frac{k(f(k-b;a,b)+a-b)}{k-b} \right\rfloor & , \text{ otherwise} \end{cases}.$$

First we prove the following technical result.

**Lemma 3.1.** *Let $a, b$ and $k$ be positive integers satisfying $k > b$ and $a - b - 1 \geq 0$, then*

$$f(k; a, b) \geq ka - 1 + \frac{k(a-b-1)}{b} \sum_{j=2}^{\lceil k/b \rceil - 1} \frac{1}{j}. \tag{3.1.1}$$

**Proof.** If $b < k \le 2b$, then the assertion of the lemma holds since $ka \ge ka - 1$.

Otherwise, let $x = \lceil k/b \rceil - 2$. Note that $x$ is the unique positive integer for which $b < k - xb \le 2b$. For every $0 \le y < x$ we have

$$\begin{aligned}
\frac{k}{k-yb} \cdot f(k-yb; a, b) &= \frac{k}{k-yb} \left\lfloor \frac{(k-yb)(f(k-(y+1)b; a,b) + a - b)}{k-(y+1)b} \right\rfloor \\
&\ge \frac{k}{k-yb} \cdot \\
&\cdot \left( \frac{(k-yb)(f(k-(y+1)b; a,b) + a - b)}{k-(y+1)b} - 1 \right) \\
&= \frac{k(a-b)}{k-(y+1)b} - \frac{k}{k-yb} \\
&\quad + \frac{k}{k-(y+1)b} \cdot f(k-(y+1)b; a, b). \qquad (3.1.2)
\end{aligned}$$

Applying the substitution rule (3.1.2) repeatedly for every $0 \le y < x$ and using the fact that $\frac{k}{k-xb} \cdot f(k - xb; a, b) = ka$, we obtain

$$\begin{aligned}
f(k; a, b) &\ge ka - 1 + \sum_{i=1}^{\lceil k/b \rceil - 2} \frac{k(a-b)}{k-ib} - \sum_{j=1}^{\lceil k/b \rceil - 3} \frac{k}{k-jb} \\
&\ge ka - 1 + k(a-b-1) \sum_{i=1}^{\lceil k/b \rceil - 2} \frac{1}{k-ib}. \qquad (3.1.3)
\end{aligned}$$

Since $\frac{1}{k-ib} \ge \frac{1}{(\lceil k/b \rceil - i)b}$ holds for every $1 \le i \le \lceil k/b \rceil - 2$, it follows by (3.1.3) that

$$\begin{aligned}
f(k; a, b) &\ge ka - 1 + k(a-b-1) \sum_{i=1}^{\lceil k/b \rceil - 2} \frac{1}{(\lceil k/b \rceil - i)\, b} \\
&= ka - 1 + k(a-b-1) \sum_{j=2}^{\lceil k/b \rceil - 1} \frac{1}{jb}
\end{aligned}$$

$$= ka - 1 + \frac{k(a-b-1)}{b} \sum_{j=2}^{\lceil k/b \rceil - 1} \frac{1}{j}.$$

□

**Lemma 3.2.** *If $t \leq f(k;a,b) + a$, then BoxMaker has a winning strategy for $B(k,t,a,b)$.*

**Proof.** We prove this lemma by induction on $k$.

If $1 \leq k \leq b$, then $t \leq f(k;a,b) + a = (k-1)(a+1) + a = k(a+1) - 1$. Since the board is a canonical hypergraph, it follows that there exists a hyperedge of size at most $a$. In his first move, BoxMaker claims all vertices of such a hyperedge and thus wins.

If $b < k \leq 2b$, then $t \leq f(k;a,b) + a = ka + a$. In his first move, BoxMaker claims $a$ vertices such that the resulting hypergraph is canonical of type $(k,t')$, where $t' = t - a \leq ka$. It follows that every hyperedge is of size at most $a$. Subsequently, in his first move, BoxBreaker claims $b < k$ vertices. Hence, there must exist an hyperedge which BoxBreaker did not touch in his first move. In his second move, BoxMaker claims all free vertices of such an hyperedge and thus wins.

Assume then that $k > 2b$ and assume that the assertion of the lemma holds for every $k_1 < k$, that is, if $t_1 \leq f(k_1;a,b) + a$, then BoxMaker has a winning strategy for $B(k_1,t_1,a,b)$. In his first move, BoxMaker claims $a$ vertices such that the resulting hypergraph is canonical of type $(k,t')$, where $t' = t - a \leq f(k;a,b)$. Subsequently, in his first move, BoxBreaker claims $b$ board elements. Let $e_1, \ldots, e_{k-b}$ be arbitrary $k-b$ winning sets which BoxBreaker did not touch in his first move. Since BoxMaker's first move results in a canonical hypergraph, it follows that $\hat{t} := \sum_{i=1}^{k-b} |e_i| \leq \frac{k-b}{k} \cdot (t'+b) \leq \frac{k-b}{k} \cdot t' + b$. Moreover, it follows by the definition of $f$ that $f(k;a,b) \leq \frac{k}{k-b}\left(f(k-b;a,b) + a - b\right)$, implying that $f(k-b;a,b) \geq \frac{k-b}{k} \cdot f(k;a,b) + b - a \geq \hat{t} - a$. Hence, in order to prove that BoxMaker has a winning strategy for $B(k,t,a,b)$, it suffices to prove that BoxMaker has a winning strategy for $B(k-b,\hat{t},a,b)$. This however follows by the induction hypothesis since $k - b < k$ and since, as noted above, $\hat{t} \leq f(k-b;a,b) + a$.

□

## 3.2 $(a : b)$ Connectivity game, Breaker's strategy

In this section, we give Breaker's strategies to win the $(a : b)$ Connectivity game $\mathcal{T}_n$ on $E(K_n)$. The upper bounds on $b_{\mathcal{T}_n}$ in Theorem 2.1 are results of Theorem 3.3, Theorem 3.4 and Theorem 3.5.

**Theorem 3.3.** *Let $\varepsilon > 0$ be a real number and let $n = n(\varepsilon)$ be a sufficiently large positive integer. If $a \leq \ln n$ and $b \geq (1+\varepsilon)\frac{an}{\ln(an)}$, then the $(a : b)$ game $\mathcal{T}_n$ is Breaker's win.*

**Proof.** Our proof relies on the approach of Chvátal and Erdős [30], who proved the special case $a = 1$.

For technical reasons we will assume that $\varepsilon < 1/3$. This does not pose a restriction by the bias monotonicity of Maker-Breaker games.

Before we describe Breaker's strategy in detail, we give its outline. Breaker's goal is to isolate some vertex $u \in V(K_n)$ in Maker's graph. His strategy consists of two phases. In the first phase, he claims all edges of a clique $C$ on $k := \left\lceil \frac{an}{(a+1)\ln(an)} \right\rceil$ vertices, such that no vertex of $C$ is touched by Maker, that is, $d_M(v) = 0$ for every $v \in C$. In the second phase, he claims all free edges which are incident with some vertex $v \in C$.

**Breaker's strategy:**

*First Phase.* For every $i \geq 1$, just before Breaker's $i$th move, let $C_i$ denote the largest clique (breaking ties arbitrarily) in Breaker's graph such that $d_M(v) = 0$ for every $v \in C_i$. Let $\ell_i$ be the largest integer for which $b_i \leq b$, where $b_i := \binom{a+\ell_i}{2} + (a+\ell_i)|C_i|$. If $|C_i| \geq k$, then the first phase is over and Breaker proceeds to the second phase of his strategy. Otherwise, in his $i$th move, Breaker picks $a+\ell_i$ vertices $u_1^i, \ldots, u_{a+\ell_i}^i$ of $V(K_n) \setminus V(C_i)$ such that $d_M(u_j^i) = 0$ for every $1 \leq j \leq a+\ell_i$, and then he claims all edges of $\{(u_{j_1}^i, u_{j_2}^i) : 1 \leq j_1 < j_2 \leq a+\ell_i\} \cup \{(u_j^i, w) : 1 \leq j \leq a+\ell_i,\ w \in V(C_i)\}$. He then claims additional $b - b_i$ arbitrary edges; we will disregard these additional edge in our analysis.

Since $b \geq \binom{a+1}{2} + (a+1)(k-1)$, it follows that $\ell_i \geq 1$ as long as $|C_i| < k$. Since Maker can touch at most $a$ vertices of $C_i$ in his $i$th move, it follows that $|C_{i+1}| \geq |C_i| + \ell_i$, assuming that, before Breaker's $i$th move, there are at least $a + \ell_i$ vertices in $V(K_n) \setminus V(C_i)$ which are isolated in Maker's graph.

It follows from the definition of $\ell_i$ that, if $|C_i| \leq \frac{b}{a+3} - \frac{a+2}{2}$, then $\ell_i \geq 3$. Similarly, if $|C_i| \leq \frac{b}{a+2} - \frac{a+1}{2}$, then $\ell_i \geq 2$ and if $|C_i| \leq k < \frac{b}{a+1} - \frac{a}{2}$, then $\ell_i \geq 1$. Hence, Breaker's clique reaches size $k$ within at most

$$\frac{b}{3(a+3)} + \frac{b}{2(a+2)(a+3)} + \frac{b}{(a+1)(a+2)}$$

moves. Since Maker can touch at most $2a$ vertices in a single move, it follows that during the first phase, the number of vertices which are either in Breaker's clique or have positive degree in Maker's graph is at most

$$\frac{b}{3(a+3)}(2a+3) + \frac{b}{2(a+2)(a+3)}(2a+2) + \frac{b}{(a+1)(a+2)}(2a+1) \leq n,$$

where this inequality follows since $a \leq \ln n$ and $\varepsilon < 1/3$.

Hence, for as long as $|C_i| < k$, there are vertices of degree 0 in $M[V(K_n) \setminus V(C_i)]$ and thus Breaker can follow the proposed strategy throughout the first phase.

*Second Phase.* Let $C$ be the clique Breaker has built in the first phase, that is, $|C| \geq k$, $d_M(v) = 0$ holds for every $v \in V(C)$, and $(u,v) \in E(B)$ holds for every $u, v \in V(C)$. In this phase Breaker will isolate some vertex $v \in V(C)$ in Maker's graph; the game ends as soon as he achieves this goal (or as soon as $E(B \cup M) = E(K_n)$, whichever happens first). In order to do so, he restricts his attention to the part of the board spanned by the free edges of $E_2 := \{(u,v) : u \in V(C), v \in V(K_n) \setminus V(C)\}$. In order to choose which edges of $E_2$ to claim in each move, he consults an auxiliary Box game $B(k, k(n-k), b, a)$ assuming the role of BoxMaker.

Since $|\{(u,v) \in E(K_n) \setminus (E(M) \cup E(B)) : v \in V(K_n)\}| \leq n - k$ holds for every $u \in V(C)$, it follows that, if BoxMaker has a winning strategy for $B(k, k(n-k), b, a)$, then Breaker, having built the clique $C$, has a winning strategy for the $(a : b)$ Connectivity game on $K_n$.

Finally, since $k > a$ and $b - a - 1 \geq 0$, it follows by Lemmas 3.1 and 3.2 that, in order to prove that BoxMaker has a winning strategy for

$B(k, k(n-k), b, a)$, it suffices to prove that

$$k(n-k) \leq kb - 1 + \frac{k(b-a-1)}{a} \sum_{i=2}^{\lceil k/a \rceil - 1} \frac{1}{i} + b. \qquad (3.2.1)$$

The latter inequality can be easily verified given our choice of $k$, the assumed bounds on $a$ and $b$, and by applying (1.7.1) to $\sum_{i=2}^{\lceil k/a \rceil - 1} \frac{1}{i}$ while using the inequality $\ln n + 1/2 > \ln(n+1)$ which holds for every $n \geq 2$. □

Note that as $a$ approaches $\ln n$, the lower bound on $b$ in Theorem 3.3 exceeds $n$ and is therefore trivial. Our next theorem improves that bound for large values of $a$.

**Theorem 3.4.** *Let $\varepsilon > 0$ be a real number. If $(1+\varepsilon)\ln n \leq a < \frac{n}{2e}$ and*

$$b \geq \frac{2a\left(n - 2 + \ln\left\lceil \frac{n}{2a} \right\rceil\right) + \ln\left\lceil \frac{n}{2a} \right\rceil - 1 + \frac{2a}{n}}{2a + \ln\left\lceil \frac{n}{2a} \right\rceil - 1 + \frac{2a}{n}},$$

*then the $(a:b)$ game $\mathcal{T}_n$ is Breaker's win.*

**Proof.** Breaker aims to win Connectivity game on $K_n$ by isolating a vertex in Maker's graph. While playing this game, Breaker plays (in his mind) an auxiliary Box game $B(n, n(n-1), b, 2a)$, assuming the role of BoxMaker. Let $V(K_n) = \{v_1, \ldots, v_n\}$ and let $e_1, e_2, \ldots, e_n$ be an arbitrary ordering of the winning sets of $B(n, n(n-1), b, 2a)$. In every move, Breaker claims $b$ free edges of $K_n$ according to his strategy for $B(n, n(n-1), b, 2a)$. That is, whenever he is supposed to claim an element of $e_i$, he claims an arbitrary free edge $(v_i, v_j)$; if no such free edge exists, then he claims an arbitrary free edge. Whenever Maker claims an edge $(v_i, v_j)$ for some $1 \leq i < j \leq n$, Breaker (in his mind) gives BoxBreaker an arbitrary free element of $e_i$ and an arbitrary free element of $e_j$. Note that every edge of $K_n$ which Maker claims translates to two board elements of $B(n, n(n-1), b, 2a)$. This is why BoxBreaker's bias is set to be $2a$. It is thus evident that if BoxMaker has a winning strategy for $B(n, n(n-1), b, 2a)$, then Breaker has a winning strategy for the $(a : b)$ Connectivity game on $K_n$.

By Lemma 3.2, in order to prove that BoxMaker has a winning strategy for $B(n, n(n-1), b, 2a)$, it suffices to prove that $n(n-1) \leq f(n; b, 2a) + b$.

Since $b - 2a - 1 \geq 0$ and $n > 2a$, it follows by Lemma 3.1, (1.7.1) and the fact that for $n \geq 2$, $\ln n + 1/2 > \ln(n+1)$ that

$$f(n; b, 2a) \geq nb - 1 + \frac{n(b-2a-1)}{2a}\left(\ln\left\lceil \frac{n}{2a} \right\rceil - 1\right).$$

Hence, it suffices to prove that

$$n(n-1) \leq \frac{n(b-2a-1)}{2a}\left(\ln\left\lceil \frac{n}{2a} \right\rceil - 1\right) + nb - 1 + b.$$

It is straightforward to verify that the above inequality holds for

$$b \geq \frac{2a(n-2+\ln\left\lceil \frac{n}{2a} \right\rceil) + \ln\left\lceil \frac{n}{2a} \right\rceil - 1 + \frac{2a}{n}}{2a + \ln\left\lceil \frac{n}{2a} \right\rceil - 1 + \frac{2a}{n}}.$$

□

Finally, for very large values of $a$ we obtain a nontrivial bound on $b$ which suffices to ensure Breaker's win.

**Theorem 3.5.** *If $a < \frac{n}{2}$ and $b \geq n-2$, then the $(a : b)$ game $\mathcal{T}_n$ is Breaker's win.*

**Proof.** In his first move, Breaker claims the edges of some graph of positive minimum degree. This is easily done as follows. If $n$ is even, then Breaker claims the edges of some perfect matching of $K_n$ and then he claims additional $b - n/2$ arbitrary free edges. If $n$ is odd, then Breaker claims the edges of a matching of $K_n$ which covers all vertices of $K_n$ but one, say $u$. He then claims a free edge $(u, x)$ for some $x \in V(K_n)$ and additional $b - (n-1)/2 - 1$ arbitrary free edges.

In his first move, Maker cannot touch all vertices of $K_n$ since $2a < n$. Let $w \in V(K_n)$ be an isolated vertex in Maker's graph after his first move. Since $d_B(w) \geq 1$ and $b \geq n-2$, Breaker can claim all free edges which are incident with $w$ in his second move and thus win. □

## 3.3 $(a : b)$ Connectivity game, Maker's strategy

For Maker's win we prove the following sufficient conditions, covering the whole range of possible values of $a$, and gives the upper bounds on $b_{\mathcal{T}_n}$ in Theorem 2.1.

**Theorem 3.6.** *If* $a = o\left(\sqrt{\frac{n}{\ln n}}\right)$ *and*

$$b < \frac{a\left(n - \frac{n}{a \ln n} + \frac{a-1}{2} \ln\left(\frac{n}{a^2 \ln n}\right)\right)}{\ln n + a + \ln\ln n + 4},$$

*or if* $a = \Omega\left(\sqrt{\frac{n}{\ln n}}\right)$, $a \leq \frac{n-1}{2}$ *and* $b < \frac{an}{a+2\ln n - 2\ln a + 4}$, *then the* $(a : b)$ *game* $\mathcal{T}_n$ *is Maker's win.*

**Proof.** Our proof relies on the approach of Gebauer and Szabó [46], who proved the case $a = 1$. First, let us introduce some terminology. For a vertex $v \in V(K_n)$ let $C(v)$ denote the connected component in Maker's graph which contains the vertex $v$. A connected component in Maker's graph is said to be *dangerous* if it contains at most $2b/a$ vertices. We define a *danger function* on $V(K_n)$ in the same way it was defined in [46],

$$\mathcal{D}(v) = \begin{cases} d_B(v), & \text{if } C(v) \text{ is dangerous} \\ -1, & \text{otherwise} \end{cases}.$$

We are now ready to describe Maker's strategy.

**Maker's strategy:** Throughout the game, Maker maintains a set $A \subseteq V(K_n)$ of *active* vertices. Initially, $A = V(K_n)$.

For as long as Maker's graph is not a spanning tree, Maker plays as follows. For every $i \geq 1$, Maker's $i$th move consists of $a$ steps. For every $1 \leq j \leq a$, in the $j$th step of his $i$th move Maker chooses an active vertex $v_i^{(j)}$ whose danger is maximal among all active vertices (breaking ties arbitrarily). He then claims a free edge $(x, y)$ for arbitrary vertices $x \in C(v_i^{(j)})$ and $y \in V(K_n) \setminus C(v_i^{(j)})$. Subsequently, Maker deactivates $v_i^{(j)}$, that is, he removes $v_i^{(j)}$ from $A$. If at any point during the game Maker is unable to follow the proposed strategy, then he forfeits the game.

Note that by Maker's strategy his graph is a forest at any point during the game. Hence, there are at most $n-1$ steps in the entire game. It follows that the game lasts at most $\lceil \frac{n-1}{a} \rceil$ rounds.

In order to prove Theorem 3.6, it clearly suffices to prove that Maker is able to follow the proposed strategy without ever having to forfeit the game. First, we prove the following claim.

**Claim 3.7.** At any point during the game there is exactly one active vertex in every connected component of Maker's graph.

**Proof.** Our proof is by induction on the number of steps $r$ which Maker makes throughout the game.

Before the game starts, every vertex of $K_n$ is a connected component of Maker's graph, and every vertex is active by definition. Hence, the assertion of the claim holds for $r = 0$.

Let $r \geq 1$ and assume that the assertion of the claim holds for every $r' < r$. In the $r$th step, Maker chooses an active vertex $v$ and then claims an edge $(x, y)$ such that $x \in C(v)$ and $y \notin C(v)$ hold prior to this move. By the induction hypothesis there is exactly one active vertex $z \in C(y)$ and $v$ is the sole active vertex in $C(v)$. Since Maker deactivates $v$ after claiming $(x, y)$, it follows that $z$ is the unique active vertex in $C(x) \cup C(y)$ after Maker's $r$th step. Clearly, every other component still has exactly one active vertex. □

We are now ready to prove that Maker can follow his strategy (without forfeiting the game) for $n-1$ steps. Assume for the sake of contradiction that at some point during the game Maker chooses an active vertex $v \in C$ and then tries to connect $C$ with some component of $M \setminus C$, but fails. It follows that Breaker has already claimed all the edges of $K_n$ with one endpoint in $C$ and the other in $V \setminus C$. Assume that Breaker has claimed the last edge of this cut in his $s$th move. As noted above, $s \leq \lceil \frac{n-1}{a} \rceil$ must hold. It follows that $|C| \leq 2b/a$ as otherwise Breaker would have had to claim at least $\frac{2b}{a}(n - \frac{2b}{a}) > sb$ edges in $s$ moves. It follows that at any point during the first $s$ rounds of the game there is always at least one dangerous connected component.

In his $s$th move, Breaker claims at most $b$ edges. Hence, just before Breaker's $s$th move, $e_B(V(C), V(K_n) \setminus V(C)) \geq |C|(n - |C|) - b$ must hold. In particular, $d_B(v) \geq n - \frac{2b}{a} - b$, where $v$ is the unique active vertex of $C$. Since Maker did not connect $C$ with $M \setminus C$ in his $(s-1)$st move, it follows that, just before this move, there must have been at least $a+1$ active vertices $v, v_1, \ldots, v_a$ such that the components $C, C(v_1), \ldots, C(v_a)$ were dangerous and $d_B(u) \geq n - \frac{2b}{a} - b$ for every $u \in \{v, v_1, \ldots, v_a\}$.

For every $1 \leq i \leq s$, let $M_i$ and $B_i$ denote the $i$th move of Maker and of Breaker, respectively. By Maker's strategy $v_1^{(1)}, \ldots, v_1^{(a)}, v_2^{(1)}, \ldots, v_2^{(a)}, \ldots,$ $v_{s-1}^{(1)}, \ldots, v_{s-1}^{(a)}$ are of maximum degree in Breaker's graph among all active vertices at the appropriate time, that is, just before Maker's $j$th step of his $i$th move, $d_B(v_i^{(j)})$ is maximal among all active vertices. Let $v_s$ be an active vertex of maximum degree in Breaker's graph just before Maker's $s$th move. Note that, for every $u \in \{v_1^{(1)}, \ldots, v_1^{(a)}, v_2^{(1)}, \ldots, v_2^{(a)}, \ldots, v_{s-1}^{(1)}, \ldots, v_{s-1}^{(a)}, v_s\}$, if $u$ is active, then $C(u)$ is a dangerous component. For every $1 \leq i \leq s-1$, let $A_{s-i} = \{v_{s-i}^{(1)}, \ldots, v_{s-i}^{(a)}, \ldots, v_{s-1}^{(1)}, \ldots, v_{s-1}^{(a)}, v_s\}$ denote the subset of vertices of $\{v_1^{(1)}, \ldots, v_1^{(a)}, v_2^{(1)}, \ldots, v_2^{(a)}, \ldots, v_{s-1}^{(1)}, \ldots, v_{s-1}^{(a)}, v_s\}$ that are still active just before Maker's $(s-i)$th move and let $A_s = \{v_s\}$. For every $A \subseteq V$, let $\overline{\mathcal{D}}_{B_i}(A) = \frac{\sum_{v \in A} \mathcal{D}(v)}{|A|}$ denote the average danger value of the vertices of $A$, immediately before Breaker's move $B_i$. The average danger $\overline{\mathcal{D}}_{M_i}(A)$ is defined analogously.

Since Maker always deactivates vertices of maximum danger, thus reducing the average danger value of active vertices, we have the following claim.

**Claim 3.8.** $\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) \geq \overline{\mathcal{D}}_{B_{s-i+1}}(A_{s-i+1})$ holds for every $1 \leq i \leq s-1$.

**Proof.** Let $u \in A_{s-i+1} = \{v_{s-i+1}^{(1)}, \ldots, v_{s-i+1}^{(a)}, \ldots, v_{s-1}^{(1)}, \ldots, v_{s-1}^{(a)}, v_s\}$ be an arbitrary vertex. Since $C(u)$ is a dangerous component immediately before $B_{s-i+1}$, the danger $\mathcal{D}(u)$ does not change during $M_{s-i}$. It follows that $\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i+1}) = \overline{\mathcal{D}}_{B_{s-i+1}}(A_{s-i+1})$.

Note that the vertices contained in $A_{s-i+1}$ were still active before $M_{s-i}$.

Following his strategy, Maker deactivated all the vertices of $\{v_{s-i}^{(1)}, \ldots, v_{s-i}^{(a)}\}$, because their danger values were the largest among all the active vertices of $\{v_{s-i}^{(1)},\ldots,v_{s-i}^{(a)},\ldots,v_{s-1}^{(1)},\ldots,v_{s-1}^{(a)},v_s\}$. It follows that

$$\min\{\mathcal{D}(v_{s-i}^{(1)}),\ldots,\mathcal{D}(v_{s-i}^{(a)})\} \geq$$
$$\max\{\mathcal{D}(v_{s-i+1}^{(1)}),\ldots,\mathcal{D}(v_{s-i+1}^{(a)}),\ldots,\mathcal{D}(v_{s-1}^{(1)}),\ldots,\mathcal{D}(v_{s-1}^{(a)}),\mathcal{D}(v_s)\},$$

and thus $\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) \geq \overline{\mathcal{D}}_{M_{s-i}}(A_{s-i+1})$, as claimed. □

The following claim gives two estimates on the change of the danger value caused by Breaker's moves.

**Claim 3.9.** The following two inequalities hold for every $1 \leq i \leq s-1$.

(*i*) $\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i}}(A_{s-i}) \leq \frac{2b}{ai+1} < \frac{2b}{ai}$.

(*ii*) Define a function $g:\{1,\ldots,s\}\to\mathbb{N}$ by setting $g(i)$ to be the number of edges with both endpoints in $A_i$ which Breaker has claimed during the first $i-1$ moves of the game. Then,

$$\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i}}(A_{s-i}) \leq \frac{b+a^2(i-1)+a+\binom{a}{2}+g(s-i+1)-g(s-i)}{ai+1}.$$

**Proof.**

(*i*) The components $C(v_{s-i}^{(1)}),\ldots,C(v_{s-i}^{(a)}),\ldots,C(v_{s-1}^{(1)}),\ldots,C(v_{s-1}^{(a)})$, $C(v_s)$ are dangerous before Maker's $(s-i)$th move. During Breaker's moves, the components of Maker's graph do not change. Hence, the change of the danger values of the vertices of $A_{s-i}$, caused by Breaker's $(s-i)$th move, depend solely on the change of their degrees in Breaker's graph. In his $(s-i)$th move, Breaker claims $b$ edges and thus the increase of the sum of the degrees of the vertices of $\{v_{s-i}^{(1)},\ldots,v_{s-i}^{(a)},\ldots,v_{s-1}^{(1)},\ldots,v_{s-1}^{(a)},v_s\}$ is at most $2b$. The size of $A_{s-i}$ is $ai+1$. Thus $\overline{\mathcal{D}}_{B_{s-i}}(A_{s-i})$ increases by at most $\frac{2b}{ai+1}$ during $B_{s-i}$.

