VaR optimal portfolio with transaction costs

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Abstract

We consider the problem of portfolio optimization under VaR risk measure taking into account transaction costs. Fixed costs as well as impact costs as a nonlinear function of trading activity are incorporated in the optimal portfolio model. Thus the obtained model is a nonlinear optimization problem with nonsmooth objective function. The model is solved by an iterative method based on a smoothing VaR technique. We prove the convergence of the considered iterative procedure and demonstrate the nontrivial influence of transaction costs on the optimal portfolio weights.

Key words: Portfolio optimization, VaR, transaction costs, market impact, smoothing methods, SVaR

1 Introduction

Portfolio optimization problem is an attractive and important research topic since the pioneering Markowitz work on optimal portfolio selection, [19]. Generally speaking one is interested in determining the optimal combination of n risky assets in such a way that the obtained portfolio has minimal risk and maximal expected gain. Based on investors preferences and risk tolerance several alternative formulations are possible, see [6] and [23] for example. In this paper we will consider

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portfolio consisting of n different assets traded at stock market. The key question in formulating the problem of portfolio optimization is how to measure the risk. A number of risk measures is available in literature with the most popular ones being the portfolio variance, Value at Risk, VaR, and Conditional Value at Risk, CVaR. Mutual relationship between these three risk measures is well known and all of them are quite popular in financial industry, see [12]. Value-at-Risk is incorporated into several regulatory requirements, like Basel Accord II, and hence plays a particularly important role in modern risk analysis.

Theoretical models very often consider an ideal situation where optimization of a portfolio is performed without considering transaction costs. More realistic models where transaction costs are included are analyzed in [20, 7, 8, 18, 22]. Roughly speaking transaction costs decrease the expected return and therefore can not be neglected in real situation. In the above cited papers the transaction costs are modeled as linear or piece-wise linear functions.

There are two main sources of transaction costs - fixed costs and impact costs. Fixed costs are different kinds of fees (brokerage fee, bank fee etc.) and in general they are linear or piece-wise linear functions of transaction size. Due to rapid development of electronic trading in the last decade and a large number of participants at modern markets, fixed costs are becoming relatively small, and in the case of large institutional investor they are not playing the dominant role anymore. But they continue to be significant for small investors. The impact costs are more complicated to model and in general nonlinear. Roughly speaking the impact costs are changes in the price of considered asset that are caused by our own trading of that particular asset. Since the price of any asset is assumed to be an equilibrium state based on demand and supply it is intuitively clear that if we are buying some amount of a particular stock we are increasing the demand and thus influencing, i.e. increasing its price. The opposite is happening if we are selling so our own trading is decreasing the price. This effect is negligible if the quantity we are buying is small but in the case of a large institutional investor can be significant. Therefore impact costs can not be excluded from portfolio optimization problem.

Modeling impact cost is a nontrivial task and there are several models proposed in the literature. Financial institutions are using their own models, based on academic work but not available publicly. One of the most detailed studies is done by Almgren and Chriss, [2], Almgren [1] and Almgren et al., [3]. Another approaches are presented in Bouchard et al. [10] and Lillo et al. [14]. In this paper we will adopt the market impact model from Almgren [1] and Almgren and Chriss, [2]. The problem we will consider in this paper is VaR optimal portfolio with fixed and impact costs. Thus the optimization problem will have a nonsmooth objective function and nonlinear constraints due to the presence of impact costs.

Minimizing VaR measure is a complicated task due to its' nonsmoothness. A number of approaches is considered in the literature. A sequence of CVaR problems is used to approximate VaR problem in [17]. Minimization of VaR function can be treated as a special case of Order-Value-Optimization, see Andreani et al., [4, 5]. One smoothing approach is analyzed in Pang, Leyffer [21]. The optimal portfolio problem with VaR constraints and transaction costs is solved using the Order-Value-Optimization approach in [9]. A smoothing method for the CVaR portfolio optimization is considered in [24]. The Smoothing VaR (SVaR) method, proposed by Gaivoronsky and Pflug [12], transforms the original VaR objective function into a smooth one, SVaR and thus the SVaR optimal portfolio is obtained by standard optimization methods. In this paper we are defining an iterative procedure based on SVaR methodology but yielding VaR optimal portfolio. A sequence of SVaR problems with decreasing smoothing parameters is solved and an approximate solution of VaR problem is thus obtained. The procedure is applied to the model of portfolio optimization with transaction costs. The transaction costs include both fixed and impact costs and yield nonlinear constraints. Although more realistic, such model is more complicated than the models dealing with linear and piece-wise transaction costs. The aim of this paper is to solve VaR optimal portfolio problem with transaction costs using a sequence of SVaR problems and to analyze the influence of transaction costs on the optimal portfolio weights. The influence of transaction costs on optimal portfolio weights is nontrivial as we will show on numerical examples. Further more, there is a clear difference between influence of fixed costs and impact costs for different kind of investors as well as between different models for impact costs. Thus the additional effort needed for solving the transaction costs model with nonlinear impact is well justified.

This paper is organized as follows. In Section 2 of this paper we state the VaR optimal portfolio problem and introduce the model of transaction costs. The optimization procedure is presented in Section 3. A sequence of SVaR optimization problems is explained and the convergence of the iterative procedure based on a decreasing sequence of smoothing parameters is proved. Section 4 contains two examples with real trade data from London Stock Exchange (LSE). The numerical results demonstrate the influence of transaction costs on VaR optimal portfolio weights as well as on the portfolio VaR value.