(*ii*) Let $p$ denote the number of edges $(x,y)$ claimed by Breaker during $B_{s-i}$ such that $\{x,y\}\subseteq A_{s-i}$ and let $q=b-p$. Hence, the increase of the sum $\sum_{u \in A_{s-i}} d_B(u)$ during $B_{s-i}$ is at most $2p + q = p + b$. It follows that $\overline{\mathcal{D}}_{B_{s-i}}(A_{s-i})$ increases by at most $\frac{b+p}{ai+1}$ during $B_{s-i}$. It remains to prove that $p \leq a^2(i-1) + a + \binom{a}{2} + g(s-i+1) - g(s-i)$. During his first $s-i-1$ moves, Breaker has claimed exactly $g(s-i)$ edges with both their endpoints in $A_{s-i}$. Hence, during his first $s-i$ moves, Breaker has claimed exactly $g(s-i)+p$ edges with both their endpoints in $A_{s-i}$. Exactly $g(s-i+1)$ of these edges have both their endpoints in $A_{s-i+1} = A_{s-i} \setminus \{v_{s-i}^{(1)}, \ldots, v_{s-i}^{(a)}\}$. There can be at most $\binom{a}{2}$ edges connecting two vertices of $\{v_{s-i}^{(1)}, \ldots, v_{s-i}^{(a)}\}$. Moreover, each vertex of $\{v_{s-i}^{(1)}, \ldots, v_{s-i}^{(a)}\}$ is adjacent to at most $a(i-1)+1$ vertices of $A_{s-i+1}$. Combining all of these observations, we conclude that $g(s-i)+p \leq g(s-i+1) + a^2(i-1) + a + \binom{a}{2}$, entailing $p \leq \binom{a}{2} + a^2(i-1) + a + g(s-i+1) - g(s-i)$ as claimed. □

Clearly, before the game starts $\mathcal{D}(u) = d_B(u) = 0$ holds for every vertex $u$. In particular $\overline{\mathcal{D}}_{B_1}(A_1) = 0$. Using our assumption that Breaker wins the game, we will obtain a contradiction by showing that $\overline{\mathcal{D}}_{B_1}(A_1) > 0$.

Note that, as previously observed, $\overline{\mathcal{D}}_{B_s}(A_s) \geq n - \frac{2b}{a} - b$. We will use this fact, Claim 3.8, Claim 3.9, the inequalities $\frac{1}{ai+1} < \frac{1}{ai}$ and $b + a^2(i-1) + a + \binom{a}{2} + g(s-i+1) - g(s-i) \geq 0$ (which hold for every $1 \leq i \leq s-1$), and (1.7.1), in order to reach the aforementioned contradiction.

Let $k := \left\lfloor \frac{n}{a^2 \ln n} \right\rfloor$. First, assume that $a = o(\sqrt{n/\ln n})$. We split the game into two parts: the main game and the last $k$ moves. In these last moves, we will use a more delicate estimate on the effect of Breaker's move on the average danger.

We distinguish between two cases.

**Case 1:** $s < k$.

$$\begin{aligned}\overline{\mathcal{D}}_{B_1}(A_1) = \overline{\mathcal{D}}_{B_s}(A_s) &+ \sum_{i=1}^{s-1} \left(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{s-i+1})\right) \\ &- \sum_{i=1}^{s-1} \left(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i}}(A_{s-i})\right)\end{aligned}$$

$$
\begin{aligned}
&\geq n - \frac{2b}{a} - b + \sum_{i=1}^{s-1} 0 \\
&- \sum_{i=1}^{s-1} \frac{b + a^2(i-1) + a + \binom{a}{2} + g(s-i+1) - g(s-i)}{ai} \\
&\geq n - \frac{b}{a}(H_{s-1} + 2 + a) - a(s-1) + aH_{s-1} - H_{s-1} - \frac{a-1}{2}H_{s-1} \\
&- \frac{g(s)}{a} + \sum_{i=1}^{s-2} \frac{g(s-i)}{ai(i+1)} + \frac{g(1)}{a(s-1)} \\
&\geq n - \frac{b}{a}(H_{s-1} + 2 + a) + \frac{a-1}{2}H_{s-1} - a(s-1) \\
&\qquad\qquad\qquad\qquad [\text{since } g(s) = 0 \text{ and } g(s-i) \geq 0] \\
&> n - \frac{b}{a}(H_s + 2 + a) + \frac{a-1}{2}H_s - as \\
&> n - \frac{b}{a}(\ln k + 3 + a) + \frac{a-1}{2}\ln k - ak \\
&\qquad\qquad\qquad\qquad [\text{since } s < k] \\
&> n - \frac{n - \frac{n}{a\ln n} + \frac{a-1}{2} \cdot \ln\left(\frac{n}{a^2 \ln n}\right)}{\ln n + a + \ln\ln n + 4}(\ln n - \ln a - \ln\ln n + a + 3) \\
&- \frac{n}{a\ln n} + \frac{a-1}{2} \cdot \ln\left(\frac{n}{a^2\ln n}\right) \\
&> 0.
\end{aligned}
$$

**Case 2:** $s \geq k$.

$$
\begin{aligned}
\overline{\mathcal{D}}_{B_1}(A_1) &= \overline{\mathcal{D}}_{B_s}(A_s) + \sum_{i=1}^{s-1} \left(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{s-i+1})\right) \\
&- \sum_{i=1}^{k-1} \left(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i}}(A_{s-i})\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=k}^{s-1}\big(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i})-\overline{\mathcal{D}}_{B_{s-i}}(A_{s-i})\big)\\
&\geq n-\frac{2b}{a}-b+\sum_{i=1}^{s-1}0\\
&-\sum_{i=1}^{k-1}\frac{b+a^2(i-1)+a+\binom{a}{2}+g(s-i+1)-g(s-i)}{ai}-\sum_{i=k}^{s-1}\frac{2b}{ai}\\
&\geq n-\frac{2b}{a}-b-\frac{b}{a}H_{k-1}+\frac{a-1}{2}H_{k-1}-a(k-1)\\
&-\frac{g(s)}{a}+\sum_{i=1}^{k-2}\frac{g(s-i)}{ai(i+1)}+\frac{g(s-k+1)}{a(k-1)}-\frac{2b}{a}(H_{s-1}-H_{k-1})\\
&\geq n-\frac{b}{a}(2H_s-H_{k-1}+2+a)+\frac{a-1}{2}H_{k-1}-a(k-1)\\
&\qquad\qquad\qquad\qquad[\text{since } g(s)=0 \text{ and } g(s-i)\geq 0]\\
&> n-\frac{b}{a}(2\ln\left(\frac{n-1}{a}\right)-\ln k+a+4)+\frac{a-1}{2}\cdot\ln k-ak\\
&\geq n-\frac{b}{a}(\ln n+\ln\ln n+a+4)+\frac{a-1}{2}\cdot\ln\left(\frac{n}{a^2\ln n}\right)-\frac{n}{a\ln n}\\
&> n-\frac{n-\frac{n}{a\ln n}+\frac{a-1}{2}\cdot\ln\left(\frac{n}{a^2\ln n}\right)}{\ln n+a+\ln\ln n+4}(\ln n+\ln\ln n+4+a)\\
&+\frac{a-1}{2}\cdot\ln\left(\frac{n}{a^2\ln n}\right)-\frac{n}{a\ln n}\\
&=0\,.
\end{aligned}
$$

Next, assume that $a=\Omega(\sqrt{n/\ln n})$ and $a\leq\frac{n-1}{2}$. In this case, the game does not last long.

$$
\begin{aligned}
\overline{\mathcal{D}}_{B_1}(A_1)=\overline{\mathcal{D}}_{B_s}(A_s)&+\sum_{i=1}^{s-1}\big(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i})-\overline{\mathcal{D}}_{B_{s-i+1}}(A_{s-i+1})\big)\\
&-\sum_{i=1}^{s-1}\big(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i})-\overline{\mathcal{D}}_{B_{s-i}}(A_{s-i})\big)
\end{aligned}
$$

$$
\begin{aligned}
&> n - \frac{2b}{a} - b + \sum_{i=1}^{s-1} 0 - \sum_{i=1}^{s-1} \frac{2b}{ai} \\
&> n - \frac{2b}{a} - b - \frac{2b}{a}(\ln s + 1) \\
&= n - \frac{2b}{a}(2 + \frac{a}{2} + \ln s) \\
&\geq n - \frac{2b}{a}\left(2 + \frac{a}{2} + \ln\left(\frac{n-1}{a}\right)\right) \\
&> n - \frac{n}{\ln n - \ln a + 2 + \frac{a}{2}}\left(2 + \frac{a}{2} + \ln\left(\frac{n-1}{a}\right)\right) \\
&\geq 0\,.
\end{aligned}
$$

□

## 3.4 $(a : b)$ Hamiltonicity game

Breaker's winning strategy for the $(a : b)$ Connectivity game is also a winning strategy in $(a : b)$ Hamiltonicity game, $\mathcal{H}_n$, since a disconnected graph cannot contain a Hamilton cycle. This gives the upper bounds on $b_{\mathcal{H}_n}$ in Theorem 2.2. Thus, in order to obtain the lower bounds on $b_{\mathcal{H}_n}$ from Theorem 2.2, we have to prove the give the sufficient condition for Maker's win in Hamiltonicity game, which is stated in the following theorem.

**Theorem 3.10.** *If $a \geq 1$ and $b \leq \frac{an}{a+\ln n}\left(1 - \frac{30}{\ln^{1/4} n}\right)$, Maker has the winning strategy in $(a : b)$ game $\mathcal{H}_n$.*

**Proof.** We follow the approach of Krivelevich [62] who gave the winning strategy for Maker in $(1 : b)$ Hamilton cycle game, for $b \leq \left(1 - \frac{30}{\ln^{1/4} n}\right) \frac{n}{\ln n}$.

Before we describe our strategy, we introduce some terminology. A graph is a *$k$-expander* if it holds that $|N_G(U)| \geq 2|U|$, for every subset $U \subset V(G)$ such that $|U| \leq k$.

Let

$$\delta = \frac{15}{\ln^{1/4} n},$$
$$k_0 = \frac{6n}{\ln^{1/2} n},$$
$$\varepsilon = 2\delta = \frac{30}{\ln^{1/4} n}.$$

Maker's strategy in the $(a : b)$ Hamiltonicity game consists of three stages.

**Stage 1**

In this stage, Maker creates a $k_0$-expander in his graph in at most $\frac{12n}{a}$ moves.

**Stage 2**

In the second stage, Maker creates a connected expander, by repeatedly claiming an edge between some two components, in at most $\frac{n}{a}$ moves.

**Stage 3**

Finally, in the third stage, Maker creates a Hamilton cycle from this connected expander in at most $\frac{n}{a}$ moves.

It is clear that if Maker can follow the proposed strategy, at the end of the game his graph will contain a Hamilton cycle.

We show that he can follow the strategy for each stage separately.

To be able to prove that Maker can follow his strategy in stage 1, we prove the following.

**Claim 3.11.** Let $1 \leq a \leq \frac{n}{2}$ and let $b \leq \frac{an(1-\varepsilon)}{a+\ln n}$. In $(a : b)$ Maker-Breaker game on $K_n$, Maker can achieve degree 12 at each vertex $v$ before Breaker claims at least $(1 - \delta)n$ edges incident to this vertex.

**Proof.** The proof of this claim is very similar to the proof of Theorem 1.2 in [46]. Thus we omit some calculations.

Maker plays the $(a : b)$ Minimum degree 12 game by playing in his mind the $(1 : \frac{b}{a})$ Minimum degree 12 game. He does so by splitting each his move into $a$ steps. In each step he claims 1 edge as a response to $b' = \frac{b}{a}$ steps of Breaker. In general, it is not possible for Maker to split his move into steps and play in each of these steps as a response to $b'$ edges of Breaker, because the first step of Maker comes only after all $b$ steps of Breaker and an edge that Maker wants to claim in, say, his first step could be already claimed by Breaker in some of Breaker's later steps of that move. Here, however, this is not a problem, as Maker only wants to claim an arbitrary free edge incident to some vertex. The only problem would be that Breaker already claimed $(1-\delta)n$ edges incident to some vertex $v$, whose degree in Maker's graph is less than 12, when Maker's strategy tells him to claim an edge incident to $v$ in some step, but we show later that this cannot happen. So, from now on, we consider these steps as moves in $(1 : b')$ Minimum degree 12 game.

For every vertex $v \in V(K_n)$, the danger function $\mathcal{D}$ is defined in the same way as it is in [46]

$$\mathcal{D}(v) := d_B(v) - 2b' d_M(v).$$

For a subset $X \subseteq V(K_n)$, $\overline{\mathcal{D}}(X) = \frac{\sum_{v \in X} \mathcal{D}(v)}{|X|}$ denotes the average danger value of vertices in $X$. A vertex $v \in V(K_n)$ is called *dangerous* if $d_M(v) < 12$. The value $\mathcal{D}$ is calculated before every move of Maker. The game ends either when Maker won and all vertices have degree at least 12, or Breaker won and therefore there exists a dangerous vertex $v \in V(K_n)$ whose danger value is

$$\mathcal{D}(v) \geq (1-\delta)n - 2 \cdot 11b' = (1-\delta)n - \frac{22b}{a}.$$

This leads us to conclusion that the game cannot last more than $g = 12n$ moves.

**Maker's strategy $\mathcal{S}_M$:**

*In the $i$th move of the game, as long as there exists at least one vertex whose degree is less than 12, Maker selects the most dangerous vertex $v_i$ among all dangerous vertices and claims an arbitrary free edge incident to $v_i$(ties broken arbitrarily).*

Suppose for a contradiction that Breaker has a strategy $\mathcal{S}_B$ to win against Maker playing with strategy $\mathcal{S}_M$. That means, playing by $\mathcal{S}_B$ Breaker ensures that at some point of the game, after his $s$th move, $1 \leq s \leq g$, a vertex $v_s$ of degree at least $(1-\delta)n$ appears in Breaker's graph when $d_M(v_s) < 12$. The game is over, by our assumption. We will give here just the relevant calculations that if this happened, then at the beginning of the game, before Breaker's first move, the average danger value of all the vertices was positive which is impossible. Let $A = \{v_1, v_2, \dots, v_s\}$ be the set of active vertices which contains all the vertices in Maker's graph of degree less than 12 that Maker selected as the most dangerous. Note here that vertices $v_1, v_2, \dots, v_s$ do not have to be distinct vertices from $V(K_n)$, since it takes 12 moves to remove a vertex from the set of active vertices. By strategy $\mathcal{S}_M$, in his $i$th move, Maker claims an edge incident with $v_i$ (for all $i$ except for $i = s$, as the game is considered to be over before his $s$th move). For $0 \leq i \leq s-1$, let $A_i = \{v_{s-i}, v_{s-i+1}, \dots, v_s\}$.

Following the notation in [46], let $\mathcal{D}_{M_i}(v)$ and $\mathcal{D}_{B_i}(v)$ denote danger values of vertex $v \in V(K_n)$ immediately before the $i$th move of Maker, respectively Breaker. Observe that Breaker in the $(a : b)$ game claims $b$ (not $b'$) edges all at once, so just before his last move in the $(1 : b')$ game, $\mathcal{D}_{B_s}(v_s) \geq (1-\delta)n - b - 22b'$.

Next lemma is useful for estimating the change in average danger value after Maker's move.

**Lemma 3.12** ([46], Lemma 3.3). *Let $i$ be an integer, $1 \leq i \leq s-1$.*

*(i) If $A_i \neq A_{i-1}$, then $\overline{\mathcal{D}}_{M_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{i-1}) \geq 0$.*

*(ii) If $A_i = A_{i-1}$, then $\overline{\mathcal{D}}_{M_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{i-1}) \geq \frac{2b'}{|A_i|}$.*

To estimate the change in average danger value after Breaker's move, we use the following lemma.

**Lemma 3.13** ([46], Lemma 3.4). *Let $i$ be an integer, $1 \leq i \leq s-1$. Then*

*(i) $\overline{\mathcal{D}}_{M_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i}}(A_i) \leq \frac{2b'}{|A_i|}$.*

*(ii) $\overline{\mathcal{D}}_{M_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i}}(A_i) \leq \frac{b'+|A_i|-1+a(i-1)-a(i)}{|A_i|}$, where $a(i)$ denotes the number of edges with both endpoints in $A_i$ which Breaker took in the first $s-i+1$ rounds.*

Combining Lemmas 3.12 and 3.13 we obtain the following corollary which estimates the change in average danger value after a whole round is played.

**Corollary 3.14** ([46], Corollary 3.5)**.** *Let $i$ be an integer, $1 \leq i \leq s-1$.*

*(i) If $A_i = A_{i-1}$, then $\overline{\mathcal{D}}_{B_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{i-1}) \geq 0$.*

*(ii) If $A_i \neq A_{i-1}$, then $\overline{\mathcal{D}}_{B_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{i-1}) \geq -\frac{2b'}{|A_i|}$,*

*(iii) If $A_i \neq A_{i-1}$, then $\overline{\mathcal{D}}_{B_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{i-1}) \geq -\frac{b'+|A_i|-1+a(i-1)-a(i)}{|A_i|}$, where $a(i)$ denotes the number of edges with both endpoints in $A_i$ which Breaker took in the first $s-i+1$ rounds.*

Let $r$ denote the number of distinct vertices in $A_1$ and let $i_1 < i_2 < \cdots < i_{r-1}$ be the indices for which $A_{i_j} \neq A_{i_j-1}$ holds, for $1 \leq j \leq r-1$. It holds that $|A_{i_j}| = j+1$. Also, by definition $A_{i_j-1} = A_{i_{j-1}}$ and $i_{j-1} \leq i_j - 1$ and so $a(i_{j-1}) \geq a(i_j - 1)$.
Before the $s$th move of Breaker $d_B(v_s) = (1-\delta)n - b$ and $d_M(v_s) < 12$. Thus $\mathcal{D}_{B_s} \geq (1-\delta)n - b - 2b' \cdot d_M(v_s) = (1-\delta)n - b - 22b'$. Fix $k := \lfloor \frac{n}{\ln n} \rfloor$. To complete the proof, we want to show that before Breaker's first move $\mathcal{D}_{B_1}(A_{s-1}) > 0$, thus obtaining a contradiction. We estimate the average danger value before the first move of Breaker by analysing two cases.

**Case 1:** $r < k$.

$$
\begin{aligned}
\overline{\mathcal{D}}_{B_1}(A_{s-1}) &= \overline{\mathcal{D}}_{B_s}(A_0) + \sum_{i=1}^{s-1} \left(\overline{\mathcal{D}}_{B_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{i-1})\right) \\
&\geq \overline{\mathcal{D}}_{B_s}(A_0) + \sum_{j=1}^{r-1} \left(\overline{\mathcal{D}}_{B_{s-i_j}}(A_{i_j}) - \overline{\mathcal{D}}_{B_{s-i_j+1}}(A_{i_j-1})\right) \\
&\qquad \text{[by Corollary 3.14 } (i)] \\
&\geq \overline{\mathcal{D}}_{B_s}(A_0) - \sum_{j=1}^{r-1} -\frac{b' + |A_{i_j}| - 1 + a(i_j - 1) - a(i_j)}{|A_{i_j}|} \\
&\qquad \text{[by Corollary 3.14 } (iii)] \\
&\geq (1-\delta)n - b - \frac{22b}{a} - \sum_{j=1}^{r-1} \frac{b' + j + a(i_j - 1) - a(i_j)}{j+1}
\end{aligned}
$$

$$
\begin{aligned}
&\geq (1-\delta)n - b - \frac{22b}{a} - b'H_r - r - \frac{a(0)}{2} + \\
&+ \sum_{j=2}^{r-1} \frac{a(i_{j-1})}{j(j+1)} - \frac{a(i_{r-1})}{r} \\
&\geq (1-\delta)n - b - \frac{22b}{a} - \frac{b}{a}(\ln k + 1) - k \\
&\qquad\qquad [\text{since } r < k \text{ and } a(0) = 0] \\
&\geq (1-\delta)n - \frac{b}{a}(a + 23 + \ln k) - k \\
&\geq (1-\delta)n - \frac{b}{a}\left(a + 23 + \ln\left(\frac{n}{\ln n}\right)\right) - \frac{n}{\ln n} \\
&\geq (1-\delta)n - \frac{(1-2\delta)an}{a(a+\ln n)}(a + \ln n + 23 - \ln\ln n) - \frac{n}{\ln n} \\
&\geq \delta n - \frac{(1-2\delta)(23 - \ln\ln n)n}{a + \ln n} - \frac{n}{\ln n} \\
&> 0.
\end{aligned}
$$

**Case 2:** $r \geq k$.

$$
\begin{aligned}
\overline{\mathcal{D}}_{B_1}(A_{s-1}) &= \overline{\mathcal{D}}_{B_s}(A_0) + \sum_{i=1}^{s-1} \left(\overline{\mathcal{D}}_{B_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{i-1})\right) \\
&\geq \overline{\mathcal{D}}_{B_s}(A_0) + \sum_{j=1}^{r-1} \left(\overline{\mathcal{D}}_{B_{s-i_j}}(A_{i_j}) - \overline{\mathcal{D}}_{B_{s-i_j+1}}(A_{i_j-1})\right) \\
&\qquad\qquad [\text{by Corollary 3.14 } (i)] \\
&\geq \overline{\mathcal{D}}_{B_s}(A_0) + \sum_{j=1}^{k-1} \left(\overline{\mathcal{D}}_{B_{s-i_j}}(A_{i_j}) - \overline{\mathcal{D}}_{B_{s-i_j+1}}(A_{i_j-1})\right) + \\
&+ \sum_{j=k}^{r-1} \left(\overline{\mathcal{D}}_{B_{s-i_j}}(A_{i_j}) - \overline{\mathcal{D}}_{B_{s-i_j+1}}(A_{i_j-1})\right)
\end{aligned}
$$

$$\geq (1-\delta)n - b - \frac{22b}{a} - \sum_{j=1}^{k-1} \frac{b' + j + a(i_j - 1) - a(i_j)}{j+1} -$$
$$- \sum_{j=k}^{r} \frac{2b'}{j+1}$$
[by Corollary 3.14$(ii)$ and 3.14 $(iii)$]
$$\geq (1-\delta)n - b - \frac{22b}{a} - b'(2H_r - H_k) - k - \frac{a(0)}{2} +$$
$$+ \sum_{j=2}^{k-1} \frac{a(i_{j-1})}{j(j+1)} - \frac{a(i_{k-1})}{k}$$
$$\geq (1-\delta)n - b - \frac{22b}{a} - b'(2H_r - H_k) - k$$
$$\geq (1-\delta)n - b - \frac{22b}{a} - \frac{b}{a}(2\ln n + 2 - \ln k) - k$$
$$\geq (1-\delta)n - b - \frac{22b}{a} - \frac{b}{a}(2\ln n + 2 - \ln n + \ln\ln n) - k$$
[since $r \leq n$]
$$\geq (1-\delta)n - \frac{b}{a}(a + 24 + \ln n + \ln\ln n) - \frac{n}{\ln n}$$
$$\geq (1-\delta)n - \frac{(1-2\delta)an}{a(a+\ln n)}(a + 24 + \ln n + \ln\ln n) - \frac{n}{\ln n}$$
$$\geq (1-\delta)n - (1-2\delta)n - \frac{(1-2\delta)n}{a+\ln n}(24 + \ln\ln n) - \frac{n}{\ln n}$$
$$\geq \delta n - \frac{(1-2\delta)(\ln\ln n + 24)n}{a+\ln n} - \frac{n}{\ln n}$$
$$> 0 \quad \text{, for sufficiently large } n.$$

This completes the proof of Claim 3.11. □

Krivelevich in [62] showed that the strategy $\mathcal{S}_M$ can be modified to obtain $k_0$-expander in the $(1:b)$ game, for $b \leq \left(1 - \frac{30}{\ln^{1/4} n}\right) \frac{n}{\ln n}$. Note that in our setting, for $b' = b/a$, in the $(1 : b')$ game $b' < \left(1 - \frac{30}{\ln^{1/4} n}\right) \frac{n}{\ln n}$. The result is given in the following lemma, which we give without proof.

**Lemma 3.15** ([62], Lemma 4)**.** *Maker has a strategy to create a $k_0$-expander in $12n$ moves in the $(1 : b')$ game.*

By Claim 3.11, Maker can achieve minimum degree at least 12 for every vertex $v$ in his graph before Breaker claims at least $(1 - \delta)n$ edges incident to $v$ in the $(a : b)$ game. Now, Lemma 3.15 in our setting of the $(a : b)$ Minimum degree game gives the $k_0$-expander in Maker's graph. In the $(1 : b')$ Minimum degree 12 game, Maker needs at most $12n$ moves to achieve this. Thus, Maker can finish this stage when playing the $(a : b)$ game in at most $\frac{12n}{a}$ moves.

A *booster* is a non edge of a graph $G$ whose addition to $G$ creates a graph $G_1$ which contains a Hamilton cycle or whose maximum path is longer than the maximum path in $G$.

We use the following lemmas.

**Lemma 3.16** ([62], Lemma 1)**.** *Let $G$ be a connected non-Hamiltonian $k$-expander. Then at least $(k+1)^2/2$ nonedges of $G$ are boosters.*

**Lemma 3.17** ([62], Lemma 2)**.** *Let $G = (V, E)$ be a $k$-expander. Then every connected component of $G$ has size at least $3k$.*

To see that Maker can follow his strategy in stage 2, we use Lemma 3.17. In Maker's graph, every connected component is of size at least $3k_0$. This gives more than $9k_0^2 = \frac{324n^2}{\ln n}$ edges between any two of these components. Maker's aim is to claim an edge between every two components, and for this he needs at most $\frac{n}{3ak_0}$ moves. During whole stage 1 and in the following $\frac{n}{3ak_0}$ moves, Breaker could claim altogether

$$\left(\frac{12n}{a} + \frac{n}{3ak_0}\right) b = \frac{nb}{a}\left(12 + \frac{1}{3k_0}\right) = (1 - 2\delta)\frac{216n^2 + n\sqrt{\ln n}}{18(a + \ln n)} < \frac{13n^2}{\ln n}$$

edges. This implies that Breaker cannot claim all the edges between some two connected components, so he cannot prevent Maker from creating a connected expander.

In stage 3, Maker creates a Hamilton cycle by adding boosters one by one. By Lemma 3.16 there are at least $\frac{k_0^2}{2} = \frac{18n^2}{\ln n}$ nonedges that are boosters. Maker has to add at most $n$ boosters, which he can do in at most $\frac{n}{a}$ moves. In all three stages both players can claim at most $\left(\frac{12n}{a} + \frac{n}{3k_0 a} + \frac{n}{a}\right)(a+b) < \frac{14n(a+b)}{a} < \frac{18n^2}{\ln n}$ edges. This completes the proof of Theorem 3.10. □

## 3.5 Concluding remarks and open problems

**Determining the threshold bias.** In this chapter we have tried to determine the winner of the $(a : b)$ Connectivity game and $(a : b)$ Hamiltonicity game on $E(K_n)$ for all values of $a$ and $b$. We have established lower and upper bounds on the threshold biases $b_{\mathcal{T}_n}(a)$ and $b_{\mathcal{H}_n}$ for every value of $a$. For most values, these bounds are quite sharp. However, for $a = c \ln n$, where $c > 0$ is fixed, the first order terms in the upper bound and the lower bound differ. For that reason, we feel that an improvement of the bounds in this case would be particularly interesting. Also, in $(a : b)$ Hamiltonicity game, it would be interesting to narrow the gap between lower and upper threshold bias when $a = \omega(\ln^{5/4} n)$.

**Analysing other games.** There are many well-studied Maker-Breaker games played on the edge set of the complete graph for which, in the biased $(a : b)$ version, the identity of the winner is known for $a = 1$ and almost all values of $b$. One example is the $H$-game, where $H$ is some fixed graph (see [20]). It would be interesting to analyse this game for other values of $a$ (and corresponding values of $b$) as well.

We note that all the results obtained for the Connectivity game also hold for the Positive minimum degree game, where Maker's goal is to touch all $n$ vertices of the board $K_n$, and Breaker's goal is to prevent Maker from doing so. Indeed, if Maker wins the Connectivity game, then he clearly wins the Positive minimum degree game with the same parameters as well. On the other hand, in all our results that guarantee Breaker's win in the Connectivity game, we in fact prove that Breaker can isolate a vertex in Maker's graph, which clearly also ensures Breaker's win in the Positive minimum degree game.

# Chapter 4

# Avoider-Enforcer star games

Our main objective in this chapter is to provide explicit winning strategies for both players under both sets of rules in the $(1 : b)$ Avoider-Enforcer $k$-star game and in monotone $K_{2,2}$-game played on $E(K_n)$. Theorem 2.3 is a direct consequence of Theorem 4.3 that we will prove in Section 4.2. Theorem 2.4 will be proved in Section 4.3.

## 4.1 Preliminaries

The act of claiming one free edge by one of the players is called a *step*. In the strict game, Enforcer's $b$ (Avoider's 1) successive steps are called a *move*. In the monotone game, each move consists of at least $b$, respectively 1 steps. A *round* in the game consists of one move of the first player (Avoider), followed by one move of the second player (Enforcer). Whenever one of the players claims an edge incident to some vertex $u$, we say that the player *touched* $u$. By $A_i$ and $F_i$ we denote the graphs with vertex set $V$, whose edges were claimed by Avoider, respectively Enforcer, in the first $i$ rounds. For any vertex $v \in V$, by $d_{A_i}(v)$ and $d_{F_i}(v)$ we denote the degree of $v$ in $A_i$, respectively $F_i$. We sometimes omit the sub index $i$ when its value is clear, unknown, or unimportant. In these cases we also refer to $d_{A_i}(v)$ as the $A$-degree of $v$. The union of the two graphs $A_i$ and $F_i$ is called the *global graph* and is denoted by $G_i$.

Let $r = r(n,b)$ be the integer for which $1 \leq r \leq b+1$ and $\binom{n}{2} \equiv r \mod (b+1)$ hold. The value of $r = r(n,b)$ is the number of free edges before the last round of the strict game, and it may be very significant in determining the identity of the winner in this game. Note that $r$ is the number of edges which remain for Avoider to choose from in his last move when playing the strict $(1 : b)$ game. We will need the following two number theoretic statements, by Bednarska-Bzdęga [19].

**Fact 4.1.** *Let $r < 2$ be a rational number and $c > 0$ be an integer. Then:*

*(i) There are infinitely many natural numbers $n$ such that $q | \binom{n}{2}$ for some $q$ with $cn^r < q < 2cn^r$;*

*(ii) There are infinitely many natural numbers $n$ such that $q | \left(\binom{n}{2} - 1\right)$ for some $q$ with $cn^r < q < 4cn^r$.*

**Fact 4.2.** *For every $\delta \in (0,1)$ there exists an integer $N_\delta$ such that if $N \geq N_\delta$ and $N^\delta < q < \frac{\delta N}{2 \ln N}$ hold, then there exists an integer $k$ such that $q \leq k \leq 2q \ln q/\delta$ and the remainder of the division of $N$ by $k$ is at least $q$.*

## 4.2 The $k$-star game

In order to state this general result about the $k$-star game, we need to introduce some functions: let

$$e^+_{n,k} = \max\left\{ b \leq 0.4 n^{\frac{k}{k-1}} : r < \frac{1}{2}\frac{n^{k+1}}{(2b)^{k-1}} \right\}, \text{ and}$$

$$e^-_{n,k} = \max\left\{ b \leq 0.4 n^{\frac{k}{k-1}} : r < \frac{1}{2}\frac{n^{k+1}}{(2b')^{k-1}} \text{ for every } 1 \leq b' \leq b \right\}.$$

**Theorem 4.3.** *Let $k \geq 3$. In the $(1 : b)$ $k$-star game $\mathcal{K}_{\mathcal{S}_k}$ we have*

*(i) $f^{mon}_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k}{k-1}})$;*

*(ii) $f^+_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k}{k-1}})$ holds for infinitely many values of $n$, and $e^+_{n,k} \leq f^+_{\mathcal{K}_{\mathcal{S}_k}} = O(n^{\frac{k}{k-1}})$ holds for all values of $n$;*

*(iii)* $f^-_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k+1}{k}})$ *holds for infinitely many values of* $n$*, and*
$\Omega(n^{\frac{k+1}{k}}) = e^-_{n,k} \leq f^-_{\mathcal{K}_{\mathcal{S}_k}} = O(n^{\frac{k+1}{k}} \ln n)$ *holds for all values of* $n$*.*

**Remark.** We would like to mention the correlation between these results and the probabilistic intuition. For different values of $b$ let us compare the outcome of the Avoider-Enforcer $(1 : b)$ $k$-star game $\mathcal{K}_{\mathcal{S}_k}$ played on $E(K_n)$ to the corresponding random graph $G \sim G(n, \frac{1}{b})$. All the following statements about $G$ hold w.h.p. (i.e. with probability tending to 1 as $n$ tends to infinity). For details the reader may refer to [26], Theorem 3.1.

- For $b = \omega(n^{\frac{k}{k-1}})$ the maximal degree in $G$ is at most $k-2$, and Avoider wins the biased $(1 : b)$ game on $E(K_n)$.

- At $b = \Theta(n^{\frac{k}{k-1}})$ vertices of degree $k-1$ emerge in $G$. If Avoider claims the last edge in the biased $(1 : b)$ game on $E(K_n)$, the appearance of a vertex of degree $k-1$ in his graph before the last round means he loses, and this is indeed the order of magnitude of $f^{mon}_{\mathcal{S}_k}$ and $f^+_{\mathcal{S}_k}$, where presumably Avoider claims the last edge.

- In the range $\omega(n^{\frac{k+1}{k}}) \leq b \leq o(n^{\frac{k}{k-1}})$ the maximal degree in $G$ is exactly $k-1$. The outcome of the strict $(1 : b)$ game on $E(K_n)$ heavily depends on the number of free edges Avoider will be able to choose from in his last move, and so the outcome oscillates.

- Finally, for $b \leq Cn^{\frac{k+1}{k}}$, where $C$ is a sufficiently small constant, vertices of degree $k$ emerge in $G$, and Enforcer wins the biased $(1 : b)$ game on $E(K_n)$.

### 4.2.1 Enforcer's strategies

In this subsection we give lower bounds on the threshold biases $f^{mon}_{\mathcal{K}_{\mathcal{S}_k}}, f^+_{\mathcal{K}_{\mathcal{S}_k}}$ and $f^-_{\mathcal{K}_{\mathcal{S}_k}}$.

**Proof of Theorem 4.3** (*i*)**, lower bound.** Let $b = 0.4n^{\frac{k}{k-1}}$. First, we describe a strategy for Enforcer in the $(1 : b)$ monotone game, then prove it is a winning strategy. Enforcer's strategy is basically to enforce a vertex with $A$-degree $k-1$ and then to claim all free edges but one incident to that vertex. In every move before such a vertex appears, Enforcer maintains a dynamic partition $V = I \cup C$ in such a way that the following two properties hold: $d_F(v) = 0$ for every $v \in I$ (i.e. all Enforcer's edges are inside $C$), and $G[C]$, the global graph induced on $C$, is a clique (i.e. there are no free edges inside $C$).

At the beginning of the game, we set $I = V$ and $C = \emptyset$. At any point during the game, if he cannot follow the proposed strategy he forfeits the game. If at any point during the game Avoider creates a vertex of $A$-degree at least $k$, he loses. Therefore, for simplicity, we assume that in this case Avoider immediately forfeits.

Explicitly, Enforcer's strategy is as follows:
Let $i > 0$ and let $I_i$ and $C_i$ represent the sets $I$ and $C$, respectively, at the end of the $i$th round of the game. Suppose that $i-1$ rounds of the game have been played and that Enforcer was able to ensure the existence of $I_{i-1}$ and $C_{i-1}$ as described above. In his $i$th move, Enforcer checks if there is a vertex $v \in V$ of $A$-degree $k-1$, and if there is, he claims all free edges in the graph but one, incident to $v$. If there is no such vertex, Enforcer enumerates the vertices $v_1^i, v_2^i, \ldots, v_{|I_{i-1}|}^i$ in a non-decreasing $A$-degree order and determines the smallest integer $n_i$ such that the number of free edges inside $C_{i-1} \cup \{v_1^i, \ldots, v_{n_i}^i\}$ is at least $b$. Then he claims all these edges and sets $C_i := C_{i-1} \cup \{v_1^i, \ldots, v_{n_i}^i\}$ and $I_i := I_{i-1} \setminus \{v_1^i, \ldots, v_{n_i}^i\}$. This new partition clearly possesses the two required properties. Note that by the definition of $n_i$, as long as there is no vertex of $A$-degree $k-1$, Enforcer does not claim more than $b + n = (1 + o(1))b$ edges per move.

In order to prove that the proposed strategy is indeed a winning strategy for Enforcer, we have to show that no matter how Avoider plays, if Enforcer plays according to this strategy, a vertex of $A$-degree $k-1$ appears and at this point (if Avoider had not already lost) there is still a free edge incident to that vertex and the number of free edges is strictly larger than $b$.

The proof goes by contradiction. If Avoider claims at least $\frac{kn}{2}$ edges throughout the game, the average degree in his graph will be at least $k$, and so it must contain a vertex of degree $k$ and thus Avoider lost. So, we suppose that his graph has less than $\frac{kn}{2}$ edges. If at any point of the game (during any of the stages), the $A$-degree of some vertex is at least $k$, Avoider has lost the game. To obtain a contradiction, we assume that during the whole game there are less than $\frac{kn}{2}$ edges in Avoider's graph and there is no vertex of $A$-degree at least $k$ in it. So, the number of edges claimed by both players in each round is at most $(1 + o(1))b$.

For the analysis we divide the course of the game into stages: the game begins at stage 1; for every $1 \leq j \leq k-2$, stage $j$ ends (and stage $j+1$ begins) at the end of the $i$th round, if $\frac{1}{|I_i|}\sum_{v \in I_i} d_{A_i}(v) < j$ and $\frac{1}{|I_{i+1}|}\sum_{v \in I_{i+1}} d_{A_{i+1}}(v) \geq j$ hold, i.e. the average $A$-degree in $I$ became at least $j$ during the $(i+1)$st round. This is well defined, as this value cannot decrease during the game: Avoider's moves increase this value, and Enforcer removes vertices of minimal $A$-degree from $I$, so he does not decrease it. It is possible that several stages will end at the same time if the average $A$-degree in $I$ increases by more than one between two consecutive rounds. In this case there will be stages of length zero. Note that if at the end of round $i$ the game is in stage $j$, then every vertex in $C_i$ was of $A$-degree strictly less than $j$ when it was added to $C$. Indeed, Enforcer's strategy implies that otherwise every vertex in $I_i$ is of $A$-degree at least $j$ and so stage $j$ would have ended.

The following claim estimates the size of $I$ at the end of stage $j$, for $1 \leq j \leq k-2$. It takes into account the only thing we know about Avoider's play: that he claims at least one edge per move, thus increasing the $A$-degree of at least one vertex in $I$ by at least one.

**Claim 4.4.** For every $1 \leq j \leq k-2$, when stage $j$ ends, the inequality $|I| \geq 0.9n^{1-\frac{j}{k-1}}$ holds.

**Proof.** First, let us consider the size of $I$ just before the beginning of the last round in the game (after which there will be no free edges left on the board). It cannot be that $|I| = \Theta(n)$ because that would imply that there are $\Theta(n^2) = \omega(b)$ free edges remaining, in contradiction to the number of edges claimed by both players in each round. Therefore, before the last round $|I| = o(n)$ and $|C| = (1 - o(1))n$ and so there are at least $(1 - o(1))|I|n$ free edges. Since this number must be at most $(1 + o(1))b$ we get $|I| \leq 0.5n^{\frac{1}{k-1}}$ just before the game ends.

We prove the claim by induction on $j$. Suppose for contradiction that the statement of the claim does not hold for $j = 1$. Since $I$ gets small enough, it means that for some $i$, after the $i$th round we have $|I_i| < 0.9n^{1-\frac{1}{k-1}}$, but the inequality $\frac{1}{|I_i|}\sum_{v \in I_i} d_{A_i}(v) < 1$ holds. Together we get $\sum_{v \in I_i} d_{A_i}(v) < 0.9n^{1-\frac{1}{k-1}}$. Due to Enforcer's strategy, in each move $1 \leq l \leq i$, Avoider must claim an edge having at least one of its endpoints, say $u$, in $I_l$, increasing $u$'s $A$-degree by one. As during these rounds Enforcer moves from $I_l$ to $C_{l+1}$ only vertices with $A$-degree zero (otherwise it would mean that the average size of $I$ is at least 1, a contradiction), $u \notin C_{l+1}$. Therefore, the sum $\sum_{v \in I_l} d_{A_l}(v)$ increases by at least one in each round and thus $i < 0.9n^{1-\frac{1}{k-1}}$. Knowing that during the first stage all the vertices that Enforcer adds to $C_i$ are of $A$-degree 0, the number of Enforcer's edges in $C_i$ is $\binom{|C_i|}{2}$. On the other hand, Enforcer claims only $(1 + o(1))b$ edges per move, hence $\binom{|C_i|}{2} < i(1 + o(1))b < 0.4n^2$. Thus, $|C_i| < 0.9n$ and $|I_i| = n - |C_i| > 0.1n$, a contradiction.

Before proceeding to the second part of the proof, we make some observations. Let $g$ be the last round of the first stage. We assume that $I_g = o(n)$, as clearly Enforcer's win under this assumption implies Enforcer's win under the assumption $I_g = \Theta(n)$ (in case $I_g = \Theta(n)$, Enforcer can reduce its size to $I_g = o(n)$ in the following $o(n)$ moves during which the average $A$-degree can only grow). Since all of Avoider's edges and all the free edges have at least one endpoint in $I_g$, and since $k$ is a constant, throughout the game Avoider claims at most $k|I_g| = o(n)$ edges, unless he loses. Therefore, for every $i > g$ and for every vertex $v \in I_i$ there are at least $(1 - o(1))n$ free edges between $v$ and $C_i$. We conclude that while there is no vertex of $A$-degree $k - 1$, Enforcer, in each of his moves after the $g$th, moves at most $(1 + o(1))\frac{b}{n}$ vertices from $I$ to $C$.

We now proceed with our proof. Suppose for contradiction that for some $1 < j \leq k-2$, the claim holds for stage $j-1$, but not for stage $j$, i.e. there exists an integer $i$ such that $|I_i| < 0.9n^{1-\frac{j}{k-1}}$, but $\sum_{v \in I_i} d_{A_i}(v) < j|I_i|$ holds. Once again, such an $i$ must exist since $I$ gets small enough. Denote by $m$ the number of rounds that have been played in stage $j$ up to and including round $i$, excluding the first round of the stage. For any of those rounds $l$ let $w(l) = \sum_{v \in I_l} (d_{A_l}(v) - (j-1))$. This is a non-negative integer after the first round of stage $j$. Avoider touches at least one vertex of $I$ in each of his moves, and so he increases this sum in each move. Enforcer, however, only removes from $I$ vertices of degree less than $j$ (otherwise it would imply that the average size of $I$ is at least $j$, a contradiction), so he does not decrease this sum in his moves. Hence, the sum increases by at least one in each round during stage $j$, so we get $m \leq w(i)$. Note that if $w(l) \geq |I_l|$ holds, then the average $A$-degree in $I_l$ is at least $j$, thus stage $j$ would have ended. This yields $w(i) < |I_i| < 0.9n^{1-\frac{j}{k-1}}$. It follows that during stage $j$ Enforcer has removed at most $(m+1)(1+o(1))\frac{b}{n} < 0.4n^{1-\frac{j-1}{k-1}}$ vertices from $I$, but using the induction hypothesis we get that $|I_i| \geq 0.9n^{1-\frac{j-1}{k-1}} - 0.4n^{1-\frac{j-1}{k-1}} = 0.5n^{1-\frac{j-1}{k-1}}$, a contradiction. □

By Claim 4.4 stage $k-2$ ends, and at its end there are at least $0.9n^{\frac{1}{k-1}}$ vertices in $I$, and therefore at least $(1-o(1))n|I| > 0.89n^{\frac{k}{k-1}} > 2.2b$ free edges remain. If after Avoider's first move in stage $k-1$ a vertex $v$ of $A$-degree $k-1$ appears, Enforcer in his next move may proceed according to his strategy and claim all free edges but one adjacent to $v$ and win, a contradiction. Otherwise, he plays his standard move, leaving at least $1.1b$ free edges after his move, and at this point all vertices in $I$ must be of $A$-degree exactly $k-2$. In his next move Avoider must create a vertex $v$ with $A$-degree $k-1$, and then, once again, Enforcer in his next move may proceed according to his strategy and claim all free edges but one adjacent to $v$ and win, a contradiction. In both cases Avoider cannot claim more than $|I| = o(b)$ edges without creating a vertex of degree $k$, so he can do nothing to stop Enforcer. This completes the proof for the lower bound in Theorem 4.3 $(i)$. □

The above strategy cannot be used in exactly the same manner in the strict game for two reasons. First of all, as Enforcer must claim exactly $b$ edges per move, he cannot maintain the clique $C$ in the global graph and only claim edges inside it. More importantly, even if Avoider creates a vertex $v$ of $A$-degree $k-1$, Enforcer cannot make sure (in general) that Avoider eventually will claim another edge incident to $v$ and lose.

**Proof of Theorem 4.3** $(ii)$**, lower bound.** Note that if $b = o(n)$ then the length of the game, and therefore the number of Avoider's edges at the end of the game, is $\Theta(n^2/b) = \omega(n)$, so in this case Enforcer wins no matter how he plays. If $b = \Theta(n)$ Enforcer does the following: before the game starts he chooses an arbitrary set $U \in V$ of size $|U| = b^{\frac{k^2}{k^2+1}} < n$, and in each step he claims some arbitrary free edge with at least one endpoint outside $U$ until he can no longer do so, i.e. until all free edges lie completely inside $U$. Then he pretends to start a new game on $n' := b^{\frac{k^2}{k^2+1}}$ vertices with bias $b = n'^{\frac{k^2+1}{k^2}}$ according to the strategy for the case $b = \omega(n)$. This is not exactly a new game because there may be some edges inside $U$ already claimed by Avoider, and the "new" game may start during Enforcer's move. However, since Avoider can claim only a constant number of edges incident to each vertex, and since Enforcer makes at most $b$ additional steps before Avoider's first move, these factors have no significant effect. They can only affect the basis of the induction in the proof of Claim 4.5 to follow, and it is easy to see that the analysis there is still valid. The number of free edges before the last round is also affected, but for this value of $b$ Enforcer wins regardless of that number, so it does not matter.

We assume that $b = \omega(n)$ and $b \leq 0.4n^{\frac{k}{k-1}}$. Recall that $r$ denotes the integer which satisfies $1 \leq r \leq b+1$ and $\binom{n}{2} \equiv r \bmod (b+1)$. This is the number of free edges before the last round of the game, i.e. the number of edges Avoider will be able to choose from before his last move. Enforcer will then claim the remaining $r-1$ edges in his last move. Enforcer strategy for the strict game is very similar to his strategy for the monotone game. If at any point during the game he cannot follow it, he immediately forfeits the game, and we assume that if Avoider has increased the maximum degree in his graph to at least $k$ (and thus lost), he also forfeits. We say that a free edge is a *threat* if it is adjacent to a vertex of $A$-degree $k-1$. If at any point during the game there exist at least $r$ threats, Enforcer switches his strategy and plays arbitrarily until the end of the game, with the only rule of not claiming a threat unless he has to. Note that if this happens, Avoider loses in his next move, so the appearance of $r$ threats ensures Enforcer's win.

Until $r$ threats appear, Enforcer, like in the monotone game, maintains a partition $V = I \cup C$ (where $I_i$ and $C_i$ represent the respective sets after the $i$th round) such that $F_i[I_i]$, Enforcer's graph induced on $I_i$, is empty and $G_i[C_i]$, the global graph induced on $C_i$, is a clique. The only difference here is that Enforcer may sometimes claim edges between $C$ and $I$. Once again, initially $I_0 = V$ and $C_0 = \emptyset$. In his $i$th move Enforcer enumerates the vertices $v_1^i, v_2^i, \ldots, v_{|I_{i-1}|}^i$ in a non-decreasing $A$-degree order, with a possible tie breaking rule that will be presented shortly, and determines the largest integer $n_i$ such that the number of free edges inside $C_{i-1} \cup \{v_1^i, \ldots, v_{n_i}^i\}$ is at most $b$. Then he claims all these edges and sets $C_i := C_{i-1} \cup \{v_1^i, \ldots, v_{n_i}^i\}$ and $I_i := I_{i-1} \setminus \{v_1^i, \ldots, v_{n_i}^i\}$. This new partition possesses the two required properties. As opposed to the monotone game, Enforcer must claim exactly $b$ edges, so there are $l_i$ more edges Enforcer must claim in order to complete his move. Enforcer chooses these edges in the following way: he picks the next $4k$ vertices of the enumeration of $I_{i-1}$, $v_{n_i+1}^i, v_{n_i+2}^i, \ldots, v_{n_i+4k}^i$, and for each $1 \leq h \leq 4k$ he claims arbitrarily $\lfloor \frac{l_i+h-1}{4k} \rfloor$ free edges (we call them *extra edges*) joining $v_{n_i+h}^i$ to vertices of $C_i$, to get a total of $l_i$ edges. When enumerating the vertices of $I_i$ in his $(i+1)$st move, Enforcer uses extra edges as a tie breaker: he always places vertices which received extra edges in his previous move as early as possible, that is, among all vertices with $A$-degree $d$, first come those which received extra edges in his previous move and afterwards those which did not. At this point, remaining ties are broken arbitrarily.

Now we prove that Enforcer is able to play according to the proposed strategy without ever having to forfeit the game, and that at some point $r$ threats will appear. Let us see first that Enforcer can claim the extra edges in each move as described above. Let $v$ be a vertex that was picked by Enforcer to receive extra edges in his $i$th move. If Avoider does not touch $v$ in his $(i+1)$st move, $v$ will be among the first $4k$ vertices in Enforcer's $(i+1)$st move enumeration. Indeed, every vertex that was placed after the first $4k$ vertices of $I_i$ in the $i$th enumeration had an $A$-degree at least as large as that of $v$. Since $v$ was not touched by Avoider in his last move it still holds, and in case they have equal $A$-degree the tie breaker puts $v$ earlier. Enforcer will then add $v$ to $C_{i+1}$ since his bias is large enough. So $v$ remains in $I$ only if Avoider touched it immediately after Enforcer. Thus, any vertex $v \in V$ can receive extra edges at most $k$ times, as otherwise Avoider must have already claimed an $\mathcal{S}_k$.

Suppose now that we are in the $(i+1)$st move of Enforcer for some $i \geq 0$, just before he adds $l_{i+1}$ extra edges to the $4k$ selected vertices from $I_{i+1}$. Note that $l_{i+1} < |C_{i+1}|$ by the choice of $n_i$, and clearly $|C_j| \leq |C_{i+1}|$ holds for any $j \leq i$. Thus, if this is the $m$th time that a vertex $v$ is picked to receive extra edges in Enforcer's graph, the total number of extra edges incident to $v$ is at most $m\lceil\frac{|C_{i+1}|}{4k}\rceil \leq k\lceil\frac{|C_{i+1}|}{4k}\rceil = (0.25+o(1))|C_{i+1}|$, as $m \leq k$ and $|C_i| = \omega(1)$ for all $i > 0$.

In addition, $d_A(v) < k$ always holds, so for every $i$ there are at least $(0.75-o(1))|C_i|$ free edges between any vertex $v \in I_i$ and $C_i$, so Enforcer is able to follow the above strategy. We now need the following strict analogue of Claim 4.4.

**Claim 4.5.** For every $1 \leq j \leq k-2$, when stage $j$ ends, the inequality $|I| \geq 0.9n\left(\frac{n}{2b}\right)^j$ holds.

**Proof.** The proof, by induction on $j$, is identical to the one of Claim 4.4, except for some slightly different calculations as follows. Just before the beginning of the last round, the size of $I$ cannot be linear in $n$ because that would imply $\Theta(n^2)$ free edges to be claimed in the last round. Therefore, before the last round $|C| = (1-o(1))n$ and there are at least $(0.75-o(1))|I|n \leq b+1$ free edges, which implies $|I| < 1.5\frac{b}{n} < 0.9n^{\frac{1}{k-1}} \leq 0.9n\left(\frac{n}{2b}\right)^{k-2}$, so $I$ cannot be too large just before the game ends.

For $j=1$ we assume for contradiction that for some $i$ both inequalities

$|I_i| < 0.45n^2/b$ and $\sum_{v \in I_i} d_{A_i}(v) < |I_i|$ hold, and since the sum $\sum_{v \in I} d_A(v)$ increases by at least one in every round we get $i < |I_i|$. By counting Enforcer's edges in $C_i$ and the number of edges he has claimed in the first $i$ rounds we get the inequality $\frac{1}{2}(|C_i| - k)^2 < ib < 0.45n^2$. Thus, $|C_i| < 0.95n$ and $|I_i| = n - |C_i| > n/20$, a contradiction.

We observe that the length of the game, and therefore the number of Avoider's edges in his final graph is $\Theta(n^2/b) = o(n)$. If $g$ denotes the last round of the first stage we get $|I_g| = o(n)$. Since for every $i > g$ there are at least $(0.75 - o(1))|C_i| \geq 2n/3$ free edges between any vertex $v \in I_i$ and $C_i$, Enforcer, in each move after the $g$th, moves at most $\frac{3b}{2n}$ vertices from $I$ to $C$.

To complete the proof we assume for contradiction that for some $1 < j \leq k-2$ the claim holds for stage $j-1$ but not for stage $j$, i.e. there exists an integer $i$ such that $|I_i| < 0.9n\left(\frac{n}{2b}\right)^j$, but $\sum_{v \in I_i}(d_{A_i}(v)) < j|I_i|$, or $\sum_{v \in I_i}(d_{A_i}(v) - (j-1)) < |I_i|$. This sum is non-negative after the first round of the stage and increases by at least one in every round, so there were at most $|I_i|$ rounds in this stage. Since the claim holds for $j-1$ it follows that $|I_i| \geq 0.9n\left(\frac{n}{2b}\right)^{j-1} - |I_i|\frac{3b}{2n} \geq \left(0.9n\left(\frac{n}{2b}\right)^{j-1}\right)(1 - 0.75)$, a contradiction. $\square$

Let us consider the beginning of stage $k-1$. Denote by $g$ the first round of this stage. As already shown, there are at least $(0.75 - o(1))|C| = (0.75 - o(1))n$ free edges between any vertex $v \in I$ and $C$. Thus every vertex in $I$ of degree $k-1$ creates at least $(0.75 - o(1))n$ unique threats. So for any $\varepsilon > 0$, if we denote $r' := r/((0.75 - \varepsilon)n)$, then if there are, at any point before the last round, $r'$ vertices of degree $k-1$, there are at least $r$ threats and Enforcer wins. The following claim shows that if the game lasts more than $g + r'$ rounds than at least $r'$ vertices of degree $k-1$ will appear in $I$ before the last round.

**Claim 4.6.** After Avoider's $(g+l)$th move either Avoider's graph contains an $\mathcal{S}_k$ or there are at least $l$ vertices in $I$ of $A$-degree $k-1$.

**Proof.** Let $v_1, v_2, \ldots, v_t$ denote the vertices of $I$ after the $g$th round with $A$-degree at most $k-3$ and let us write $m = \sum_{i=1}^{t}(k - 2 - d_{A_g}(v_i))$. If

Avoider has not yet created an $\mathcal{S}_k$, all vertices have $A$-degree at most $k-1$, thus after the $g$th round there are at least $m$ vertices in $I$ with $A$-degree $k-1$, as the average $A$-degree in $I_g$ is at least $k-2$. Since all edges claimed by Avoider have at least one endpoint in $I$, in every move after the $g$th but at most $m$ he creates a new vertex of $A$-degree $k-1$. $\square$

In his $g$th move Enforcer removes at most $(1.33+o(1))\frac{b}{n}$ vertices from $I$. A simple calculation yields

$$\frac{2b}{n} = \left(\frac{2b}{n}\right)^{k-1}\left(\frac{n}{2b}\right)^{k-2} \leq \left(0.8n^{\frac{1}{k-1}}\right)^{k-1}\left(\frac{n}{2b}\right)^{k-2} \leq 0.64n\left(\frac{n}{2b}\right)^{k-2},$$

and by using Claim 4.5 we get $|I_g| \geq |I_{g-1}|-(1.33+o(1))\frac{b}{n} \geq 0.47n\left(\frac{n}{2b}\right)^{k-2}$, and therefore the number of free edges after the $g$th round is at least $|I_g|(0.75 - o(1))n \geq 0.35\frac{n^k}{(2b)^{k-2}}$. Thus, if the inequality $r'(b+1) < 0.35\frac{n^k}{(2b)^{k-2}}$ holds, the game will last more than $g+r'$ rounds and Enforcer will win. This inequality may be simplified to

$$r < \frac{1}{2}\frac{n^{k+1}}{(2b)^{k-1}}. \tag{4.2.1}$$

From all that is said above, it is clear that $e^+_{n,k} \leq f^+_{\mathcal{K}_{\mathcal{S}_k}}$.

Finally, applying Fact 4.1 $(ii)$ with $r = \frac{k}{k-1}$ and $c = 1$, we obtain infinitely many integers $n$ such that there exists an integer $q$ with $n^{\frac{k}{k-1}} < q < 4n^{\frac{k}{k-1}}$ and $q \mid \left(\binom{n}{2}-1\right)$. For each such $n, q$, by setting $b := \lfloor q/10 \rfloor - 1$ we get that $0.09n^{\frac{k}{k-1}} < b < 0.4n^{\frac{k}{k-1}}$. Denote $s := \left(\binom{n}{2}-1\right)/q$ and $\alpha := q \bmod 10$. Both players claim together $b+1 = \lfloor q/10 \rfloor = (q-\alpha)/10$ edges in each round, so after $10s$ rounds $qs - \alpha s = \binom{n}{2} - 1 - \alpha s$ edges will be claimed, so there will be $1+\alpha s$ free edges left on the board. Note that $s = \Theta(n^2/q) = o(n)$, and since $b = \omega(n)$ the next round will be the last round in the game, so $r = 1+\alpha s$. Regardless of the value of $\alpha$ we get that $r = o(n)$. However, $\frac{n^{k+1}}{(2b)^{k-1}} = \Theta(n)$, so the inequality $r < \frac{1}{2}\frac{n^{k+1}}{(2b)^{k-1}}$ holds and Enforcer wins. This shows that for these values of $n$ we have $e^+_{n,k} = \Omega(n^{\frac{k}{k-1}})$. $\square$

**Proof of Theorem 4.3** (*iii*)**, lower bound.** The proof is identical to the proof of Theorem 4.3 (*ii*) for the lower bound. Exactly the same calculation and explanation give that $e^-_{n,k} \le f^-_{\mathcal{K}_{\mathcal{S}_k}}$. Note that since $r \le b+1$, the inequality (4.2.1) holds if $b < \frac{1}{2}\frac{n^{k+1}}{(2b)^{k-1}} \iff (2b)^k < n^{k+1} \iff b < 0.5n^{\frac{k+1}{k}}$ (it suffices to use $b$ instead of $b+1$ since the constant $\frac{1}{2}$ used in the inequality is not tight). So, if $b < 0.5n^{\frac{k+1}{k}}$ Enforcer wins regardless of $r$, which shows that for all $n$, $e^-_{n,k} = \Omega(n^{\frac{k+1}{k}})$.

□

### 4.2.2 Avoider's strategy

In this subsection we establish upper bounds on the threshold biases $f^{mon}_{\mathcal{K}_{\mathcal{S}_k}}$, $f^+_{\mathcal{K}_{\mathcal{S}_k}}$ and $f^-_{\mathcal{K}_{\mathcal{S}_k}}$.

**Proof of Theorem 4.3** (*i*)**, upper bound.** As shown in Section 4.2.1, in the monotone game Avoider is doomed at the moment he creates a vertex of $A$-degree $k-1$ which still has at least one free edge incident to it. To prevent this situation we provide Avoider with a simple strategy which keeps the maximum degree of his graph low. In this strategy Avoider claims exactly one edge in each move.

**Avoider's strategy**:
Let $E_i$ be the set of free edges before Avoider's $i$th move, then for any edge $(u, v) \in E_i$ let $d_{max}(u, v) = \max\{d_{A_{i-1}}(u), d_{A_{i-1}}(v)\}$. Avoider claims an arbitrary edge from the set of free edges with minimum $d_{max}$-value.

To obtain the upper bound of Theorem 4.3 *(i)* it is enough to show that if Avoider plays according to this strategy, he wins independently of Enforcer's strategy provided Enforcer's bias is at least $2n^{\frac{k}{k-1}}$. For the analysis, we divide the course of the game into stages according to the degrees in Avoider's graph, similarly to the proof of Enforcer's strategies. This time, stage $j$ starts with Avoider's move in which he creates the first vertex with A-degree $j$ and ends right after Enforcer's last move for which Avoider's graph still has maximum degree $j$.

**Claim 4.7.** At the end of stage $j$, the vertices of $A$-degree at most $j-1$ form a clique in the global graph.

**Proof.** According to his strategy, Avoider would otherwise claim an edge without creating a vertex of $A$-degree $j+1$. $\square$

**Claim 4.8.** For any positive integer $j$, at the end of stage $j$, the number of vertices of $A$-degree $j$ is at most $(2^{j-1}+o(1))\frac{n^{j+1}}{b^j}$ and the number of free edges remaining in the game is at most $(2^{j-1}+o(1))\frac{n^{j+2}}{b^j}$.

**Proof.** We prove the claim by induction on $j$. As in every round the total number of edges taken by Avoider and Enforcer is at least $b+1$, the whole game, and thus stage 1, cannot last longer than $\lceil\binom{n}{2}/(b+1)\rceil$ rounds. As in each move during stage 1 Avoider creates two vertices of degree 1, the number of such vertices at the end of the stage is at most $\lceil n(n-1)/(b+1)\rceil = (1+o(1))\frac{n^2}{b}$. By Claim 4.7, none of the edges between two vertices of $A$-degree 0 is still available, therefore the number of free edges is at most $(1+o(1))\frac{n^2}{b}\cdot n = (1+o(1))\frac{n^3}{b}$. This proves the statement of the claim for the case $j=1$.

Assume now that the statement of the claim holds for $j$ and thus the number of free edges at the end of stage $j$ is at most $(2^{j-1}+o(1))\frac{n^{j+2}}{b^j}$. Therefore stage $j+1$ cannot last longer than $(2^{j-1}+o(1))\frac{n^{j+2}}{b^j(b+1)} = (2^{j-1}+o(1))\frac{n^{j+2}}{b^{j+1}}$. In each move Avoider claims exactly one edge, therefore in one move at most two more vertices of $A$-degree $j+1$ can appear and thus the number of such vertices by the end of stage $j+1$ cannot exceed $(2^j+o(1))\frac{n^{j+2}}{b^{j+1}}$. Thus, by Claim 4.7, the number of free edges at the end of stage $j+1$ is at most $(2^j+o(1))\frac{n^{j+2}}{b^{j+1}}\cdot n = (2^j+o(1))\frac{n^{j+3}}{b^{j+1}}$. $\square$

Claim 4.8 easily implies the upper bounds of Theorem 4.3 on $f^{mon}_{\mathcal{K}_{\mathcal{S}_k}}$. Indeed, if Enforcer's bias is at least $2n^{\frac{k}{k-1}}$, then if Avoider uses the above strategy, then either stage $k-2$ does not end during the course of the game and thus not even a vertex of $A$-degree $k-1$ is created, or at the end of stage $k-2$ the number of free edges is strictly less than $b+1$. In this case, no matter which edge Avoider claims in his first move in stage $k-1$, Enforcer will be forced to claim all remaining free edges in his move right after that, and Avoider wins. $\square$

**Proof of Theorem 4.3** $(ii)$**, upper bound.** Let $b \geq 2n^{\frac{k}{k-1}}$. Avoider's strategy in the strict biased $(1 : b)$ game is the same as in the proof of Theorem 4.3 $(i)$ for the upper bound. This strategy is valid as Avoider claims exactly one edge in each move. The proof that Avoider can follow this strategy is exactly the same as the proof of Theorem 4.3 $(i)$ for the upper bound. $\square$

**Proof of Theorem 4.3** $(iii)$**, upper bound.** Recall the proof of Theorem 4.3 $(i)$ for the upper bound. Avoider uses the same strategy as there, which is possible since it assumes claiming exactly one edge in each move.

To see that infinitely many times $f^-_{\mathcal{K}_{\mathcal{S}_k}} = O(n^{\frac{k+1}{k}})$, suppose that the number of vertices $n$, and Enforcer's bias $b$, satisfy $2n^{\frac{k+1}{k}} < b < 4n^{\frac{k+1}{k}}$ and $\binom{n}{2} \equiv 0 \mod (b+1)$. According to part *(i)* of Fact 4.1, with $r = \frac{k+1}{k}$ and $c = 2$, there exist infinitely many integers $n$ with such $b$. Then, by applying Claim 4.8 from the proof of Theorem 4.3 $(i)$, upper bound we get that regardless of Enforcer's strategy, at the end of stage $k-2$ the number of vertices of $A$-degree $k-2$ is at most $\frac{1}{2}n^{\frac{2}{k}}$ and the number of free edges is at most $\frac{1}{2}n^{\frac{k+2}{k}}$. Therefore there remain at most $\frac{1}{4}n^{\frac{1}{k}}$ rounds in the game. In each move Avoider creates at most 2 vertices of $A$-degree $k-1$, thus before Avoider's final move, there are at most $\frac{1}{2}n^{\frac{1}{k}}$ of them, creating at most $\frac{1}{2}n^{\frac{k+1}{k}}$ threats (free edges incident to vertices of $A$-degree $k-1$). But as $\binom{n}{2} \equiv 0 \mod (b+1)$ holds, Avoider in his last move has the possibility to choose from $b+1 \geq 2n^{\frac{k+1}{k}}$ free edges, so he can choose a free edge which is not a threat and win.

Finally, to see the general upper bound on $f^-_{\mathcal{K}_{\mathcal{S}_k}}$, by applying Fact 4.2 with $\delta = \frac{k+1}{2k}$, $N = \binom{n}{2}$ and $q = 2n^{\frac{k+1}{k}}$, we obtain that for any sufficiently large $n$ there exists an integer $b$ with $2n^{\frac{k+1}{k}} \leq b \leq 8n^{\frac{k+1}{k}} \ln n$ such that the remainder $r = r(n, b)$ (the number of free edges before the last round) is at least $2n^{\frac{k+1}{k}}$. A computation identical to the one above yields the general statement about the upper bound on $f^{-}_{\mathcal{K}_{\mathcal{S}_k}}$. □

## 4.3 The $K_{2,2}$-game

**Proof of Theorem 2.4.** We will first give a winning strategy of Enforcer, thus establishing the lower bound.

Throughout the game, let $A$ be Avoider's graph, $F$ Enforcer's graph and let $t = n^{\frac{2}{3}}$. $A$ and $F$ are dynamically updated in each move. Let $V_A$ denote the set of vertices spanned by edges of $A$. A vertex $v$ is called *saturated* if Enforcer has claimed all the edges incident to it. We denote the set of such vertices by $S$. Initially, $S = \emptyset$. Vertices that are not saturated are called *unsaturated*. Since the game is monotone, it is enough to prove the case $b = \frac{1}{4}n^{\frac{4}{3}}$.

First, we give the strategy and show this is indeed a winning strategy, and then show that Enforcer can follow it. Enforcer's strategy is divided into two main stages.

**Stage 1.** Stage 1 lasts while there are less than $t$ vertices of degree at least 1 in $A$ and there exists no $P_4$ in $A$.

In each move of stage 1, Enforcer chooses an unsaturated vertex $v \in V$, s.t. $d_A(v) = 0$ and claims all the edges incident to $v$. After that, $S = S \cup \{v\}$. He repeats this process until he claims at least $b$ edges. Note that in every move of this stage Enforcer claims at least $b$ edges, but not more than $b+n$ edges. Also note that after each move of Enforcer during this stage, $F[V \backslash S]$ is empty.

**Stage 2.** Stage 2 begins when either there exists a $P_4$ in $A$ or $A$ has at least $t$ vertices of degree at least 1 after Avoider's move. Depending on which case initiated the beginning of stage 2, Enforcer differentiates between two strategies.

*Case 1:* There exists a $P_4$ in $A$.

Let $x$ and $y$ be the endpoints of Avoider's $P_4$. In his move, Enforcer claims all the edges except $(x, y)$ and Avoider loses in his next move. We will show that $(x, y)$ is still free before Avoider's next move.

*Case 2:* There is no $P_4$ in $A$ and there are at least $t$ vertices of degree at

least 1 in $A$.
In this case, Enforcer first saturates all unsaturated vertices that are still untouched by Avoider. Then, he chooses a set of vertices $R \subset V_A$, such that $t-1 \leq |V_A \backslash R| \leq t$ holds and $A[V_A \backslash R]$ contains no isolated vertex. It can be easily verified that this can be done for all graphs $A[V_A]$ without isolated vertices, which is the case here. Note here that connected components of $A$ can be stars, $K_{1,q}$, $q \geq 1$ or triangles, $K_3$. For each vertex $v \in R$, Enforcer claims all the remaining edges incident to $v$. Also, if $A[V_A \backslash R]$ contains some number $p$, $1 \leq p \leq \frac{t}{3}$, of connected components that are $P_3$, then for each such component, Enforcer claims the edge $e$ such that $P_3 + e$ is a $K_3$ in the global graph. After Avoider's next move, there has to be at least one $P_4$ in $A$ (we will show that afterwards). Enforcer spots a $P_4$. Let $x$ and $y$ be its endpoints. Enforcer claims all the unclaimed edges except the edge $(x, y)$, for which we will prove it is still free at that point. Avoider thus loses in his next move.

Now, we will show that Enforcer can follow the proposed strategy. We consider each stage separately.

Firstly, we prove that Enforcer can always make a move during stage 1, i.e. that there are always enough edges for him to claim.
Avoider can claim at most $t-1$ edges without touching at least $t$ vertices or creating a $P_4$. Thus, stage 1 can last at most $t-1$ moves and there are less than $t$ vertices of degree at least 1 in Avoider's graph. So, at any point of stage 1, there are more than $n-t$ isolated vertices in $A$. On the other hand, Enforcer claims at most $b+n$ edges per move, and so at most $(t-1)(b+n)$ edges during the stage 1. Hence, there are more than $\binom{n-t}{2} - t(b+n) > b+n$ unclaimed edges among the isolated vertices in $A$. This implies that in each move there are enough free edges at Enforcer's disposal to claim throughout whole stage 1.

Stage 2 begins with the first move of Avoider which creates at least $t$ vertices of degree at least one in $A$ or a $P_4$ in $A$. Let us suppose that stage 2 starts at round $m$, $1 \leq m \leq t$. For each case, we show that Enforcer can play by the given strategy.
*Case 1:* There exists a $P_4$ in $A$.

Let $x$ and $y$ be the endpoints of a $P_4$. During $m-1$ moves, $m \leq t$, following his strategy, Enforcer has claimed no more than $(t-1)(b+n)$ edges, none of which spans any two vertices in $A$. This implies that, unless there is a $K_{2,2}$ in $A$, the edge $(x,y)$ is still unclaimed before Enforcer's $m$th move. Since $ex(n, K_{2,2}) \approx \frac{1}{2}n^{\frac{3}{2}} < n^{\frac{3}{2}}$, unless he has lost already, Avoider could have claimed no more than $n^{\frac{3}{2}}$ edges all together. This gives at least $\binom{n}{2} - n^{\frac{3}{2}} - t(b+n) = \dfrac{n^2}{4}(1-o(1)) > b$ edges still unclaimed in the graph, so Enforcer can claim all edges except the edge $(x,y)$ in his move, thus making Avoider lose in his next move.

For showing that Enforcer can play according to this strategy in *case 2*, we will use the following simple claim.

**Claim 4.9.** At the end of stage 1, $|S| < \frac{n}{2}$.

**Proof.** Suppose for a contradiction that $|S| \geq \frac{n}{2}$. This gives at least $\binom{\frac{n}{2}}{2} + \dfrac{n^2}{4} = \dfrac{3n^2}{8} - \dfrac{n}{4}$ edges claimed by Enforcer in less than $t$ moves. However, in less than $t$ moves Enforcer, following his strategy, could claim no more than $t(b+n) \leq \frac{n^2}{4} + n^{\frac{5}{3}} < \frac{3n^2}{8} - \frac{n}{4}$ edges. A contradiction. □

*Case 2:* There is no $P_4$ in $A$ and there are at least $t$ vertices of degree at least 1 in $A$.

Let $a = |V_A|$ after the $m$th move of Avoider. Since there are $|S|$ saturated vertices it must be that $t \leq a \leq n - |S|$. The fact that there is no $P_4$ in $A$ implies that every connected component of $A$ is either a star, a $K_{1,q}$, $q \geq 1$, or a $K_3$. Having this in mind, the total number of edges in $A$ cannot be more than $a$, and the total number of connected components cannot be greater than $\frac{a}{2}$. By Claim 4.9 there are $n - |S| > \frac{n}{2}$ unsaturated vertices in Enforcer's graph.

In his next move, Enforcer saturates all $n - |S| - a$ vertices.

If $a = t$, then he claims $\binom{n-|S|-a}{2} + (n-|S|-a)a > \binom{\frac{n}{2} - n^{\frac{2}{3}}}{2} + \left(\dfrac{n}{2} - n^{\frac{2}{3}}\right) n^{\frac{2}{3}} > b$ edges.

Otherwise, we give an algorithm by which Enforcer chooses which vertices of $V_A$ will be added to $R$ to match the requirement of his strategy, i.e. that $t-1 \leq |V_A \setminus R| \leq t$.

Let $C_1, C_2, ..., C_g$, $1 \leq g \leq \frac{a}{2}$, denote the connected components of $A$, enumerated in a non-decreasing order by the number of vertices in them and let $n_i = |C_i|$, $1 \leq i \leq g$.

By choosing vertices of $C_i$ to $R$ in a *smart way*, Enforcer takes care not to choose the center of the star if $C_i$ is a star $K_{1,q}$, $q \geq 2$, but $|V_A \backslash R| - t$ arbitrarily chosen leaves of $C_i$. If $C_i$ is a triangle ($K_3$), then he adds to $R$ an arbitrary vertex from that $K_3$.

*Algorithm:*

```
j=1;
R = ∅;
while |V_A\R| - t ≥ n_j
  R = R ∪ V(C_j);
  j=j+1;
end;
if |V_A\R| - t = n_j - 1 then
  R = R ∪ V(C_j)
else
  choose |V_A\R| - t vertices of C_j in a smart way;
```

When $|V_A \backslash R| - t = n_j - 1$, Enforcer adds $n_j$ vertices to $R$ and only then $|V_A \backslash R| = t - 1$. In all other cases, $|V_A \backslash R| = t$. In all cases, the requirements of the strategy are satisfied, $t-1 \leq |V_A \backslash R| \leq t$ holds and $A[V_A \backslash R]$ contains no isolated vertex.

To saturate all $n - |S| - a$ vertices and to claim all the edges incident to every vertex $v \in R$, Enforcer needs at least $\binom{n-|S|-t}{2} + (n-|S|-t)t - (a-t) > \binom{\frac{n}{2} - n^{\frac{2}{3}}}{2} + \left(\frac{n}{2} - n^{\frac{2}{3}}\right) n^{\frac{2}{3}} - n > b$ edges. If there is some number of connected components that are $K_{1,2} = P_3$ remaining in $A[V_A \backslash R]$, Enforcer, following his strategy, claims at most $\frac{t}{3}$ additional edges.

Now, we need to prove that after Avoider's next move, unless Enforcer has already won, there will be more than $b$ unclaimed edges left among which there will be a free edge between the endpoints of a $P_4$. Avoider has to claim at least one edge $e$ in his $(m+1)$st move. Each edge Avoider claims creates a $P_4$ in his graph, since there are no isolated vertices in $A$. Suppose there is exactly one $P_4$ that Avoider created, otherwise, Enforcer chooses one arbitrary $P_4$. Let $x$ and $y$ be the endpoints of the chosen $P_4$. We need to show that in this case the edge $(x, y)$ is free. There can be some Enforcer's edges among vertices from $V_A \setminus R$ and they lie in the connected components of $A$ that are $P_3$. Suppose that $(u, w)$ is an edge incident to a $P_3$ in $A$ that Avoider claimed in his $(m+1)$st move. Let $\{v_1, v_2, v_3\}$ be the set of vertices of a $P_3$ with $d_A(v_2) = 2$ and $u \in \{v_1, v_2, v_3\}$ and let $(v_1, v_3)$ be the edge claimed by Enforcer. Also, let $(w, z)$ be the edge adjacent to $(u, w)$, such that $(w, z) \notin E(P_3)$.

If $u$ is one of $\{v_1, v_3\}$, then let $q$ be the vertex from $\{v_1, v_3\}$, different from $u$. In this case, there are two $P_4$ in $A$: $zwuv_2$ and $wuv_2q$. Let $P := zwuv_2$. In this case $x = v_2$ and $y = z$ and $(x, y) \neq (v_1, v_3)$. Analogue analysis is for the other $P_4$.

If $u = v_2$ then there are also two $P_4$ created: $zwuv_1$ and $zwuv_3$. Let $P := zwuv_1$ be one of them. In this case $x = v_1$, $y = z$ and again $(x, y) \neq (v_1, v_3)$. Analogue analysis is for the other $P_4$.

In all other cases, Avoider claims edges within a component $C_i$ or between components $C_i$ and $C_l$, none of which is a $P_3$. All the edges there are only claimed by him during the course of game. So, when a $P_4$ is created in $A$, the edge $(x, y)$ between its endpoints is still unclaimed.

It remains to be shown that there are enough free edges left on the board, after $(m+1)$st move of Avoider. Suppose that $A$ is $K_{2,2}$-free after Avoider's $(m+1)$st move. Since $ex(t, K_{2,2}) \approx \frac{1}{2}t^{\frac{3}{2}} < t^{\frac{3}{2}}$, there cannot be more than $t^{\frac{3}{2}}$ Avoider's edges in $A[V_A \setminus R]$. Therefore, there are more than $\binom{t-1}{2} - \frac{t}{3} - t^{\frac{3}{2}} = \frac{t^2}{2}(1 - o(1)) = \frac{n^{\frac{4}{3}}}{2}(1 - o(1)) > b$ still unclaimed edges. Enforcer claims all unclaimed edges except $(x, y)$. Avoider has to play the next move and claim the remaining edge, thus losing the game.

For Avoider's strategy we can use the strategy for the strict non-planarity game given in [53, Theorem 2.3] with small modifications. Avoider plays slowly, by claiming exactly one edge per move. His strategy is divided into three stages, since we do not need stage 3 from the original proof. Let $b \geq n^{\frac{4}{3}}$. We divide the game into three main stages.

**Stage 1.** In stage 1 Avoider creates a matching $M$, by claiming in each move an edge $e$ such that $e \cap e' = \emptyset$, for every edge $e' \in M$ and sets $M = M \cup \{e\}$. By $V_M$ we denote the vertices of edges in $M$. He continues playing like that until there are no free edges among vertices in $V \setminus V_M$.
**Stage 2.** In this stage, Avoider claims edges $e = (u, v)$ such that $u \in V_M$ and $v \in V \setminus V_M$ and no vertex in $V \setminus V_M$ is connected to more than one vertex in $V_M$. He does this until it is no longer possible.
**Stage 3.** In this stage Avoider claims just one edge whose endpoints are inside $V_M$.

If at any point of the game Avoider cannot play according to the given strategy, he immediately forfeits the game. This is the strategy and now we will show that his graph doesn't contain a $K_{2,2}$. After the first stage, Avoider's graph is a matching. Avoider's graph after stages 1 and 2 will be a matching with some hanging edges. So far Avoider has not created any $K_{2,2}$. There can be some $P_4$ in his graph, but the endpoints of these $P_4$ are vertices of $V \backslash V_M$, and $F[V \backslash V_M]$ is a clique. Thus, these $P_4$ are not dangerous for Avoider. We show that if Avoider plays at most one move in the third stage, his graph will not contain a $K_{2,2}$ and he will win. Indeed, whichever edge Avoider claims right after stage 2, it cannot be the edge between the endpoints of any $P_4$ in Avoider's graph. The claimed edge can just create some new path of length three or more in his graph.

Now, we show that he can follow the proposed strategy.

By $e$ we denote the total number of edges that Avoider claimed in the game, so $e \leq \frac{\binom{n}{2}}{b+1}$. Avoider's matching consist of at most $e$ edges, so $|\,[V \backslash V_M]\,| \geq n - 2e$. This implies that Enforcer must have claimed at least $\binom{n-2e}{2}$ edges. So, there are at most $\binom{n}{2} - \binom{n-2e}{2} \leq 2en$ edges left in the graph, out of which Avoider can claim at most $\frac{2en}{b}$ till the end of the game. After stage 2, there are at most $2e \cdot \frac{2en}{b} = \frac{4e^2n}{b}$ unclaimed edges in $[V_M, V \backslash V_M]$. Also, at most $\binom{2e}{2} - e$ edges are unclaimed among vertices in $V_M$. Hence, we need to prove there will be at most one more Avoider's move. After stage 2, the total number of unclaimed edges in the graph is at most $\binom{2e}{2} - e + \frac{4e^2n}{b} < \frac{n^4}{2b^2} + \frac{n^5}{b^3} < b$. This completes the proof for Avoider's strategy in the $K_{2,2}$ game. □

## 4.4 Concluding remarks and open problems

We show that for any sufficiently large $n$ and for every $k \geq 3$, the threshold biases $f^-_{\mathcal{K}_{\mathcal{S}_k}}$ and $f^+_{\mathcal{K}_{\mathcal{S}_k}}$ are not of the same order. However, for each of them we gave general upper and lower bounds for every $n$, and the exact order of magnitude only for infinitely many values of $n$. The general upper bound on $f^-_{\mathcal{K}_{\mathcal{S}_k}} = O(n^{\frac{k+1}{k}} \ln n)$ that we obtained matches the general upper bound on $f^-_{\mathcal{K}_H}$ for an arbitrary graph $H$ that Bednarska-Bzdęga obtained in [19]. The general lower bound on $f^+_{\mathcal{K}_{\mathcal{S}_k}}$ is only implicit. Recall that in order to obtain the general upper bound on $f^-_{\mathcal{K}_{\mathcal{S}_k}}$ we used Fact 4.2, which for every $k$ shows the existence of a bias $b$ of the appropriate order of magnitude such that the corresponding remainder $r = r(n, b)$ is close to $b$. In order to obtain an explicit lower bound on $f^+_{\mathcal{K}_{\mathcal{S}_k}}$ we would need an analogous number theoretic statement that ensures the existence of a bias $b$ of the right order, such that the remainder $r(n, b)$ is small enough. Nevertheless, we believe that in fact $f^+_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k}{k-1}})$ and $f^-_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k+1}{k}})$ hold for any sufficiently large $n$ and not only infinitely often. Moreover, if indeed these equalities hold for every $n$, another question arises: can we find an upper bound or a lower bound of the same order, and how good can the leading constants be?

Note that our results for the $k$-star game only hold for a constant $k$. Not only that some parts of our proofs rely on the fact that $k$ is a constant, but also even if they were not relying on that, Theorem 4.3 would give that for any $k = \omega(1)$, all threshold biases, $f^{mon}$, $f^-$ and $f^+$, equal $\Theta(n)$. This is unlikely to be the correct threshold bias for **every** such $k$. It will be interesting to analyse this game for the non constant case.

# Chapter 5

# Fast biased Maker-Breaker games

In the following sections, we study the $(1 : b)$ Perfect matching game, $\mathcal{M}_n$, and the Hamiltonicity game, $\mathcal{H}_n$, played on $E(K_n)$, and give fast winning strategies for Maker, as well as the strategy for Breaker to slow Maker down. The upper bound on $\tau_M(\mathcal{M}_n)$ of Theorem 2.5 is a direct consequence of Theorem 5.11, which will be proved in Section 5.2, and the lower bound is a direct consequence of Theorem 5.14*(i)*, which will be proved in Section 5.4. The upper bound on $\tau_M(\mathcal{H}_n)$ of Theorem 2.6 follows directly from Theorem 5.13, which will be proved in Section 5.3, and its lower bound follows directly from Theorem 5.14*(ii)* which will be proved in Section 5.4.

## 5.1 Preliminaries

Assume that some Maker-Breaker game, played on the edge set of some graph $G$, is in progress. At any given moment during this game, we denote the graph consisting of Maker's edges by $M$ and the graph consisting of Breaker's edges by $B$; the edges of $G \setminus (M \cup B)$ are called *free*.

In this part we give theorems which we use as tools for proving Theorem 2.5 and Theorem 2.6. The fundamental Theorem 1.6 is a useful sufficient condition for Breaker's win in the $(a : b)$ game $(X, \mathcal{F})$. While it is useful in proving that Breaker wins a certain game, it does not show that he wins this game quickly. The following theorem is helpful in this respect.

**Theorem 5.1** ([17], *The trick of fake moves*)**.** *Let $(X, \mathcal{F})$ be a hypergraph of the game and let $b, b'$ be positive integers. If Maker has a winning strategy in $(1 : b)$ game on $(X, \mathcal{F})$, then he can also win $(1 : b')$ game on $(X, \mathcal{F})$, for $b' < b$ in at most $1 + \frac{|X|}{b+1}$ moves.*

The idea of the proof of this theorem relies on the fact that Maker in his mind gives Breaker additional $b - b'$ elements of the board after each Breaker's move (and does not consider them any more). The exact details of the proof can be found in [17].

The following theorem shows that in a $(1 : b)$ Maker-Breaker game, played on the edge set of some graph $G$ in which the minimum degree is not too small, Maker can build a spanning subgraph with large minimum degree fast, while making sure that throughout the game, as long as a vertex is not of large degree in Maker's graph, the proportion between Maker's and Breaker's edges touching that vertex is "good". It will be used in proving that the upper bound on $\tau_{\mathcal{M}_n}$ in Theorem 2.5 holds.

**Theorem 5.2.** *For every sufficiently large integer $n$ the following holds. If*

*(i) $G$ is a graph with $|V(G)| = n$, and*

*(ii) $b \leq \frac{\delta(G)}{4 \ln n}$, and*

*(iii) $c$ is an integer such that $c(2b + 1) \leq \frac{\delta(G)}{3}$,*

*then, in the $(1 : b)$ Maker-Breaker game played on $E(G)$, Maker has a strategy to build a graph with minimum degree $c$. Moreover, Maker can do so within $cn$ moves and in such a way that for every $v \in V(G)$, as long as $d_M(v) \leq c$, for each $v \in V(G)$:*

$$d_B(v) - 2b \cdot d_M(v) \leq b(2 \ln n + 1).$$

**Proof.** The proof is very similar (in fact, almost identical) to the proof of [46, Theorem 1.2], so we omit some of the calculations. Since claiming an extra edge is never a disadvantage for any of the players, we can assume that Breaker is the first player to move. At any point of the game, for every vertex $v \in V(G)$, let $\mathcal{D}(v) := d_B(v) - 2b \cdot d_M(v)$ be the *danger value* of $v$. For a subset $X \subseteq V(G)$, define $\overline{\mathcal{D}}(X) = \frac{\sum_{v \in X} \mathcal{D}(v)}{|X|}$, the average danger of vertices in $X$. A vertex $v \in V(G)$ is called *dangerous* if $d_M(v) \leq c - 1$.

The game ends when either all the vertices have degree at least $c$ in Maker's graph (and Maker won) or there exists a dangerous vertex $v \in V(G)$ for which $\mathcal{D}(v) > b(2 \ln n + 1)$ (and Maker failed the degree condition) or $d_B(v) \geq d_G(v) - c + 1$ (and Breaker won). Note that since

$$d_G(v) - c + 1 - 2b \cdot (c-1) \geq d_G(v) - c - 2bc > b(2 \ln n + 1), \qquad (5.1.1)$$

it is enough to say that Maker fails if $\mathcal{D}(v) > b(2 \ln n + 1)$ for some vertex $v \in V(G)$ with $d_M(v) \leq c - 1$.

**Maker's strategy $S_M$:** Before his $i$th move Maker identifies a dangerous vertex $v_i$ with

$$\mathcal{D}(v_i) = \max\{\mathcal{D}(v) : v \in V(G) \text{ and } v \text{ is dangerous}\},$$

and claims an arbitrary free edge $(v_i, u_i)$, where ties are broken arbitrarily.

Suppose towards a contradiction that Breaker has a strategy $S_B$ by which Maker, who plays according to the strategy $S_M$ as suggested above, fails. That is, playing according $S_B$, Breaker can ensure that at some point during the game, there exists a dangerous vertex $v \in V(G)$ for which $\mathcal{D}(v) > b(2 \ln n + 1)$.

Let $s$ be the length of this game and let $A = \{v_1, v_2, \ldots, v_s\}$ be the set of active vertices which contains all the vertices in Maker's graph of degree less than $c$ that Maker selected as the most dangerous. Note here that vertices $v_1, v_2, \ldots, v_s$ do not have to be distinct vertices from $V(K_n)$, since it takes $c$ moves to remove a vertex from the set of active vertices. So, $A$ can have less than $s$ elements. By strategy $\mathcal{S}_M$, in his $i$th move, Maker claims an edge incident with $v_i$ (for all $i$ except for $i = s$, as the game is considered to be over before his $s$th move). For $0 \leq i \leq s-1$, let $A_i = \{v_{s-i}, v_{s-i+1}, \ldots, v_s\}$.

Following the notation in [46], let $\mathcal{D}_{M_i}(v)$ and $\mathcal{D}_{B_i}(v)$ denote danger values of vertex $v \in V(K_n)$ immediately before $i$th move of Maker, respectively Breaker.

Analogously to the proof of Theorem 1.2 in [46], we state the following lemmas. Next lemma is useful for estimating the change in average danger value after Maker's move.

**Lemma 5.3** ([46], Lemma 3.3)**.** *Let $i$ be an integer, $1 \leq i \leq s-1$.*

*(i) If $A_i \neq A_{i-1}$, then $\overline{\mathcal{D}}_{M_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{i-1}) \geq 0$.*

*(ii) If $A_i = A_{i-1}$, then $\overline{\mathcal{D}}_{M_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{i-1}) \geq \frac{2b}{|A_i|}$.*

To estimate the change in average danger value after Breaker's move, we use the following lemma.

**Lemma 5.4** ([46], Lemma 3.4(i))**.** *Let $i$ be an integer, $1 \leq i \leq s-1$. Then*

$$\overline{\mathcal{D}}_{M_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i}}(A_i) \leq \frac{2b}{|A_i|}.$$

Combining Lemmas 5.3 and 5.4 we obtain the following corollary which estimates the change in average danger value after a whole round is played.

**Corollary 5.5** ([46], Corollary 3.5)**.** *Let $i$ be an integer, $1 \leq i \leq s-1$.*

*(i) If $A_i = A_{i-1}$, then $\overline{\mathcal{D}}_{B_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{i-1}) \geq 0$.*

*(ii) If $A_i \neq A_{i-1}$, then $\overline{\mathcal{D}}_{B_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{i-1}) \geq -\frac{2b}{|A_i|}$,*

To complete the proof, we want to show that before Breaker's first move $\mathcal{D}_{B_1}(A_{s-1}) > 0$, thus obtaining a contradiction.
Let $r$ denote the number of distinct vertices in $A_1$ and let $i_1 < i_2 < \cdots < i_{r-1}$ be the indices for which $A_{i_j} \neq A_{i_j-1}$ holds, for $1 \leq j \leq r-1$. Then $|A_{i_j}| = j+1$.

Recall that since Maker fails in his $s$th move, the danger value of $v_s$ immediately before $B_s$ is

$$\mathcal{D}_{B_s}(v_s) > 2b \ln n. \tag{5.1.2}$$

We have that

$$\begin{aligned}
\overline{\mathcal{D}}_{B_1}(A_{s-1}) &= \overline{\mathcal{D}}_{B_s}(A_0) + \sum_{i=1}^{s-1} \left(\overline{\mathcal{D}}_{B_{s-i}}(A_i) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{i-1})\right) \\
&\geq \overline{\mathcal{D}}_{B_s}(A_0) + \sum_{j=1}^{r-1} \Big(\overline{\mathcal{D}}_{B_{s-i_j}}(A_{i_j}) - \overline{\mathcal{D}}_{B_{s-i_j+1}}(A_{i_j-1})\Big) \\
&\qquad \text{[by Corollary 5.5 } (i)\text{]} \\
&\geq \overline{\mathcal{D}}_{B_s}(A_0) - \sum_{j=1}^{r-1} \frac{2b}{j+1} \\
&\qquad \text{[by Corollary 5.5}(ii)\text{]} \\
&\geq \overline{\mathcal{D}}_{B_s}(A_0) - 2b \ln n \\
&> 0. 
\end{aligned} \tag{5.1.3}$$

□

Graph $G$ is *Hamilton connected* if for every $v, w \in V(G)$ there exists a Hamilton path between $v$ and $w$. The following theorem gives the sufficient condition for a graph to be Hamilton connected (see [59, Theorem 1.2] for the general statement).

**Theorem 5.6** ([39], Theorem 2.10)**.** *Let $G = (V, E)$ be graph such that $|V| = k$ and let $D = \ln \ln k$. If $G$ satisfies the following properties:*

*(P1) For every $S \subset V$, s.t. $|S| \leq \frac{k}{\ln k}$ we have $|N_G(S)| \geq D|S|$, and*

*(P2) There exists at least one edge in $G$ between any two disjoint subsets $A, B \subseteq V$ s.t. $|A|, |B| \geq \frac{k}{\ln k}$,*

*then $G$ is Hamilton connected, for sufficiently large integer $k$.*

Since making a Hamilton connected subgraph is in the basis of Maker's strategy in the fast Hamiltonicity game, we will include also the proof of the following theorem.

**Theorem 5.7** ([39], Proposition 2.9)**.** *Let $k$ be a sufficiently large integer and let $b \leq \frac{k}{\ln^2 k}$. If $G$ is a graph with $k$ vertices whose minimum degree $\delta$ is at least $k - g(k)$, where $g(k) = o(k/\ln k)$, then Maker can build a Hamilton-connected graph playing $(1 : b)$ game on $E(G)$ in $O(k \ln^2 k)$ moves.*

**Proof.** Let $D = \ln \ln k$ and let $\mathcal{G}_1$ be the hypergraph whose vertices are the edges of $G$ and whose set of hyperedges is

$$\left\{ E_G(A, B) : A, B \subseteq V, A \cap B = \emptyset, |A| \leq \frac{k}{\ln k}, |B| = k - (D+1)|A| \right\}.$$

Taking into account the minimum degree in $G$, the number of edges in $E_G(A, B)$ is

$$e_G(A, B) \geq |A|(|B| - g(k)) \geq (1 - o(1))|A|k$$

for every $A$ and $B$ that satisfy the assumption from the definition of $\mathcal{G}_1$. Let $\mathcal{G}_2$ be the hypergraph whose vertices are all the edges from $G$ and whose set of hyperedges is

$$\left\{ E_G(A, B) : A, B \subseteq V, A \cap B = \emptyset, |A| = |B| \geq \frac{k}{\ln k} \right\}.$$

Taking into account the minimum degree in $G$, the number of edges in $E_G(A, B)$ for all $A$ and $B$ that satisfy the assumption from the definition of $\mathcal{G}_2$ is

$$e_G(A, B) = |A| \cdot (|B| - g(k)) \geq \frac{k}{\ln k} \left( \frac{k}{\ln k} - g(k) \right) \geq \left( \frac{k}{2 \ln k} \right)^2.$$

In order to show that Maker can win the $(1 : b)$ Hamilton connected graph game in $O(k \ln^2 k)$ moves, it is enough to show that Breaker can win $(b : 1)$ game on $\mathcal{G}_1 \cup \mathcal{G}_2$, for $b = \frac{k}{\ln^2 k}$. Breaker's win on $\mathcal{G}_1$ would satisfy the property $(P1)$ from the Theorem 5.6, as every $A \subseteq V$ with $|A| \leq \frac{k}{\ln k}$ would have neighbourhood of at least $D|A|$ vertices. Similarly, Breaker's win on $\mathcal{G}_2$ would satisfy property $(P2)$ from Theorem 5.6.

To see that Breaker can win on $\mathcal{G}_1 \cup \mathcal{G}_2$, we will use Theorem 1.6. We would like to show that

$$\sum_{F \in \mathcal{G}_1 \cup \mathcal{G}_2} 2^{\frac{-|F|}{b}} < 1.$$

Let $a = |A|$. For every $1 \leq a \leq \frac{k}{\ln k}$ it holds that

$$\begin{aligned}\binom{k}{a}\binom{k}{k-(D+1)a} 2^{-(1-o(1))\frac{ak}{b}} &\leq k^a k^{a(D+1)} 2^{-(1-o(1))\frac{ak}{b}} \\ &\leq e^{a(D+2)\ln k - (1-o(1))\ln 2ak\frac{\ln^2 k}{k}} \\ &= o(1/k).\end{aligned}$$

Thus,

$$\sum_{F \in \mathcal{G}_1} 2^{\frac{-|F|}{b}} \leq \sum_{i=1}^{\frac{k}{\ln k}} \binom{k}{a}\binom{k}{k-(D+1)a} 2^{-(1-o(1))\frac{ak}{b}} = o(1).$$

Analogously,

$$\begin{aligned}\sum_{F \in \mathcal{G}_2} 2^{\frac{-|F|}{b}} &\leq \binom{k}{\frac{k}{\ln k}}^2 e^{-\ln 2\left(\frac{k}{2\ln k}\right)^2 \frac{\ln^2 k}{k}} \\ &\leq e^{\frac{(2+o(1))k \ln\ln k}{\ln k} - \ln 2\left(\frac{k}{2\ln k}\right)^2\frac{\ln^2 k}{k}} \\ &= o(1).\end{aligned}$$

So,

$$\sum_{F \in \mathcal{G}_1 \cup \mathcal{G}_2} 2^{\frac{-|F|}{b}} = \sum_{F \in \mathcal{G}_1} 2^{\frac{-|F|}{b}} + \sum_{F \in \mathcal{G}_2} 2^{\frac{-|F|}{b}} = o(1).$$

From this, we get that Maker can win the $(1 : b)$ Hamilton connected graph game. From Theorem 5.1 we obtain that he can do it in at most $1 + \frac{|E|}{b+1} = 1 + \frac{|E|}{k/\ln^2 k} = O(k \ln^2 k)$ moves. □

In order to give the upper bound on $\tau_{\mathcal{H}_n}$ in Theorem 2.6 we need the previous theorem as well as the following.

**Theorem 5.8** ([48], Hajnal-Szemerédi Theorem)**.** *If $G$ is a graph with $n$ vertices and maximum degree $\Delta$, then $G$ can be coloured with $\Delta + 1$ colours and moreover, each colour class is of size $\frac{n}{\Delta+1}$.*

In [62], Krivelevich proved the following theorem.

**Theorem 5.9** ([62], Theorem 1)**.** *Maker has a strategy to win the $(1 : b)$ Hamiltonicity game played on $E(K_n)$ in at most $14n$ moves, for every $b \leq \left(1 - \frac{30}{\ln^{1/4} n}\right) \frac{n}{\ln n}$, for all large enough $n$.*

Following the proof of Theorem 5.9 line by line we obtain the following theorem, which is used in the proof for the upper bound on $\tau_{\mathcal{M}_n}$ in Theorem 2.5.

**Theorem 5.10** (Hamiltonicity game)**.** *For every $\varepsilon > 0$ there exists $\delta > 0$ and an integer $n_0 := n_0(\delta, \varepsilon)$ such that the following holds. If*

*(i) $H$ is a graph with $|V(H)| = n \geq n_0$, and*

*(ii) $\Delta(H) \leq \delta n$, and*

*(iii) $|E(H)| \leq \frac{n^2}{\ln n}$,*

*then for every $b \leq (1 - \varepsilon) \frac{n}{\ln n}$, in the $(1 : b)$ Maker-Breaker game played on $E(K_n \setminus H)$, Maker can build a Hamilton cycle of $K_n \setminus H$ in $O(n)$ moves.*

**Proof.** [Sketch] The proof is almost identical to the proof of [62, Theorem 1], so we omit most of the details. Throughout the proof we assume that the edges of $H$ were claimed by Breaker.

Maker's strategy consists of the following three main stages.

**Stage 1.** Creating an expander. In this stage, Maker creates a "good" expander in at most $12n$ moves. That is, after this stage, Maker's graph $M$ satisfies the following property:

$|N_M(X) \setminus X| \geq 2|X|$ for every $X \subseteq V(K_n)$ of size $|X| \leq \frac{n}{\ln^{0.49} n}$.

**Stage 2.** Maker turns his expander into a connected graph within $n$ moves.

**Stage 3.** Turning the connected expander into a Hamiltonian graph.

Following the proof of [62, Theorem 1], we can prove that Maker can complete all the stages. For given $\varepsilon$, and $\delta(\varepsilon)$, the same calculations as there give us that Maker's graph satisfies the required property after stage 1. Notice that after stage 1, we consider that Breaker has claimed at most $12nb + |E(H)| = O\left(\frac{n^2}{\ln n}\right)$ edges. Since every connected component of Maker's graph is of size $\Theta\left(\frac{n}{\ln^{0.49} n}\right)$, we conclude that most of the edges between any two components have not been claimed by Breaker so far. Therefore, in his next $n$ moves, Maker can merge all the components into one component and complete stage 2. After that, in the following at most $n$ moves, Maker creates a Hamilton cycle. $\square$

## 5.2 Perfect matching game, Maker's strategy

In this section we give a strategy of Maker to win quickly in the $(1 : b)$ Perfect matching game $\mathcal{M}_n$, played on the edge set of the complete graph on $n$ vertices, $K_n$, thus obtaining the upper bound on $\tau_{\mathcal{M}_n}$ in Theorem 2.5.

**Theorem 5.11.** *There exist $\delta > 0$ and $C > 0$ such that for every sufficiently large integer $n$ and every $b \leq \frac{\delta n}{100 \ln n}$, Maker has a strategy to win the $(1 : b)$ Perfect matching game played on $E(K_n)$ within $\frac{n}{2} + Cb \ln b$ moves.*

**Proof.** Since an extra move cannot harm Maker, we can assume that Breaker starts the game. First we describe a strategy for Maker and then prove it is a winning strategy. At any point throughout the game, if Maker is unable to follow the proposed strategy (including the time limits), then he forfeits the game.

Let $\delta > 0$ be a small positive constant and let $n_0 := n_0(\delta)$ be a positive integer as obtained by Theorem 5.10, applied with (say) $\varepsilon = 99/100$ (we make no effort to optimize $\varepsilon$). Let $b \leq \frac{\delta n}{100 \ln n}$ and let $n_1$ be such that $b = \frac{n_1}{100 \ln n_1}$. Set $m := \frac{n_0 + n_1}{\delta}$. Maker's strategy is divided into the following two main stages.

**Stage 1.** In this stage, Maker's aim is to build a matching $M' \subseteq E(K_n)$ of size $\ell := \frac{n-m}{2}$ in $|M'|$ moves. For each $1 \leq i \leq \ell$, after Maker's $i$th move his graph consists of a partial matching $M_i \subseteq E(K_n)$ (with $M' = M_\ell$) and a set of isolated vertices $U_i \subseteq V(K_n)$, where $U_i = \{V(K_n) \setminus V(M_i)\}$. Initially, $M_0 := \emptyset$ and $U_0 := V(K_n)$. Now, for each $1 \leq i \leq \ell$, in his $i$th move Maker claims an arbitrary free edge $(v_i, w_i) \in E(K_n[U_{i-1}])$ such that:

$(i)$ $d_B(v_i, U_{i-1}) = \max\{d_B(v, U_{i-1}) : v \in U_{i-1}\}$, and

$(ii)$ $d_B(w_i, U_{i-1}) = \max\{d_B(w, U_{i-1}) : w \in U_{i-1}$ and $(w, v_i)$ is free $\}$.

As soon as stage 1 ends, Maker proceeds to stage 2.

**Stage 2.** In this stage, Maker builds an expander on $E((K_n \setminus B)[U_l])$. Moreover, Maker does so in $O(b \ln b)$ moves.

Maker's graph after stage 1 contains a matching and a set of isolated vertices. In stage 2, Maker builds an expander on these isolated vertices which is connected and contains a Hamilton cycle, which implies the existence of a perfect matching. If Maker can follow stages 1 and 2 of the proposed strategy without forfeiting the game then he wins. It thus suffices to show that indeed Maker can follow the proposed strategy without forfeiting the game. We consider each stage separately.

Let us first look at stage 1. For $0 \leq i \leq \ell$, let $S_i = \sum_{v \in U_i} d_B(v, U_i)$ and $D_i = \frac{S_i}{|U_i|}$ denote the sum and the average of the degrees of vertices in $B[U_i]$ before Breaker's $(i+1)$st move, respectively, and let $d_i = \Delta(B[U_i])$.

We want to show that for each $1 \leq i \leq \ell$, Maker can always make his $i$th move according to the strategy proposed in stage 1. In order to show that, it is enough to prove that $d_i \leq \delta|U_i|$ holds for each such $i$. This will follow from the following claim.

**Claim 5.12.** The following two properties hold for each $1 \leq i \leq \ell$:

$(i)$ $D_i \leq 2b$, and

$(ii)$ $d_i \leq \delta|U_i|$.

**Proof.**

$(i)$ Notice that since Breaker's bias is $b$, it follows that for each $i$, in his $(i+1)$st move Breaker can increase $S_i$ by at most $2b$. Moreover, playing according to the proposed strategy for stage 1, by claiming the edge $v_{i+1}w_{i+1}$, Maker decreases $S_i$ by

$$2d_B(v_{i+1}, U_i) + 2d_B(w_{i+1}, U_i) = 2d_i + 2d_B(w_{i+1}, U_i).$$

Therefore, we have that

$$\begin{aligned} D_{i+1} &\leq \frac{D_i|U_i| + 2b - 2d_i - 2d_B(w_{i+1}, U_i)}{|U_{i+1}|} \\ &= \frac{D_i(|U_i| - 2 + 2) + 2b - 2d_i - 2d_B(w_{i+1}, U_i)}{|U_{i+1}|} \\ &= D_i + 2 \cdot \frac{D_i + b - d_i - d_B(w_{i+1}, U_i)}{|U_i| - 2}. \end{aligned} \tag{5.2.1}$$

Now, by induction on $i$ we prove that $D_i \leq 2b$ holds for each $0 \leq i \leq \ell$.

For $i = 0$ we trivially have that $D_i = 0 \leq 2b$.

Assume that $D_i \leq 2b$ holds, we want to show that $D_{i+1} \leq 2b$ as well. We distinguish between the following two cases.

*Case 1:* $D_i \leq 3b/2$. In this case, using the estimate (5.2.1) and the fact that $|U_i| \geq 12$ we have that $D_{i+1} \leq 3b/2 + \frac{5b}{12-2} = 2b$ as desired.

*Case 2:* $D_i > 3b/2$. Notice that from (5.2.1) it is enough to show that

$$d_i + d_B(w_{i+1}, U_i) \geq D_i + b. \tag{5.2.2}$$

Indeed, if it is true then we obtain that $D_{i+1} \leq D_i$ which by the induction hypothesis is bounded by $2b$.

If $d_i \geq 3b$ then (5.2.2) trivially holds, as $d_i + d_B(w_{i+1}, U_i) \geq d_i \geq 2b + b \geq D_i + b$. Otherwise, we have that $3b/2 < D_i \leq d_i < 3b$. Let

$x$ be the number of vertices in $U_i$ with degree at least $b$ in Breaker's graph. Notice that since $3b/2 < D_i \leq \frac{3bx+(|U_i|-x)b}{|U_i|}$, it follows that $x > \frac{|U_i|}{4}$. Now, since $\frac{|U_i|}{4} > 3b$, it follows that there exists a vertex $w \in U_i$ for which the edge $v_{i+1}w$ is free and $d_B(w, U_i) \geq b$. Therefore, $d_B(w_{i+1}, U_i) \geq b$. Finally, combining it with the fact that $d_i \geq D_i$, we conclude that (5.2.2) holds.

(*ii*) Notice first that while $\delta|U_i| \geq 2b(1 + \ln 2n)$, the claim is true as a consequence of Theorem 5.2. The conditions of Theorem 5.2 are satisfied as $c = 1$ which satisfies condition (*iii*), and $\delta|U_i| \geq 2b(1 + \ln 2n)$ implies that $b \leq \frac{\delta|U_i|}{2(1+\ln 2n)} \leq \frac{|U_i|(1-\delta)}{4\ln|U_i|}$, which satisfies (*ii*). The strategies of Maker in both games are the same, to touch the vertex of the largest degree. Therefore, it is enough to prove the claim for $i$'s such that $|U_i| < \frac{2b(1+\ln 2n)}{\delta} \leq \frac{n}{10}$.

Let $s = n - \frac{b(1+\ln 2n)}{\delta}$. Assume towards a contradiction that for some $s \leq i_0 \leq \ell$, after Maker's $i_0$th move, there exists a vertex $v \in U_{i_0}$ for which $d^* := d_B(v, U_{i_0}) > \delta|U_{i_0}|$. Now, for each $k \geq 1$ we inductively find a set $R_k$ for which the following holds:

(a) $R_k \subseteq U_{i_0-k}$,

(b) $|R_k| = k + 1$, and

(c) for each $k \geq 1$, after $(i_0 - k)$th round,

$$\sum_{u \in R_k} d_B(u, U_{i_0-k}) \geq (k+1)\left(d^* - 2b \cdot \sum_{j=2}^{k+1} \frac{1}{j}\right). \qquad (5.2.3)$$

For $k = 1$, let $R_1 := \{v, v_{i_0}\} \subseteq U_{i_0-1}$, where $v$ is a vertex with $d_B(v, U_{i_0}) = d^*$ and $v_{i_0} \in U_{i_0-1}$ is the vertex that Maker has touched in his $i_0$th move. Since $d_B(v_{i_0}, U_{i_0-1}) = d_{i_0} \geq d^*$ and Breaker, claiming $b$ edges per move, could not increase the degrees of these two vertices by more than $2b$ in his $i_0$th move, inequality (5.2.3) trivially holds.

Assume we built $R_k$, satisfying $(a), (b)$ and $(c)$, we want to build $R_{k+1}$. Let $v_{i_0-k} \in U_{i_0-k-1}$ be the vertex that Maker has touched in his $(i_0-k)$th move. Notice that $v_{i_0-k} \notin R_k$ (otherwise $R_k$ cannot be a subset of $U_{i_0-k}$) and $U_{i_0-k} \subseteq U_{i_0-k-1}$.

Hence, we conclude that before Maker's $(i_0-k)$th move

$$\begin{aligned} d_B(v_{i_0-k}, U_{i_0-k-1}) &\geq \frac{1}{|R_k|} \sum_{u\in R_k} d_B(u, U_{i_0-k}) \\ &= \frac{1}{k+1} \sum_{u\in R_k} d_B(u, U_{i_0-k}). \end{aligned} \tag{5.2.4}$$

Define $R_{k+1} := R_k \cup \{v_{i_0-k}\}$. We have that $R_k \subseteq U_{i_0-k} \subseteq U_{i_0-k-1}$ and $v_{i_0-k} \in U_{i_0-k-1}$, which together imply that $R_{k+1} \subseteq U_{i_0-k-1}$, satisfying $(a)$. Also, $|R_{k+1}| = |R_k|+1 = k+2$ satisfies $(b)$. Combining (5.2.3), (5.2.4) and the fact that Breaker can increase the sum of all degrees in $U_{i_0-k-1}$ by at most $2b$ in one move we obtain that

$$\begin{aligned} \sum_{u\in R_{k+1}} d_B(u, U_{i_0-k-1}) \geq& \sum_{u\in R_k} d_B(u, U_{i_0-k}) + \\ &+ \frac{1}{k+1} \sum_{u\in R_k} d_B(u, U_{i_0-k}) - 2b \\ &= \frac{k+2}{k+1} \cdot \sum_{u\in R_k} d_B(u, U_{i_0-k}) - 2b \\ &\geq (k+2) \cdot \\ &\cdot \left( \frac{1}{k+1} \cdot (k+1) \left( d^* - 2b \cdot \sum_{j=2}^{k+1} \frac{1}{j} \right) - \frac{2b}{k+2} \right) \\ &= (k+2) \left( d^* - 2b \cdot \sum_{j=2}^{k+2} \frac{1}{j} \right), \end{aligned}$$

and so the property $(c)$ is also satisfied for $R_{k+1}$. This completes the inductive step. Now, for $k = |U_{i_0}| - 1$ we obtain that

$$\begin{aligned}
D_{i_0-k} &= \frac{\sum\limits_{u \in U_{i_0-k}} d_B(u, U_{i_0-k})}{|U_{i_0-k}|} \\
&\geq \frac{\sum\limits_{u \in R_k} d_B(u, U_{i_0-k})}{|U_{i_0-k}|} \\
&\geq \frac{(k+1)\left(d^* - 2b \cdot \sum\limits_{j=2}^{k+1} \frac{1}{j}\right)}{3k+1} \\
&\geq \frac{d^* - 2b \ln |U_{i_0}|}{3} \\
&\geq \frac{\delta |U_{i_0}| - 2b \ln |U_{i_0}|}{3} \\
&> 2b,
\end{aligned}$$

which is clearly a contradiction to $(i)$. This completes the proof of Claim 5.12.

□

Now we show that Maker can follow stage 2 of the proposed strategy. Let $H = B[U_l]$. When stage 1 is over, Claim 5.12 gives that $\Delta(H) = \delta |U_l|$ and $|E(H)| \leq \frac{|U_l|^2}{\ln |U_l|}$. This satisfies the conditions of Theorem 5.10, so Maker can make an expander on $V(K_n \setminus B)[U_l]$ in $O(|U_l|) = O(b \ln b)$ moves. □

## 5.3 Hamiltonicity game, Maker's strategy

In this section, we give a fast winning strategy for Maker in the $(1 : b)$ Hamiltonicity game $\mathcal{H}_n$, played on the edge set of the complete graph on $n$ vertices, $K_n$, thus obtaining the upper bound on $\tau_{\mathcal{H}_n}$ in Theorem 2.6.

**Theorem 5.13.** *There exists a constant* $\delta > 0$ *such that for every sufficiently large integer* $n$ *and every* $b \leq \delta\sqrt{\frac{n}{\ln^5 n}}$, *Maker has a strategy to win the* $(1 : b)$ *Hamiltonicity game played on* $E(K_n)$ *in* $n + O(b^2 \ln^5 n)$ *moves.*

**Proof.** Since an extra move cannot harm Maker, we can assume that Breaker starts the game. Maker's strategy in the Hamiltonicity game is divided into three main stages.

**Stage 1.** Maker splits the vertices of the board into two sets, $X$ and $I$, such that at the beginning $X = \emptyset$ and $I = V$. Throughout the stage 1, Maker's plan for set $X$ is to contain the vertices of vertex disjoint Hamilton connected expanders, such that at any point only one expander is being built, while the others are completed, and his graph on $I$ will be a collection of paths, each of length $\geq 0$, denoted by $\mathcal{P}$. Let $End(\mathcal{P})$ denote the multiset of endpoints of paths in collection $\mathcal{P}$. Note that isolated vertices in $\mathcal{P}$ (viewed as paths of length 0) appear twice in $End(\mathcal{P})$. Both $\mathcal{P}$ and $End(\mathcal{P})$ are updated dynamically. For each path $P \in \mathcal{P}$, let $v_P^1$ and $v_P^2$ denote its endpoints (in arbitrary order). At the beginning, every $v \in I$ is considered as a path of length 0.

During this stage, Maker plays the following two games in parallel.

(1) In his odd moves, Maker builds $L = L(b, n) := 13b \ln n$ Hamilton connected expanders of order $t = t(b, n) := \frac{1}{2} b \ln^2 n$ that are vertex disjoint. He builds them one by one, repeatedly choosing new $t$ isolated vertices from $I$ that are independent in Breaker's graph and moving them to $X$, whenever the previous expander is completed.

(2) In each of his even moves, Maker chooses a vertex $v \in End(\mathcal{P})$, $v = v_P^1$ s.t. $d_B(v) = \max_{w \in End(\mathcal{P}) \setminus \{v\}} d_B(w)$ (ties broken arbitrarily) and claims a free edge between $v$ and some other vertex $u \in End(\mathcal{P}) \setminus \{v\}$, $u \neq v_P^2$.

By Theorem 5.7, Maker needs $O(t \ln^2 t)$ moves to build one Hamilton connected expander of order $t$, thus for $L$ such expanders he needs $O(Lt \ln^2 t) = O(b^2 \ln^5 n)$ moves. So, this stage lasts $Cb^2 \ln^5 n$, $C > 0$ moves.

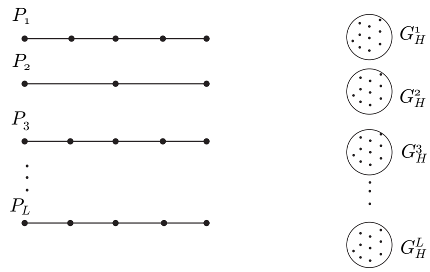


Figure 5.1: Maker's graph at the end of stage 2

Slika 5.1: Makerov graf na kraju faze 2

**Stage 2.** In this stage, with each his edge, Maker plays in the same way as in part (2) of stage 1. In each move, he chooses a vertex $v \in End(\mathcal{P})$, $v = v_P^1$ s.t. $d_B(v) = \max\limits_{w \in End(\mathcal{P})} d_B(w)$ (ties broken arbitrarily) and claims a free edge between $v$ and some other vertex $u \in End(\mathcal{P}) \setminus \{v\}$, $u \neq v_P^2$. He plays like this until $|\mathcal{P}| = L$. This stage lasts $n - |X| - L - \frac{C}{2} b^2 \ln^5 n = n - Lt - L - \frac{C}{2} b^2 \ln^5 n = n - 13b \ln n \left(\frac{1}{2} b \ln^2 n + 1\right) - \frac{C}{2} b^2 \ln^5 n$ moves.

**Stage 3.** At the beginning of this stage there are $L$ Hamilton connected expanders, $G_H^1, G_H^2, \dots, G_H^L$, and $L$ paths, $P_1, P_2, \dots, P_L$ (Figure 5.1). Before this stage begins, Maker fixes which paths (through which exact endpoints) will be joined to specific expander graphs. He uses the following rule: for each $i$, $1 \leq i \leq L$, $v_{P_i}^1$ and $v_{P_{(i \mod L)+1}}^2$ will be connected to two arbitrary different vertices in $G_H^i$ (see Figure 5.2). In each move, Maker chooses a vertex $v \in End(\mathcal{P})$ such that $d_B(v) \geq \max\limits_{w \in End(\mathcal{P}) \setminus \{v\}} d_B(w)$ (ties broken arbitrarily), connects it to the expander according to the aforementioned rule, and removes $v$ from $End(\mathcal{P})$. This stage will last $2L = 26b \ln n$ moves.

It is straightforward to conclude that following the described strategy Maker can build a Hamilton cycle. Indeed, since each expander is Hamilton connected, there exists a Hamilton path between any pair of vertices within an expander. These paths circularly connect to paths from $\mathcal{P}$ to form a

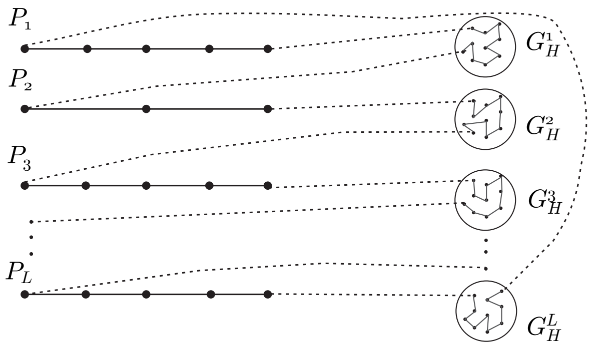


Figure 5.2: Maker's strategy in stage 3

Slika 5.2: Makerova strategija u fazi 3

Hamilton cycle.

Now we will show that Maker can follow this strategy. We perform the proof for each stage separately.

**Stage 1.** At the beginning of the game, $I = V$ and $X = \emptyset$. For each expander that he builds, Maker chooses $t$ vertices $\{v_1, v_2, \ldots, v_t\} \in I$ that are isolated in his graph and independent in Breaker's graph. Then, $I = I \setminus \{v_1, v_2, \ldots, v_t\}$ and $X = X \cup \{v_1, v_2, \ldots, v_t\}$. Since he plays on the set $X$ in every second move, this game can be treated as $(1 : 2b)$ Hamilton connected expander game. In his first move, Maker selects the first $t$ vertices $\{v_1, v_2, \ldots, v_t\} \in I$ that are independent in Breaker's graph, which are easy to find, as there are only $b$ edges claimed on the board in total. After that, $|I| = |V| - t$ and $X = X \cup \{v_1, v_2, \ldots, v_t\}$, implying $|End(\mathcal{P})| = 2(n - t)$, since each vertex is treated as two endpoints of a path of length 0. We will first take a closer look at (2).
In every second move, Maker plays according to (2). This part of the strategy can be seen as playing an auxiliary Box game, where Maker takes the role of BoxBreaker—each $v \in End(\mathcal{P})$ represents one box whose elements are all edges incident to $v$. With every edge $(p, q)$ that Maker claims ac-

cording to the strategy, BoxBreaker claims an element from box $p$ and an element from box $q$ (two elements in total), and removes them from the set of boxes. Note, however, that the same vertex can account for two boxes (if it is the double endpoint of a path of length 0), so BoxBreaker can claim up to four elements in one move. As this game is played in every second move, Breaker can claim $2b$ edges, and so there are up to $8b$ elements claimed by BoxMaker in each move. To analyse this game, we observe it as the *fake Box game*. Let $p$ be a vertex selected by Maker in the game as an endpoint of some path of the largest Breaker's degree. When Maker claims an edge $(p, q)$, the box corresponding to $p$ is labelled "killed" and the box corresponding to $q$ is labelled "hidden". So, he plays the game $B(2(n-t), 2(n-t)^2, 8b, 1)$. If this game would be played until its end (when there are no more free elements on the board), using Theorem 1.10 it is easy to verify that BoxMaker cannot win (and thus isolate one vertex in $I$) before all boxes are killed. We assume that only one box is removed from the game after each round and disregard the fact that Maker in the real game, transfers $t$ vertices from $I$ to $X$ on every $O(t \ln^2 t)$ moves (and consequently removes $2t$ boxes), because having more boxes available helps BoxMaker to claim more elements in one of them at the end.
Now, we want to estimate the size of the largest box that BoxMaker could fill until the end of the game. Note that this gives us the maximum degree in Breaker's graph at every $v \in End(\mathcal{P})$, at any point of stage 1. The size of the largest box is

$$\begin{aligned} s &= \frac{8b}{2(n-t)} + \frac{8b}{2(n-t)-1} + \cdots + \frac{8b}{1} \qquad (5.3.1) \\ &= 8b \sum_{i=1}^{2(n-t)} \frac{1}{i} \\ &\leq 8b \ln(2(n-t)) \\ &< 16b \ln n. \end{aligned}$$

This implies that when stage 1 is over, every vertex in $End(\mathcal{P})$ has Breaker's degree less than $16b \ln n$.

Now, we look at part (1). We need to prove two things: first, that Maker can build a Hamilton connected expander on $t = \frac{1}{2} b \ln^2 n$ vertices, among which no edge is claimed by either of players, and second, that Maker can find such $t$ vertices that induce no edge, whenever he decides to build each of his $L$ expanders. In order to show that Maker can build an expander on $t$ vertices when playing the $(1 : 2b)$ game, we need to verify the conditions of Theorem 5.7. The graph Maker plays the game on is $K_t$, so the degree condition is fulfilled. Also, $\frac{t}{\ln^2 t} > \frac{2b \ln^2 n}{\ln^2 n} = 2b$ for values of $b$ that we consider. This gives that Maker can build a Hamilton connected expander on $V(K_t)$ in at most $1 + \frac{E(K_t)}{t \ln^2 t} \leq cb \ln^4 n$ moves, for $0 < c < \frac{1}{2}$. We will show that Maker can find $t$ vertices that induce no edge for each expander. As building each expander requires $cb \ln^4 n$ moves and Maker should build $L = 13b \ln n$ of them, this gives in total at most $13cb^2 \ln^5 n$ moves. During this number of moves, playing according to (2), Maker could touch at most $2 \cdot 13cb^2 \ln^5 n$ vertices in $I$. Also, before selecting the $t$ vertices for his last expander, Maker has already removed $(L-1) \cdot t = \frac{13}{2} b^2 \ln^3 n - \frac{1}{2} b \ln^2 n$ vertices from $I$. So, before choosing vertices for each expander, there are at least $n' = n - 26cb^2 \ln^5 n - \frac{13}{2} b^2 \ln^3 n + \frac{1}{2} b \ln^2 n > \frac{n}{2}$ vertices in $I$ that are isolated in Maker's graph. According to Maker's strategy in (2), every vertex in $End(\mathcal{P})$ has Breaker's degree less than $16b \ln n$. Applying Theorem 5.8, we can partition $n'$ vertices into at least $16b \ln n$ independent sets, each of size at least $\frac{n'}{16b \ln n} = \Omega\left(\sqrt{n \ln^3 n}\right) > t$.

**Stage 2.** When this stage begins, $|End(\mathcal{P})| = 2n - 26cb^2 \ln^5 n - \frac{1}{2} b^2 \ln^3 n$. Here again we look at the same fake Box game, as described in stage 1 for game (2), taking the role of BoxBreaker. The boxes in this game are vertices in $End(\mathcal{P})$, which have at least $|End(\mathcal{P})| - 16b \ln n$ elements each. The difference here is that BoxMaker claims $4b$ elements of the board in each move, since Maker responds to $b$ edges of Breaker in the real game. Formally, Maker plays the game $B(|End(\mathcal{P})|, |End(\mathcal{P})| \cdot |End(\mathcal{P})| - 16b \ln n, 4b, 1)$, pretending to be BoxBreaker. Here again, a calculation similar to the one in (5.3.1) gives that playing on the board of order $|End(\mathcal{P})|$ until the end, BoxMaker cannot claim more than $8b \ln n$ elements in one box. This means that when stage 2 is over, there are $L$ paths in $\mathcal{P}$ whose endpoints have degree in Breaker's graph less than $16b\ln n + 8b\ln n = 24b\ln n$.

**Stage 3.** Maker connects $L$ paths through $L$ expanders into a Hamilton cycle. Yet again, we will look at an auxiliary Box game, where Maker pretends to be BoxBreaker. Now there are $2L$ boxes in the game, representing each of the endpoints of $L$ paths in $\mathcal{P}$. After stage 2 is over, there are less than $24b\ln n$ Breaker's edges incident to each $v \in End(\mathcal{P})$. Since Maker has to connect two endpoints to the distinct vertices of some expander, we can split the vertices of each expander arbitrarily into two sets of equal size. Each box consists of all free edges between one endpoint of the path and half of the vertices in one expander, and so, each box is of size more than $s = \frac{t}{2} - 24b\ln n = \frac{1}{4}b\ln^2 n - 24b\ln n$. Each Breaker's edge is counted as claiming one element of the board, so the game played is $B(2L, 2Ls, b, 1)$. The size of the largest box that BoxMaker could fully claim until the end of the game playing with bias $b$ is at most

$$l = \frac{b}{2L} + \frac{b}{2L-1} + \cdots + \frac{b}{1} = b\sum_{i=1}^{2L}\frac{1}{i} \leq b(1 + \ln 2L) < s.$$

This means that BoxMaker is unable to fully claim any box in this game before BoxBreaker claims an element in it. So, this stage ends in $2L$ moves and at its end, Maker's graph contains a Hamilton cycle. The total number of moves in this stage is $2L = 26b\ln n$, so the game lasts altogether at most $n + \frac{13}{2}cb^2\ln^5 n - \frac{13}{2}b^2\ln^3 n + 13b\ln n = n + O(b^2\ln^5 n)$ moves. $\square$

## 5.4 Breaker's strategies

In order to obtain the lower bound on $\tau_{\mathcal{M}_n}$ in Theorem 2.5 and $\tau_{\mathcal{H}_n}$ in Theorem 2.6, we will prove the following theorem.

**Theorem 5.14.** *In the $(1 : b)$ Maker-Breaker game on $E(K_n)$, for bias $b = o(n)$, Breaker can delay Maker's win*

*(i) in the Perfect matching game for at least $\frac{b}{4}$ moves,*

*(ii) in the Hamiltonicity game for at least $\frac{b}{2}$ moves.*

**Proof.**
*(i)* Breaker's strategy consists of claiming all the edges of some clique $C$ on $\frac{b}{2}$ vertices such that no vertex of $C$ is touched by Maker and maintaining this clique. Let $C_i$ be the clique of Breaker before his $i$th move. Let $u_i$ be the largest integer such that $b_i \le b$, where $b_i := \binom{u_i+1}{2} + (u_i+1)|C_i|$. In his $i$th move, Breaker chooses $u_i+1$ vertices $\{v_1, v_2, \ldots, v_{u_i+1}\} \in V(K_n)\backslash V(C_i)$ such that $d_M(v_j) = 0$, for $1 \le j \le u_i + 1$ and claims the edges $\{(v_j, v_k) : 1 \le j < k \le u_i + 1\} \cup \{(v_j, v) : 1 \le j \le u_i + 1, v \in V(C_i)\}$. He also claims $b - b_i$ arbitrary edges which we will disregard in our analysis. Maker, on the other hand, can touch at most one vertex from $C_i$ in his following move, so right before Breaker's $(i+1)$st move, $|C_{i+1}| \ge |C_i| + u_i$. It is easy to verify that $u_i \ge 1$ while $|C_i| \le \frac{b}{2}$, and after that $|C_{i+1}| \ge |C_i|$, provided there is at least one vertex in $V(K_n)\backslash V(C_i)$ isolated in Maker's graph. What we need to show is that there are enough vertices for Breaker to create a clique with $\frac{b}{2}$ vertices. By the definition of $u_i$, until $|C_i| \le \frac{b}{4} - 2$, $u_i \ge 3$. While $|C_i| \le \frac{b}{3} - 1$, $u_i \ge 2$ and if $|C_i| \le \frac{b}{2} - 1$, then $u_i \ge 1$. It takes at most

$$\frac{\frac{b}{4} - 2}{3} + \frac{\frac{b}{3} - 1 - \frac{b}{4} + 1}{2} + \frac{\frac{b}{2} - 1 - \frac{b}{3}}{1} = \frac{7b - 40}{24}$$

moves to achieve this and at most

$$5\frac{\frac{b}{4} - 2}{3} + 4\frac{\frac{b}{3} - 1 - \frac{b}{4} + 1}{2} + 3\frac{\frac{b}{2} - 1 - \frac{b}{3}}{1} = \frac{13b - 76}{12} < n$$

vertices are touched so far, knowing that $b = o(n)$. When it is no longer possible for Breaker to add new vertices, isolated in Maker's graph, to his clique, Maker needs at least one move to connect each vertex $w \in V(C)$ to some vertex $v \in V(K_n) \setminus V(C)$, which has degree at least one in Maker's graph. When this happens, $d_M(v) = d_M(v) + 1$. Maker has to claim at least $\frac{b}{2}$ edges to touch all vertices in Breaker's clique. In the smallest graph that contains a perfect matching all vertices have degree one. The double number of extra edges that Maker claims is thus $\sum_{v\in V(K_n)}(d_M(v) - 1) \ge \frac{b}{2}$, by the aforementioned analysis. The number of extra edges Maker has claimed in this game is thus $\frac{\sum_{v\in V(K_n)}(d_M(v)-1)}{2} \ge \frac{b}{4}$.

*(ii)* The proof is similar to the one for *(i)*. Let $i$ be the smallest integer such that $C_i$ is the clique in Breaker's graph of order $\frac{b}{2}$, created in the same way as in *(i)* and for every $w \in V(C_i)$ it holds $d_M(w) = 0$. In the Hamiltonicity game, Breaker considers a vertex $w \in C_i$ to be removed from his clique only if $d_M(w) = 2$. So, when $|C_i| = \frac{b}{2}$ and for every $w \in V(C_i)$, $d_M(w) = 0$ holds, Maker needs two moves to remove a vertex from $C$. This means that in $k$th move, $k > i$, Breaker chooses one untouched vertex $v \in V(K_n)\backslash V(C_k)$ and enlarges his clique by one. So, after each two moves, at most one vertex can be removed, and $|C_{k+2}| \geq |C_k| + 1$ holds. This is clearly possible while $|C| \leq b$. What remains to be proved is that there are enough untouched vertices until Breaker creates a clique of order $b$. From *(i)* we know that at most $\frac{13b-76}{12}$ vertices are touched until the clique of order $\frac{b}{2}$ is created in Breaker's graph. After that in at most $b$ more moves Breaker enlarges his clique to order $b$, and at that point the total of at most $\frac{37b-76}{12} < n$ vertices are touched. When $d_M(v) \geq 1$ for every $v \in V(K_n)\backslash V(C)$, Breaker still enlarges his clique by adding to it a vertex of degree 1 in Maker's graph. However, from that point on, Maker needs one move to remove such a vertex of degree one from $C_{k+1}$ and $|C_{k+1}| \geq |C_k|$. This implies that Breaker can maintain a clique in his graph of order $b$ until for all vertices $v \in V(K_n)\backslash V(C)$ it holds that $d_M(v) \geq 2$.
For every vertex $w \in V(C)$, $d_M(w) \leq 1$. In order to connect a vertex $w \in V(C)$ to some $v \in V(K_n)\backslash V(C)$ Maker needs at least one move. In a graph that contains a Hamilton cycle all the vertices have degree at least 2. So, the number of extra moves that Maker has made when the game is over is $\frac{\sum_{v \in V(K_n)}(d_M(v)-2)}{2}$. By the given strategy, the sum $\sum_{v \in V(K_n)}(d_M(v) - 2)$ grows by one for every $w \in V(C)$, and thus the number of extra moves is $\frac{\sum_{v \in V(K_n)}(d_M(v)-2)}{2} \geq \frac{b}{2}$. $\square$

## 5.5 Concluding remarks and open problems

From Theorem 5.11 and Theorem 5.14 *(i)* we obtain that the shortest duration of the $(1 : b)$ Maker-Breaker Perfect matching game is between $\frac{n}{2} + \frac{b}{4}$ and $\frac{n}{2} + O(b \ln b)$ moves, for $b \leq \frac{\delta n}{100 \ln n}$, where $\delta > 0$ is a small constant.

Also, Theorem 5.13 and Theorem 5.14 *(ii)* give that the shortest duration of $(1 : b)$ Maker-Breaker Hamiltonicity game is between $n + \frac{b}{2}$ and $n + O(b^2 \ln^5 n)$ moves, for $b \leq \delta\sqrt{\frac{n}{\ln^5 n}}$, where $\delta > 0$ is a small constant. It would be interesting to find the tighter bounds for $\tau_M$ for both games. This would be specially interesting in the Hamiltonicity game, where in the upper bound, second order term depends on both $b$ and $n$.

# Prošireni izvod

## Pozicione igre

Pozicione igre spadaju u grupu kombinatornih igara. To su konačne igre savršenih informacija koje igraju dva igrača naizmenično, bez slučajnih poteza. Pozicione igre se razlikuju od klasične teorije igara, koja je grana ekonomije, i koja proučava igre koje se igraju simultano i sa sakrivenim informacijama.

Pozicione igre su determinističke igre, tako da, ako pretpostavimo da igrači igraju koristeći svoje optimalne strategije, svemogući računar bi mogao (bar teoretski) odrediti kakav će biti ishod igre: pobeda prvog igrača, pobeda drugog igrača ili nerešeno. U principu, ishod igre bi se znao i pre nego što ona započne. Međutim, čak i današnji moćni računari su ograničenih mogućnosti u pretrazi celog, eksponencijalno velikog, stabla igre. To nas dovodi do zaključka da su matematički alati i algoritmi od presudnog značaja u analizi ovih igara.

Zanimljiva, i pomalo iznenađujuća (zbog toga što je reč o igrama savršenih informacija) činjenica je da se verovatnoća može primeniti u razvoju veoma korisnih alata za rad sa pozicionim igrama. Jožef Bek (József Beck) je to temeljno proučavao u svojoj knjizi [17]. U suštini, pozicione igre su moćan instrument u procesu derandomizacije i algoritmizacije važnih verovatnosnih alata, te tako imaju i jak uticaj na oblast teorijske informatike. Sem sa verovatnoćom, pozicione igre su usko povezane i sa drugim poljima kombinatorike, kao što su ekstremalna i Remzijeva (Ramsey) teorija.

Formalno, poziciona igra je uređen par $(X, \mathcal{F})$, gde je $X$ konačan skup, a

$\mathcal{F} = \{A_1, A_2, \ldots, A_n\} \subseteq 2^X$. Skup $X$ nazivamo *tablom* na kojoj se igra, a $\mathcal{F}$ familijom ciljnih skupova. Takođe, $(X, \mathcal{F})$ se još naziva i *hipergrafom igre*, čiji su čvorovi elementi iz skupa $X$, a *hipergrane* skupovi $A_1, A_2, \ldots, A_n$. Kada je jasno na kojoj tabli se igra $(X, \mathcal{F})$, koristimo samo $\mathcal{F}$ da označimo hipergraf igre. U igri postoje i dva parametra – $a$ i $b$. U $(a : b)$ pozicionoj igri igrači selektuju $a$, odnosno $b$, slobodnih elemenata table po potezu. Kada je $a = b = 1$, takve igre zovemo *fer* (bez biasa). U suprotnom ih zovemo *asimetričnim.* Prema načinu na koji se igraju, ove igre se mogu razvrstati u klasu *slabih* i *jakih* igara. Svaka klasa ima svoje podklase i o nekima od njih će biti reči u narednim delovima.

Pozicione igre se mogu igrati na različitim tablama: mreži, kompletnom grafu, uniformnom grafu, retkom grafu, itd. Rezultati u vezi sa različitim pozicionim igrama se mogu naći npr. u [1, 2, 7, 8, 11–16, 32, 34, 43, 45, 51, 56, 64–68, 70]. Takođe zanimljivo polje istraživanja su igre na slučajnim grafovima, o kojima se više može pročitati u npr. [10, 24, 29, 35, 41, 71].

## Jake igre

Jake igre, koje je definisao Jožef Bek u [17] pod tim imenom, su standardne, među ljudima popularne igre. Ciljni skupovi $A \in \mathcal{F}$ u ovoj varijanti se nazivaju *pobednički skupovi.* U jakim igrama oba igrača imaju isti cilj – da selektuju sve elemente nekog pobedničkog skupa. Onaj igrač koji to *prvi* uradi pobeđuje. Ako je cela tabla popunjena, a niko od igrača nije pobedio, igra se završava rezultatom *nerešeno.* Najpoznatija igra ovog tipa je *Iks-Oks*, u kojoj dva igrača ,“X”i “O”, naizmenično postavljaju po jedan svoj znak po potezu na slobodno polje table. Poznato je da se ova igra završava rezultatom nerešeno, ako oba igrača igraju optimalno.

U analizi jakih igara koristi se moćan alat *krađa strategije* (Strategy Stealing), prema kome prvi igrač nikako ne može da izgubi. Na žalost, ovaj alat ništa ne govori o tome kako treba prvi igrač da igra da bi pobedio. Tako da, argument je jak, u smislu velike primenljivosti, ali je neeksplicitan.

Drugi alat je argument Remzijevog tipa. Koristeći ga, možemo utvrditi da li svako bojenje table u dve boje daje monohromatsku kopiju nekog pobedničkog skupa iz $\mathcal{F}$. Tako da, ako $\mathcal{F}$ ima Remzijevu osobinu, tada prema alatu krađa strategije prvi igrač ima pobedničku strategiju.

Postoji takođe i jednostavan *koncept uparivanja* (the pairing strategy), koji drugom igraču omogućava nerešen rezultat ako se svi elementi table mogu upariti.

Sem ova tri, skoro da ne postoje drugi alati za analizu jakih igara. U skorije vreme pojavile su se strategije za pobedu prvog igrača u jakoj igri na grafu [36, 37]. Takođe, još neki rezultati vezani za jake igre se mogu pronaći npr. u [11, 17, 31, 49]. Sve ovo govori u prilog tome da su jake igre izuzetno zanimljive za igranje i veoma teške za analizu.

## Mejker-Brejker igre

Kada akcenat nije na takmičenju ko će pre nešto uraditi, već na samom cilju, tada govorimo o Mejker-Brejker (Maker-Breaker) igrama. Slično, kao i kod jakih igara, i ovde se ciljni skupovi nazivaju *pobednički skupovi*. Ali, pravila igre se razlikuju. Igrač kojeg zovemo *Mejker* (*Praviša*), želi da *osvoji* (selektuje sve elemente) neki pobednički skup, ali ne nužno prvi. Drugi igrač, *Brejker* (*Kvariša*), želi da ga spreči da ostvari svoj cilj, dakle, da selektuje bar jedan element iz svakog pobedničkog skupa. Kada su svi elementi table selektovani, ishod može biti ili Mejkerova pobeda, ili Brejkerova. U ovakvoj postavci nije moguće da se igra završi rezultatom nerešeno.

Neka su $a$ i $b$ pozitivni celi brojevi. U asimetričnoj $(a : b)$ Mejker-Brejker igri $(X, \mathcal{F})$, Mejker selektuje $a$, a Brejker $b$ elemenata po potezu. Parametri $a$ i $b$ zovu se *bias* Mejkera, odnosno Brejkera. Mejker pobeđuje u igri, ako ima strategiju da pobedi protiv bilo koje strategije Brejkera. Analogno se definiše i Brejkerova pobeda. Na primer, posmatrajmo igru *Iks-Oks* u Mejker-Brejker verziji. Kada se pravila igre promene, jednostavnom analizom slučajeva se lako može naći Mejkerova strategija za ovu igru. Iz ovoga možemo zaključiti da je uslov da igrač osvoji pobednički skup, ali ne nužno prvi, dovoljan da Mejkeru obezbedi pobedu u ovoj igri.

Uobičajeno je da se Mejker-Brejker igra igrà na hipergrafu $(X, \mathcal{F})$, čiji elementi $A \in \mathcal{F}$ poseduju neku monotono rastuću osobinu (kao na primer pokrivajuća stabla, Hamiltonova kontura, itd.). U tom slučaju, dovoljno je da Mejker osvoji sve elemente nekog minimalnog pobedničkog skupa da bi pobedio. Takođe, dovoljno je da Brejker osvoji bar po jedan element iz svakog minimalnog pobedničkog skupa kako bi pobedio. Ovo je dovoljno da možemo svesti igru na samo minimalne pobedničke skupove, i kada ne postoji mogućnost da dođe do zabune, sa $\mathcal{F}$ obeležavamo hipergraf koji sadrži samo minimalne pobedničke skupove.

Igre koje su predmet ove teze se igraju na tabli koja predstavlja sve grane kompletnog grafa sa $n$ čvorova, $K_n$, gde je $n$ dovoljno velik ceo broj. Pobednički skupovi su poznate grafovske osobine kao npr. pokrivajuća stabla, Hamiltonove konture, klike itd. Ove igre su intenzivno proučavane tokom poslednjih nekoliko godina i razni rezultati na ovu temu mogu da se nađu u [9, 18, 20–23, 30, 42, 44, 46, 53, 61–63].

S obzirom da je Mejkeru lako da pobedi u nekim fer igrama, Hvatal (Chvátal) i Erdoš (Erdős) su u svom radu [30] odlučili da nadoknade Brejkeru prednost koju Mejker ima, tako što su mu omogućili da bude moćniji. Tako su nastale asimetrične igre, tj. igre u kojima $a, b \geq 1$, $(a, b) \neq (1, 1)$. Pošto u $(1 : 1)$ igri pokrivajućih stabala (čiji je hipergraf označen sa $\mathcal{T}_n$) na kompletnom grafu sa $n$ čvorova Mejker lako pobeđuje, proučavali su $(1 : b)$ verziju iste igre gde je bias Brejkera veći od 1.

Hvatal i Erdoš [30] su primetili da su Mejker-Brejker igre bias monotone, što znači da ako u nekoj $(1 : b)$ Mejker-Brejker igri $(X, \mathcal{F})$ Brejker pobeđuje, onda on pobeđuje i u $(1 : b + 1)$ igri $(X, \mathcal{F})$. Kako važi da je $(1 : |X|)$ igra $(X, \mathcal{F})$ takođe pobeda za Brejkera, sledi da, sem ako je $\emptyset \in \mathcal{F}$ ili $\mathcal{F} = \emptyset$ (što zovemo degenerativnim slučajem), postoji jedinstveni nenegativan ceo broj $b_{\mathcal{F}}$ takav da u $(1 : b)$ igri $(X, \mathcal{F})$ Mejker pobeđuje ako i samo ako $b \leq b_{\mathcal{F}}$, za monotono rastuće hipergrafove $\mathcal{F}$. Vrednost $b_{\mathcal{F}}$ zovemo granični bias igre $(X, \mathcal{F})$ (v. sliku 1.2)

Ako je $(1 : 1)$ Mejker-Brejker igra $(X, \mathcal{F})$ takva da u njoj Brejker pobeđuje, tada je zanimljivo naći najmanju vrednost $a_{\mathcal{F}}$, takvu da za svako $a \geq a_{\mathcal{F}}$, Mejker pobeđuje u $(a : 1)$ igri $(X, \mathcal{F})$.

U opštem slučaju takođe važi da su $(a : b)$ Mejker-Brejker igre $(X, \mathcal{F})$ bias monotone. Ako Mejker pobeđuje u $(a : b)$ igri, tada on pobeđuje i u $(a+1 : b)$ i $(a : b-1)$ igrama. Analogno, ako Brejker pobeđuje u $(a : b)$ igri, tada on pobeđuje i u $(a - 1 : b)$ i $(a : b + 1)$ igrama. Iz ovoga zaključujemo da odabir više elemenata po potezu ne može da škodi igraču.

Slično kao u $(1 : b)$ igri $(X, \mathcal{F})$, može se definisati opšti granični bias za $(a : b)$ igru $(X, \mathcal{F})$. Ako je data nedegenerativna Mejker-Brejker igra $(X, \mathcal{F})$ i $a \geq 1$, neka $b_{\mathcal{F}}(a)$ predstavlja jedinstven nenegativan ceo broj takav da u $(a : b)$ igri Mejker pobeđuje ako i samo ako važi da je $b \leq b_{\mathcal{F}}(a)$ (v. sliku 1.3). Veoma je teško odrediti tačnu vrednost za $b_{\mathcal{F}}$, te često možemo dati gornje i donje ograničenje.

Za analizu Mejker-Brejker igara možemo koristiti i neke rezultate iz jakih igara. Pobeda prvog igrača u jakoj igri $(X, \mathcal{F})$ odmah implicira da u Mejker-Brejker verziji iste igre Mejker pobeđuje. Takođe, Brejkerova pobeda u Mejker-Brejker igri $(X, \mathcal{F})$ implicira da drugi igrač ima strategiju da ostvari nerešen rezultat u jakoj igri na istom hipergrafu.

Ako hipergraf poseduje Remzijevu osobinu, to nam govori da i u Mejker-Brejker igri postoji Mejkerova pobednička strategija, jer ova osobina znači postojanje pobedničke strategije prvog igrača u jakoj igri na istom hipergrafu. Strategija uparivanja drugog igrača u jakoj igri je u stvari implikacija Brejkerove strategije uparivanja u Mejker-Brejker igri.

Alat krađa strategije se, pak, ne može primeniti u Mejker-Brejker igrama. Razlog tome leži u različitim ciljevima igrača u Mejker-Brejker igri, pa nema svrhe uzimati strategiju kada se time ne postiže željeni cilj.

Pored nabrojanih alata koji se mogu koristiti postoje i drugi. Najstariji kriterijum kojim se određuje da li Brejker pobeđuje u $(1 : 1)$ igri je teorema Erdoš-Selfridž (Erdős-Selfridge).

**Teorema 6.1** ([32], *Teorema Erdoš-Selfridž*)**.** *Ako važi*

$$\sum_{A \in \mathcal{F}} \frac{1}{2^{|A|}} < \frac{1}{2},$$

*tada Brejker (kao drugi igrač) pobeđuje u* $(1 : 1)$ *igri* $(X, \mathcal{F})$*. Kada je Brejker prvi igrač, on pobeđuje ako važi da je* $\sum_{A \in \mathcal{F}} \frac{1}{2^{|A|}} < 1$*.*

Ova teorema, osim što daje kriterijum, daje i eksplicitnu Brejkerovu pobedničku strategiju. Štaviše, jačina ove teoreme je i u tome što ona ne zavisi ni od veličine table, ni od strukture pobedničkih skupova. Ono što je bitno je samo veličina pobedničkih skupova. Bek je u [9] dao uopštenje teoreme 6.1.

**Teorema 6.2** ([9], *Proširena Erdoš-Selfridž teorema*)**.** *Neka su $a$ i $b$ pozitivni celi brojevi i neka je $(X, \mathcal{F})$ poziciona igra. Ako važi*

$$\sum_{A \in \mathcal{F}} (1+b)^{-\frac{|A|}{a}} < \frac{1}{1+b},$$

*tada Brejker, kao drugi igrač, pobeđuje u $(a : b)$ Mejker-Brejker igri $(X, \mathcal{F})$. Ako je Brejker prvi igrač u ovoj igri, onda on pobeđuje ako važi da je $\sum_{A \in \mathcal{F}} (1+b)^{-\frac{|A|}{a}} < 1$.*

Sledeći kriterijum takođe daje postojanje Brejkerove pobedničke strategije (uparivanja). On je jednostavniji, ali nije tako primenljiv kao teorema 6.2.

**Teorema 6.3** ([17], *Kriterijum uparivanja*)**.** *Neka je $(X, \mathcal{F})$ poziciona igra, gde je $\mathcal{F}$ $n$-uniformni hipergraf, tj. $|A| = n$, za svako $A \in \mathcal{F}$. Ako je maksimalni stepen najviše $n/2$, $\Delta(\mathcal{F}) \leq n/2$, tada postoji pobednička strategija (uparivanja) za Brejkera u igri $(X, \mathcal{F})$.*

Postoje i kriterijumi kojima se može odrediti postojanje Mejkerove pobedničke strategije u igri. Na žalost, ovi kriterijumi su teško primenljivi u praksi.

**Teorema 6.4** ([17], *Kriterijum za pobedu Mejkera*)**.** *Neka je $(X, \mathcal{F})$ poziciona igra. Neka je $\Delta_2(\mathcal{F})$ oznaka za najveći stepen para čvorova iz $X$, tj. $\Delta_2(\mathcal{F}) = max\{|\{A \in \mathcal{F} : \{x, y\} \subseteq A\}| : \{x, y\} \in X\}$. Ako važi*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} > \frac{1}{8} \Delta_2(\mathcal{F}) |X|,$$

*tada Mejker pobeđuje u $(1 : 1)$ igri $(X, \mathcal{F})$.*

Teorema 6.4 se može uopštiti na sledeći način.

**Teorema 6.5** ([17], *Uopšteni kriterijum za pobedu Mejkera*)**.** *Neka su $a$ i $b$ pozitivni celi brojevi i neka je $(X, \mathcal{F})$ poziciona igra. Neka je $\Delta_2(\mathcal{F})$ oznaka za najveći stepen para čvorova iz $X$, tj. $\Delta_2(\mathcal{F}) = max\{|\{A \in \mathcal{F} : \{x, y\} \subseteq A\}| : \{x, y\} \in X\}$. Ako važi*

$$\sum_{A\in\mathcal{F}} \left(\frac{a+b}{a}\right)^{-|A|} > \frac{a^2 \cdot b^2}{(a+b)^3}\Delta_2(\mathcal{F})|X|,$$

*tada Mejker ima pobedničku strategiju u $(a : b)$ Mejker-Brejker igri $(X, \mathcal{F})$.*

### Boks igra

Ako hipergraf $(X, \mathcal{F})$ čine disjunktni skupovi, postoji još jedan koristan alat, tačnije igra, koja se često koristi kao alat. Igra *boks* (Box) je prvi put pomenuta u radu Hvatala i Erdoša [30]. Hipergraf $\mathcal{H}$ je tipa $(k, t)$ ako važi da je $|\mathcal{H}| = k$, grane $e_1, e_2, \ldots, e_k$ su uzajamno disjunktne i zbir njihovih veličina je $\sum_{i=1}^{k} |e_i| = t$. Dalje, hipergraf $\mathcal{H}$ je *kanonički* ako $||e_i| - |e_j|| \leq 1$ važi za svako $1 \leq i, j \leq k$. Tabla igre boks, $B(k, t, a, b)$, je kanonički hipergraf tipa $(k, t)$. Igru igraju dva igrača koja se zovu *Boksmejker* (BoxMaker) i *Boksbrejker* (BoxBreaker), i Boksbrejker je prvi igrač. Boksmejker bira $a$ čvorova iz $\mathcal{H}$ po potezu, dok Boksbrejker bira $b$ čvorova iz $\mathcal{H}$ po potezu. Boksmejker pobeđuje u boks igri na $\mathcal{H}$ ako osvoji sve čvorove iz neke hipergrane iz $\mathcal{H}$, u suprotnom Boksbrejker pobeđuje. Hipergrane $e_1, e_2, \ldots, e_k$ se još nazivaju *boksovi*, a čvorovi u njima – *elementi* boksa.

Hvatal i Erdoš su analizirali igru u kojoj je $a \geq 1$, a $b = 1$. Da bi mogao da se formuliše kriterijum za Boksmejkerovu pobedu u $B(k, t, a, 1)$, u [30] je definisana sledeća rekurzivna funkcija:

$$f(k, a) := \begin{cases} 0, & k = 1 \\ \lfloor \frac{k(f(k-1,a)+a)}{k-1} \rfloor, & k \geq 2. \end{cases}$$

Vrednost $f(k, a)$ se može aproksimirati kao

$$(a-1)k \sum_{i=1}^{k} \frac{1}{i} \leq f(k, a) \leq ak \sum_{i=1}^{k} \frac{1}{i}.$$

Sledeća teorema daje kriterijum za Boksmejkerovu pobedu u $B(k,t,a,1)$ igri. Dokaz teoreme u [30] je imao grešku, ali je tvrđenje tačno, što su pokazali Amidun (Hamidoune) i La Verna (Las Vergnas) u [50], gde su ispravili grešku u samom dokazu.

**Teorema 6.6** ([30], *Boksmejkerova pobeda*). *Neka su $a$, $k$ i $t$ pozitivni celi brojevi. Boksmejker ima pobedničku strategiju u igri $B(k,t,a,1)$ ako i samo ako $t \leq f(k,a)$.*

## Avojder-Enforser igre

Avojder-Enforser (Avoider-Enforcer) igre se igraju po pravilima koja su u neku ruku suprotna od onih u Mejker-Brejker igrama. Naime, u ovoj igri, igrači igraju *da izgube*, u odnosu na cilj u Mejker-Brejker igri. U Avojder-Enforser igri, igrač *Avojder* želi da izbegne *osvajanje* (selektovanje svih elemenata tog skupa) bilo kog ciljnog skupa, dok drugi igrač – *Enforser* želi da ga natera da osvoji neki ciljni skup. Rezultati vezani za ovu vrstu igara se mogu naći npr. u [3,4,6,40,55,57,65]. U $(a:b)$ Avojder-Enforser igri $(X,\mathcal{F})$ gde su $a,b \geq 1$, Avojder selektuje $a$, a Enforser $b$, elemenata table $X$ po potezu. Ukoliko ima strogo manje elemenata na tabli $X$ nego što je bias igrača koji treba da igra, igrač mora selektovati sve preostale elemente. U ovoj postavci se familija $\mathcal{F}$ naziva familijom *gubitničkih skupova*, jer Avojder gubi u igri čim osvoji skup $A \in \mathcal{F}$. U suprotnom, Avojder pobeđuje.

Važna osobina Mejker-Brejker igara je njihova bias monotonost. U opštem slučaju, to ne važi za Avojder-Enforser igre, što ih čini težim za analizu. Tačnije, nije moguće na isti način definisati granični bias kao u Mejker-Brejker igrama. Ali, može se definisati sledeće, što su uveli Hefec (Hefetz), Krivelevič (Krivelevich) i Sabo (Szabó) u [57]: Neka je $(X,\mathcal{F})$ Avojder-Enforser igra. *Donji granični bias* se definiše kao najveći ceo broj $f^-_{\mathcal{F}}$ takav da važi da za svako $b \leq f^-_{\mathcal{F}}$, Enforser pobeđuje u $(1:b)$ igri. *Gornji granični bias* $f^+_{\mathcal{F}}$ je najmanji nenegativni ceo broj takav da za svako $b > f^+_{\mathcal{F}}$ Avojder pobeđuje u $(1:b)$ igri. Sem u nekim trivijalnim slučajevima, $f^-_{\mathcal{F}}$ i $f^+_{\mathcal{F}}$ uvek postoje i uvek važi da je $f^-_{\mathcal{F}} \leq f^+_{\mathcal{F}}$ (v. sliku 1.6). Ako važi $f^-_{\mathcal{F}} = f^+_{\mathcal{F}}$, taj broj označavamo da $f_{\mathcal{F}}$ i zovemo ga *granični bias* igre.

Da bi se prevazišla ova poteškoća sa nemonotonošću, Hefec, Krivelevič, Stojaković i Sabo su u [55] predložili način da se napravi monotona verzija Avojder-Enforser igara. U monotonoj $(a : b)$ Avojder-Enforser igri, Avojder i Enforser selektuju bar $a$, odnosno bar $b$, elemenata table u svakom potezu. Ovo je zaista monotona igra, jer igrači profitiraju ako selektuju manje elemenata. Dakle, ako u $(a : b)$ igri Enforser pobeđuje, tada on pobeđuje i u $(a + 1 : b)$ i $(a : b - 1)$ igrama. Ako Avojder pobeđuje u $(a : b)$ igri, tada on pobeđuje i u $(a - 1 : b)$ i $(a : b + 1)$ igrama. Ova nova pravila su omogućila definisanje jedinstvenog *monotonog graničnog biasa* $f_{\mathcal{F}}^{mon}$ za igru $\mathcal{F}$, koji predstavlja najveći nenegativni ceo broj takav da Enforser pobeđuje u $(1 : b)$ Avojder-Enforser igri $\mathcal{F}$ ako i samo ako važi $b \leq f_{\mathcal{F}}^{mon}$ (v. sliku 1.7).

U okviru teze ova nova pravila zovemo *monotona* pravila, kako bismo ih razlikovali od originalnih, *striktnih*, pravila. Shodno tome, igre koje se igraju po monotonim pravilima se nazivaju *monotone igre*, a one koje se igraju po striktnim pravilima se zovu – *striktne*.

Može se postaviti pitanje po kojim pravilima je *bolje* igrati, međutim nema pravog odgovora. Prednost strogih igara je u njihovoj primenljivosti u Mejker-Brejker igrama (npr. [53]) i igrama proporcionalnosti *discrepancy games* (npr. [17, 58]). Ipak, ishod igre, kao i sama njena analiza zavise od ostatka pri celobrojnom deljenju $|X|$ sa $b + 1$. Sa druge strane, prednost monotone verzije je ta što postoji jedinstveni granični bias.

Postoji par kriterijuma za određivanje pobednika, ali ne toliko koliko u Mejker-Brejker igrama.

Hefec, Krivelevič i Sabo su u [57] dali opšti kriterijum za pobedu Avojdera u $(a : b)$ igri $(X, \mathcal{F})$.

**Teorema 6.7** ([57], Teorema 1.1)**.** *Ako važi*

$$\sum_{A \in \mathcal{F}} \left(1 + \frac{1}{a}\right)^{-|A|+a} < 1,$$

*tada Avojder pobeđuje u* $(a : b)$ *i striktnoj i monotonoj igri* $(X, \mathcal{F})$*, za svako* $b \geq 1$.

Ipak, ovaj kriterijum zavisi samo od biasa Avojdera $a$ i ne uzima u obzir Enforserov bias $b$, što nije baš efektno, kada je $b$ veliko.

Nedavno je Bednarska-Bzdenga (Bednarska-Bzdęga) u [19] dala drugi kriterijum za pobedu Avojdera i u monotonim i u striktnim igrama koje se igraju na hipergrafu malog ranga. *Rang* hipergrafa $\mathcal{F}$ se definiše kao $rank(\mathcal{F}) = \max_{A \in \mathcal{F}} |A|$.

**Teorema 6.8** ([19], Teorema 1.2 (i))**.** *Neka je* $(X, \mathcal{F})$ *hipergraf ranga* $r$*. Ako važi*

$$\sum_{A \in \mathcal{F}} \left(1 + \frac{b}{ar}\right)^{-|A|+a} < 1,$$

*tada Avojder ima pobedničku strategiju i u monotonoj i u striktnoj* $(a : b)$ *Avojder-Enforser igri* $(X, \mathcal{F})$*.*

## Igre na grafovima

Vrlo je prirodno igrati i Mejker-Brejker igre i Avojder-Enforser igre na granama zadatog grafa $G$. U ovom slučaju, tabla za igru je $X = E(G)$, a ciljni skupovi su sve grane podgrafa $H \subseteq G$ koji poseduju neku monotono rastuću osobinu. Posebno su zanimljive igre u kojima je $G = K_n$. Hvatal i Erdoš su proučavali igru povezanosti (*Connectivity game*), $\mathcal{T}_n$, igru Hamiltonove konture (*Hamiltonicity game*), $\mathcal{H}_n$, i igru savršenog mečinga (*Perfect matching game*), $\mathcal{M}_n$, koje se igraju na $E(K_n)$. U igri povezanosti ciljni skupovi su grane svih pokrivajućih stabala na $K_n$; u igri Hamiltonove konture, ciljni skupovi su grane svih Hamiltonovih kontura grafa $K_n$; u igri savršenog mečinga ciljni skupovi su skupovi $\lfloor |V(G)|/2 \rfloor$ nezavisnih grana grafa $K_n$. U svom radu [30], Hvatal i Erdoš su dali Brejkerovu strategiju u sve tri igre koje se igraju sa biasom $(1 : b)$ u kojoj on izoluje čvor u Mejkerovom grafu, ako je $b \geq \frac{(1+\varepsilon)n}{\ln n}$, za bilo koje $\varepsilon > 0$. Njihov rezultat, u kombinaciji sa rezultatom Gebauer (Gebauer) i Sabo [46] daje da je vrednost graničnog biasa u igri povezanosti $\Theta(\frac{n}{\ln n})$. U igri Hamiltonove konture i savršenog mečinga, granični bias je takođe $\Theta(\frac{n}{\ln n})$, gde je gornje ograničenje dao Krivelevič u [62].

Takođe je zanimljivo igrati pomenute igre prema pravilima koja važe u Avojder-Enforser igrama. Iz rezultata Hefec et al. [55] i rezultata Krivelevič, Sabo [63] dobija se da u monotonoj verziji sve tri igre na kompletnom grafu sa $n$ čvorova važi

$$f^{mon}_{\mathcal{T}_n}, f^{mon}_{\mathcal{H}_n}, f^{mon}_{\mathcal{M}_n} = \Theta\left(\frac{n}{\ln n}\right).$$

Rezultati u striktnoj verziji ovih igara se razlikuju. U $(1 : b)$ igri povezanosti je pokazano da Avojder pobeđuje ako i samo ako do kraja igre ima najviše $n-2$ selektovanih grana u svom grafu, te granični bias u ovoj igri postoji i linearan je [57]. U striktnoj verziji igre Hamiltonove konture i igre savršenog mečinga, donji granični biasi su redom $f^{-}_{\mathcal{H}_n} = (1-o(1))\frac{n}{\ln n}$ [63] i $f^{-}_{\mathcal{M}_n} = \Omega\left(\frac{n}{\ln n}\right)$ [57]. Za gornji granični bias imamo samo trivijalne granice.

U [55], posmatrane su $(1 : b)$ Avojder-Enforser igre na $E(K_n)$ u kojima Avojder želi da izbegne selektovanje kopije nekog zadatog grafa $H$. U ovom slučaju, $X = E(K_n)$, a $\mathcal{F} = \mathcal{K}_H \subseteq 2^{E(K_n)}$ se sastoji od svih kopija grafa $H$ u $K_n$. Ova igra se zove $H$-igra. Oni su pretpostavili da za svaki zadat graf $H$, biasi $f^{-}_{\mathcal{K}_H}$ i $f^{+}_{\mathcal{K}_H}$ nisu istog reda veličine. Proučavali su $H$-igre u kojima je $H = K_3$ i $H = P_3 = K_3^-$ i dobili sledeće rezultate:

$$f^{mon}_{\mathcal{K}_{P_3}} = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor - 1,\ f^{+}_{\mathcal{K}_{P_3}} = \binom{n}{2} - 2,\ f^{-}_{\mathcal{K}_{P_3}} = \Theta(n^{\frac{3}{2}}) \text{ and } f^{mon}_{\mathcal{K}_{K_3}} = \Theta(n^{\frac{3}{2}}).$$

Ovaj primer zadovoljava njihovu pretpostavku, jer $f^{+}_{\mathcal{K}_{P_3}}$ i $f^{-}_{\mathcal{K}_{P_3}}$ zaista nisu istog reda veličine. Takođe, autori su se interesovali za rezultate $H$-igara u kojima $|V(H)| > 3$. Bednarska-Bzdenga je u [19] dala donja i gornja ograničenja za $f^{+}_{\mathcal{K}_H}$, $f^{-}_{\mathcal{K}_H}$ i $f^{mon}_{\mathcal{K}_H}$ koja važe za svaki zadati graf $H$, ali ove granice nisu jako bliske.

## Brza pobeda u pozicionim igrama na grafovima

U jakim igrama, kao što smo već videli, ili prvi igrač pobeđuje, ili drugi može da odigra nerešeno. Na žalost, za ovu vrstu igara je veoma teško naći eksplicitnu pobedničku strategiju prvog igrača, jer poznati alati omogućuju utvrđivanje samo da li strategija postoji ili ne. Ova poteškoća je delimično inicirala analizu slabih igara – igara tipa Mejker-Brejker i njihove *mizer* verzije Avojder-Enforser igara.

Kako je nemoguće igrati nerešeno u slabim igrama, uvek pobeđuje neki igrač. U nekoliko (fer) standardnih igara $\mathcal{F}$ koje se igraju na $E(K_n)$ Mejker, odnosno Enforser, prilično lako pobede. Uzimajući to u obzir, zanimljivo pitanje koje se javlja je *koliko brzo* može Mejker, odnosno Enforser, da pobedi u igri $\mathcal{F}$? Parametri $\tau_M(\mathcal{F})$ i $\tau_E(\mathcal{F})$ izražavaju trajanje Mejker-Brejker, odnosno Avojder-Enforser, igre $\mathcal{F}$ i definisani su kao najmanji ceo broj $t$ takav da Mejker, odnosno Enforser, pobeđuju u igri $\mathcal{F}$ u $t$ poteza. Ukoliko Brejker, odnosno Avojder, pobeđuje u $\mathcal{F}$, uzima se da je $t = \infty$.

Brze pobedničke strategije u Mejker-Brejker igrama su proučavane u [28, 33, 36, 37, 39, 54, 61, 69], a rezultati vezani za brze strategije u Avojder-Enforser igrama se mogu naći u [3,4,6,52]. Sem toga što je zanimljivo samo po sebi tražiti brze strategije, one imaju i drugu primenu. Naime, (skoro) savršeno brz Mejker može pomoći prvom igraču u jakoj igri. Npr. znamo da u $(1:1)$ (fer) igri povezanosti, $\mathcal{T}_n$, koja se igra na granama kompletnog grafa sa $n$ čvorova, $K_n$ (čak je dovoljno da se igra na bilo kom grafu $G$ sa $n$ čvorova koji sadrži dva disjunktna pokrivajuća stabla bez jedne grane), Mejker pobeđuje u $n-1$ poteza. Dakle, $\tau_M(\mathcal{T}_n) = n-1$. Donje ograničenje je $n-1$, jer pokrivajuće stablo grafa sa $n$ čvorova mora imati $n-1$ granu, a Leman (Lehman) je u [64] postavio gornje ograničenje dajući strategiju koja ne sadrži mogućnost kreiranja kontura. Ista strategija daje pobedu prvog igrača u jakoj igri, jer drugi igrač ne može osvojiti pokrivajuće stablo u manje od $n-1$ poteza. Postoji još par primera ove pojave, igra savršenog mečinga, igra Hamiltonove konture, kao i igra $k$-povezanosti. Ferber (Ferber) i Hefec su u [36, 37] dali brze strategije za prvog igrača u pomenutim jakim igrama koristeći se činjenicom da su poznate eksplicitne pobedničke strategije za Mejkera u datim igrama.

# Rezultati

## Mejker-Brejker igre sa dvostrukim biasom

U glavi 3 prikazani su originalni rezultati u $(a : b)$ Mejker-Brejker igrama, kada za oba parametra važi $a, b \geq 1$. Posmatramo dve igre na granama kompletnog grafa sa $n$ čvorova, $K_n$ – igru povezanosti, čiji je hipergraf $\mathcal{T}_n$, i igru Hamiltonove konture, čiji je hipergraf $\mathcal{H}_n$.

Za opisivanje pobedničke strategije Brejker-a u igri povezanosti, koristimo boks igru $B(k, t, a, b)$, u kojoj oba igrača (Boksmejker i Boksbrejker) mogu selektovati više elemenata po potezu. Boks igra se igrà na kanoničkom hipergrafu tipa $(k, t)$, gde je $k$ – broj hipergrana, a $t$ – ukupan broj čvorova ovog hipergrafa. Hvatal i Erdoš su u [30] analizirali ovu igru za slučaj kada je $b = 1$. Amidun i La Verna su u [50] analizirali opšti slučaj ove igre, ali njihov rezultat nije dovoljno precizan za neke vrednosti parametara $a$ i $b$. Za date pozitivne cele brojeve $a$ i $b$ definišemo sledeću funkciju:

$$f(k; a, b) := \begin{cases} (k-1)(a+1) & , \text{ if } 1 \leq k \leq b \\ ka & , \text{ if } b < k \leq 2b \\ \left\lfloor \frac{k(f(k-b;a,b)+a-b)}{k-b} \right\rfloor & , \text{ inače} \end{cases}.$$

Zatim dokazujemo sledeće:

**Lema 6.9.** *Ako važi da je $t \leq f(k; a, b) + a$ tada Boksmejker ima pobedničku strategiju u igri $B(k, t, a, b)$.*

Brejkerova strategija u igri povezanosti se zasniva na izolaciji čvora u Mejkerovom grafu. Samim tim, graf koji Mejker može napraviti do kraja igre nije povezan, što implicira da takav graf ne može ni sadržati Hamiltonovu konturu.

U $(a : b)$ igri povezanosti, $\mathcal{T}_n$, dobijamo sledeća ograničenja za granični bias $b_{\mathcal{T}_n}$.

**Teorema 6.10.** $(i)$ *Ako je $a = o(\ln n)$, tada*
$\frac{an}{\ln n} - (1 + o(1))\frac{an(\ln\ln n + a)}{\ln^2 n} < b_{\mathcal{T}_n}(a) < \frac{an}{\ln n} - (1 - o(1))\frac{an \ln a}{\ln^2 n}$.

(*ii*) *Ako je* $a = c\ln n$ *za neko* $0 < c \le 1$, *tada*
$(1-o(1))\frac{cn}{c+1} < b_{\mathcal{T}_n}(a) < \min\left\{cn, (1+o(1))\frac{2n}{3}\right\}$.

(*iii*) *Ako je* $a = c\ln n$ *za neko* $c > 1$, *tada*
$(1-o(1))\frac{cn}{c+1} < b_{\mathcal{T}_n}(a) < (1+o(1))\frac{2cn}{2c+1}$.

(*iv*) *Ako je* $a = \omega(\ln n)$ *i* $a = o\left(\sqrt{\frac{n}{\ln n}}\right)$, *tada*
$n - \frac{n\ln n}{a} < b_{\mathcal{T}_n}(a) < n - (1-o(1))\frac{n\ln(n/a)}{2a}$.

(*v*) *Ako je* $a = \Omega\left(\sqrt{\frac{n}{\ln n}}\right)$ *i* $a = o(n)$, *tada*
$n - (1+o(1))\frac{2n\ln(n/a)}{a} < b_{\mathcal{T}_n}(a) < n - (1-o(1))\frac{n\ln(n/a)}{2a}$.

(*vi*) *Ako je* $a = cn$ *za neko* $0 < c < \frac{1}{2e}$, *tada*
$n - \frac{2\ln(1/c)+4}{c} < b_{\mathcal{T}_n}(a) < n - 2 - \frac{1-2c}{2c}\left(\ln(\frac{1}{2c}) - 1\right) + o(1)$.

(*vii*) *Ako je* $a = cn$ *za* $\frac{1}{2e} \le c < \frac{1}{2}$, *tada*
$n - \frac{2\ln(1/c)+4}{c} < b_{\mathcal{T}_n}(a) < n - 2$.

U $(a : b)$ igri Hamiltonove konture, $\mathcal{H}_n$, dobijamo sledeća ograničenja za granični bias $b_{\mathcal{H}_n}$.

**Teorema 6.11.** (*i*) *Ako je* $a = o(\ln n)$, *tada*
$\frac{an}{\ln n} - (1-o(1))\frac{an(30\ln^{3/4} n + a)}{\ln^2 n} < b_{\mathcal{H}_n}(a) < \frac{an}{\ln n} - (1-o(1))\frac{an\ln a}{\ln^2 n}$.

(*ii*) *Ako je* $a = c\ln n$ *za neko* $0 < c \le 1$, *tada*
$\frac{cn}{c+1}(1 - \frac{30}{\ln^{1/4} n}) < b_{\mathcal{H}_n}(a) < \min\left\{cn, (1+o(1))\frac{2n}{3}\right\}$.

(*iii*) *Ako je* $a = c\ln n$ *za neko* $c > 1$, *tada*
$\frac{cn}{c+1}(1 - \frac{30}{\ln^{1/4} n}) < b_{\mathcal{H}_n}(a) < (1+o(1))\frac{2cn}{2c+1}$.

(*iv*) *Ako je* $a = \omega(\ln n)$ *i* $a = o(\ln^{5/4} n)$, *tada*
$n - (1+o(1))\frac{n\ln n}{a} < b_{\mathcal{H}_n}(a) < n - (1-o(1))\frac{n\ln(n/a)}{2a}$.

(*v*) *Ako je* $a = c\ln^{5/4} n$, $c > 0$, *tada*
$n - (1-o(1))\frac{(30c+1)n}{c\ln^{1/4} n} < b_{\mathcal{H}_n}(a) < n - (1-o(1))\frac{n}{2c\ln^{1/4} n}$.

(*vi*) *Ako je* $a = \omega(\ln^{5/4} n)$ *i* $a = o(n)$, *tada*
$n - \frac{30n}{\ln^{1/4} n} - (1 + o(1))\frac{n \ln n}{a} < b_{\mathcal{H}_n}(a) < n - (1 - o(1))\frac{n \ln(n/a)}{2a}$.

(*vii*) *Ako je* $a = cn$ *za* $0 < c < \frac{1}{2e}$, *tada*
$n - \frac{30n}{\ln^{1/4} n} - \frac{\ln n}{c} < b_{\mathcal{H}_n}(a) < n - 2 - \frac{1-2c}{2c}\left(\ln(\frac{1}{2c}) - 1\right) + o(1)$.

(*viii*) *Ako je* $a = cn$ *za* $\frac{1}{2e} \leq c < 1$, *tada*
$n - \frac{30n}{\ln^{1/4} n} - \frac{\ln n}{c} < b_{\mathcal{H}_n}(a) < n - 2$.

Rezultati vezani za igru povezanosti i boks igru u ovom poglavlju su zajednički rad sa Hefecom i Stojakovićem, [60].

## Avojder-Enforser igre zvezda

U glavi 4 posmatramo $(1 : b)$ Avojder-Enforser igre na granama kompletnog grafa sa $n$ čvorova $K_n$. Prikazaćemo originalne rezultate u dve igre. Obe igre koje posmatramo imaju zajedničku karateristiku – Avojder pokušava da izbegne selektovanje kopije zadatog malog grafa $H$. Drugim rečima, familija gubitničkih skupova u ovim igrama je $\mathcal{F} = \mathcal{K}_H \subseteq 2^{E(K_n)}$ i sastoji se od svih kopija grafa $H$ u $K_n$.

Prvo posmatramo i striktnu i monotonu $(1 : b)$ igru $k$-*zvezde* ($k$-star), za konstantno $k \geq 3$. U ovoj igri, Avojder pokušava da izbegne selektovanje $k$ ili više grana incidentnih sa istim čvorom. Odnosno, u ovoj igri je $H = K_{1,k}$.

Sledeća teorema daje gornje i donje ograničenje za granični bias i u striktnoj i u monotonoj igri $k$-zvezda koju označavamo sa $\mathcal{K}_{\mathcal{S}_k}$.

**Teorema 6.12.** *Neka je* $k \geq 3$. *Za* $(1 : b)$ $k$-*zvezda igru* $\mathcal{K}_{\mathcal{S}_k}$

(*i*) $f^{mon}_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k}{k-1}})$,

(*ii*) $f^{+}_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k}{k-1}})$ *važi za beskonačno mnogo vrednosti* $n$,

(*iii*) $f^{-}_{\mathcal{K}_{\mathcal{S}_k}} = \Theta(n^{\frac{k+1}{k}})$ *važi za beskonačno mnogo vrednosti* $n$.

Posmatramo još jednu monotonu $(1:b)$ igru, gde je $H = K_{2,2}$. Ovu igru zovemo $K_{2,2}$-igra. Sledeća teorema daje red veličine za granični bias u ovoj igri, čiji hipergraf označavamo sa $\mathcal{K}_{K_{2,2}}$.

**Teorema 6.13.** *U $K_{2,2}$-igri, $\frac{1}{4}n^{\frac{4}{3}} < f^{mon}_{\mathcal{K}_{K_{2,2}}} < n^{\frac{4}{3}}$.*

Rezultat Teoreme 6.12 je zajednički rad sa Grzesikom (Grzesik), Nađom (Nagy), Naorom (Naor), Patkošem (Patkós) i Skerman (Skerman), [47].

## Brze asimetrične Mejker-Brejker igre

Glava 5 posvećena je brzim pobedničkim strategijama Mejkera u dve standardne grafovske igre – igri savršenog mečinga i igri Hamiltonove konture – koje se igraju na granama kompletnog grafa sa $n$ čvorova $K_n$. Ovde je akcenat na asimetričnim $(1:b)$ igrama, gde je $b \geq 1$.

**Teorema 6.14.** *U $(1:b)$ Mejker-Brejker igri savršenog mečinga koja se igra na $E(K_n)$, $\mathcal{M}_n$, važi da*

$$\frac{n}{2} + \frac{b}{4} \leq \tau_M(\mathcal{M}_n, b) \leq \frac{n}{2} + O(b \ln b)$$

*za sve vrednosti $b \leq \frac{\delta n}{100 \ln n}$, gde je $\delta > 0$ mala konstanta.*

**Teorema 6.15.** *U $(1:b)$ Mejker-Brejker igri Hamiltonove konture koja se igra na $E(K_n)$, $\mathcal{H}_n$, važi da*

$$n + \frac{b}{2} \leq \tau_M(\mathcal{H}_n, b) \leq n + O(b^2 \ln^5 n)$$

*za sve vrednosti $b \leq \delta\sqrt{\frac{n}{\ln^5 n}}$, gde je $\delta > 0$ mala konstanta.*

Rezultati u ovoj glavi su dobijeni zajedničkim radom sa Ferberom, Hefecom i Stojakovićem, [38].

# Short biography

Mirjana Mikalački (maiden Rakić) was born in Novi Sad, on 21st February 1984. She graduated Computer Science at the Department of Mathematics and Informatics on Faculty of Sciences, University of Novi Sad in 2007. For the excellence during her studies she has received the University award "Aleksandar Saša Popović", as the best graduate computer scientist in her generation. In the same year she went to Siemens in Erlangen, Germany for a 3.5 months internship within "Zoran Đinđić Internship programme". After that, she enrolled PhD studies of Theoretical Computer Science at the Department of Mathematics and Informatics. Within two years she has passed all her exams with the average grade 10.00 (max 10.00). During her PhD studies Mirjana participated in various summer and winter schools and took part in combinatorial conferences and seminars. In 2013, she has received DAAD scholarship for a research stay on Free University of Berlin.

Between 2007 and 2009, Mirjana worked as a Research Assistant at the Faculty of Sciences and received a scholarship of the Ministry of Science and Technological Development. Since 2009, she has been employed as a Teaching Assistant at the Faculty of Sciences and has been working as an Associate in High school "Jovan Jovanović Zmaj".

Novi Sad, September 6, 2013. Mirjana Mikalački

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We study Maker-Breaker games played on the edges of the complete graph on $n$ vertices, $K_n$, whose family of winning sets $\mathcal{F}$ consists of all edge sets of subgraphs $G \subseteq K_n$ which possess a predetermined monotone increasing property. Two players, Maker and Breaker, take turns in claiming $a$, respectively $b$, unclaimed edges per move. We are interested in finding the threshold bias $b_{\mathcal{F}}(a)$ for all values of $a$, so that for every $b$, $b \leq b_{\mathcal{F}}(a)$, Maker wins the game and for all values of $b$, such that $b > b_{\mathcal{F}}(a)$, Breaker wins the game. We are particularly interested in cases where both $a$ and $b$ can be greater than 1. We focus on the *Connectivity game*, where the winning sets are the edge sets of all spanning trees of $K_n$ and on the *Hamiltonicity game*, where the winning sets are the edge sets of all Hamilton cycles on $K_n$.

Next, we consider biased $(1 : b)$ Avoider-Enforcer games, also played on the edges of $K_n$. For every constant $k \geq 3$ we analyse the $k$-star game,

where Avoider tries to avoid claiming $k$ edges incident to the same vertex. We analyse both versions of Avoider-Enforcer games, the strict and the monotone, and for each provide explicit winning strategies for both players. Consequentially, we establish bounds on the threshold biases $f_{\mathcal{F}}^{mon}$, $f_{\mathcal{F}}^{-}$ and $f_{\mathcal{F}}^{+}$, where $\mathcal{F}$ is the hypergraph of the game (the family of target sets). We also study the monotone version of $K_{2,2}$-game, where Avoider wants to avoid claiming all the edges of some graph isomorphic to $K_{2,2}$ in $K_n$.

Finally, we search for the fast winning strategies for Maker in Perfect matching game and Hamiltonicity game, again played on the edge set of $K_n$. Here, we look at the biased $(1 : b)$ games, where Maker's bias is 1, and Breaker's bias is $b, b \geq 1$.

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**Thesis defend board:**

President: Dragan Mašulović, PhD, Full Professor, Faculty of Sciences, University of Novi Sad

Member: Miloš Stojaković, PhD, Associate Professor, Faculty of Sciences, University of Novi Sad

Member: Tibor Szabó, PhD, Full Professor, Department of Mathematics and Computer Science, Free University of Berlin

Member: Dejan Vukobratović, PhD, Assistant Professor, Department of Power, Electronics and Communication Engineering, Faculty of Technical Sciences, University of Novi Sad

**DB**

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Proučavamo takozvane Mejker-Brejker (Maker-Breaker) igre koje se igraju na granama kompletnog grafa sa $n$ čvorova, $K_n$, čija familija pobedničkih skupova $\mathcal{F}$ obuhvata sve skupove grana grafa $G \subseteq K_n$ koji imaju neku monotono rastuću osobinu. Dva igrača, *Mejker* (*Praviša*) i *Brejker* (*Kvariša*) se smenjuju u odabiru $a$, odnosno $b$, slobodnih grana po potezu. Interesuje nas da pronađemo granični bias $b_{\mathcal{F}}(a)$ za sve vrednosti parametra $a$, tako da za svako $b$, $b \leq b_{\mathcal{F}}(a)$, Mejker pobeđuje u igri, a za svako $b$, takvo da je $b > b_{\mathcal{F}}(a)$, Brejker pobeđuje. Posebno nas interesuju slučajevi u kojima oba parametra $a$ i $b$ mogu imati vrednost veću od 1. Naša pažnja je posvećena igri povezanosti, gde su pobednički skupovi grane svih pokrivajućih stabala grafa $K_n$, kao i igri Hamiltonove konture, gde su pobednički skupovi grane svih Hamiltonovih kontura grafa $K_n$.

Zatim posmatramo igre tipa Avojder-Enforser (Avoider-Enforcer), sa biasom $(1 : b)$, koje se takođe igraju na granama kompletnog grafa sa $n$ čvorova, $K_n$. Za svaku konstantu $k$, $k \geq 3$ analiziramo igru $k$-zvezde (zvezde

sa $k$ krakova), u kojoj *Avojder* pokušva da izbegne da ima $k$ svojih grana incidentnih sa istim čvorom. Posmatramo obe verzije ove igre, striktnu i monotonu, i za svaku dajemo eksplicitnu pobedničku strategiju za oba igrača. Kao rezultat, dobijamo gornje i donje ograničenje za granične biase $f_{\mathcal{F}}^{mon}$, $f_{\mathcal{F}}^{-}$ i $f_{\mathcal{F}}^{+}$, gde $\mathcal{F}$ predstavlja hipergraf igre (familija ciljnih skupova). Takođe, posmatramo i monotonu verziju $K_{2,2}$-igre, gde Avojder želi da izbegne da graf koji čine njegove grane sadrži graf izomorfan sa $K_{2,2}$.

Konačno, želimo da pronađemo strategije za brzu pobedu Mejkera u igrama savršenog mečinga i Hamiltonove konture, koje se takođe igraju na granama kompletnog grafa $K_n$. Ovde posmatramo asimetrične igre gde je bias Mejkera 1, a bias Brejkera $b$, $b \geq 1$.

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**Članovi komisije:**

Predsednik: dr Dragan Mašulović, redovni profesor, Prirodno-matematički fakultet, Univerzitet u Novom Sadu

Član: dr Miloš Stojaković, vanredni profesor, Prirodno-matematički fakultet, Univerzitet u Novom Sadu

Član: dr Tibor Sabo, redovni profesor, Departman za matematiku i informatiku, Freie Univerzitet u Belinu

Član: dr Dejan Vukobratović, docent, Departman za energetiku, elektroniku i telekomunikacije, Fakulet tehničkih nauka, Univerzitet u Novom Sadu

**KO**