2 VaR optimal portfolio and transaction costs

2.1 Value at Risk

Information about future return of a portfolio are unknown at the time of portfolio allocation, so every decision brings some risk. The Value at Risk is an important measure of the risk exposure of a given portfolio. It is now the main tool in financial industry and part of regulatory mechanism.

If W is a random variable representing the return of a risky asset with the distribution function $F_W(u) = P\{W < u\}$ then VaR of such asset is the α-quantile of return i.e.

$$Q_\alpha(W) = \sup\{u : F_W(u) \leq \alpha\}$$

for a given $\alpha \in (0, 1)$. Therefore VaR represents the largest return underperformance which is possible in α outcomes. In general $1 - \alpha$ is called the confidence level. The most common values of α are 0.1, 0.05 and 0.01. [1].

There are two main approaches to compute VaR, historical (nonparametric) VaR and model-based (parametric) VaR. In the parametric approach one assumes that the asset returns are governed by a given distribution and thus can obtain an analytical expression for VaR. Unless the return distribution is normal this approach is quite difficult for continuously distributed returns. The sampling approach allows us to work directly with a finite sample of return observations without assuming any specific distribution. The sample can be historical or simulated. We will follow the historical approach and thus assume that a sample of historical returns is available.

Let us assume that we have an universe of n risky assets $i = 1, \ldots, n$. We consider an investor who has the initial portfolio $\hat{x}$ that is composed of n assets,

$$\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n),$$

and aims to change that portfolio to the minimal VaR portfolio

$$x = (x_1, \ldots, x_n)$$

which earns at least the minimal expected return μ. This initial portfolio $\hat{x}$ is introduced here only for generality, i.e. to allow us to consider both buying and selling costs. The same reasoning applies if the portfolio x is purchased from some capital available in cash but in that case only the buying costs would be relevant. If

$$\xi = (\xi_1, \ldots, \xi_n)$$

is the vector of assets returns then the portfolio return is

$$w(x) = x^T \xi.$$

To facilitate notation in VaR definition we introduce the sequence of functions

$$f_i : \mathbf{R}^n \to \mathbf{R}, i = 1, \ldots, m,$$

as well as their ordering

$$f_{i_1}(x) \geq \ldots \geq f_{i_k}(x) \geq \ldots \geq f_{i_m}(x) \tag{1}$$

[1]There are in fact two definitions of VaR, one that we will use here and another one defining VaR as $E[w] - \inf\{w \in \mathbf{R} \mid P(W \leq w) > \alpha\}$, for details see [13]. However since we are dealing with small time period in this paper (one day) for any reasonable portfolio we have $E[w] \approx 0$, so the difference in definition of VaR does not play any role in our analysis.

for any $x \in \mathbf{R}^n$. Let $\xi^1, \ldots, \xi^m$ be a sample of historical realizations of the portfolio return vector ξ. Then the empirical VaR function for a given α is defined as

$$VaR(x) = F(m - \lfloor \alpha m \rfloor - 1, x) \quad \text{where} \quad \alpha m - 1 < \lfloor \alpha m \rfloor \leq \alpha m$$

with

$$f_i(x) = x^T \xi^i, \qquad F(k, x) = f_{i_{k+1}}(x).$$

If the sample mean is denoted by $\bar{\xi}$, $\bar{\xi} = m^{-1} \sum_{i=1}^{m} \xi^i$, then the optimal VaR portfolio problem is

$$\min_x -F(m - \lfloor \alpha m \rfloor - 1, x) \tag{2}$$

$$\text{s.t. } x^T \bar{\xi} \geq \mu \tag{3}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{4}$$

$$x \geq 0. \tag{5}$$

The constraints in (3)- (5) represent the common conditions in portfolio optimization - the expected return lower bound (3) and the budget constraint (4) while the last condition prohibits short positions. The budget constraint (4) means that all available capital is invested in the set of risky assets i.e. there is no cash in the portfolio.

This problem is nonconvex. As the ordering of the function f_i given by (1) changes with x the objective function is nonsmooth. Therefore (2) -(5) cannot be solved by standard algorithms for solving smooth nonlinear optimization problems. Several approaches for solving this problem are suggested in literature. The problem (2) - (5) can be considered in the framework of Order-Value Optimization (OVO) problems and thus methods developed in [4, 5] can be applied. Another possibility is given in [21]. In this paper we will use the SVaR technique developed in [12].

2.2 Transaction costs

Transaction costs are inevitably present in real life and yield decrease in portfolio return. Therefore there is practical need to include these costs in optimization problems dealing with portfolio allocation. In this paper we will assume that we have an initial portfolio, say $\hat{x}$ and that we want to transform that portfolio to the optimal one. In other words we are considering re-optimization of an existing portfolio. This assumption is just convenient since we wanted to have both selling and buying costs in our numerical examples, but it can be easily dropped without any changes in further considerations.

Two main types of costs we consider in this analysis are fixed costs and impact costs. Fixed costs are in general proportional to the transaction size in monetary units and thus modeled by piecewise linear functions. Such kind of cost is considered in [7, 8, 18, 22]. These costs include different fees and taxes and in general depend on investors type. Roughly speaking a small investor is facing larger fixed costs than a large institutional investor. So we will make distinction between these two types of investors.

Impact costs are complicated to measure and model. Intuitively, the impact costs are easy to understand but there is no universally accepted model for these costs and in real life it is quite difficult to distinguish the impact costs from noise that is always present. In this paper we will adopt the models presented in [2] and [1]. According to these models the total impact of our own trading can be divided into temporary and permanent impact. Permanent impact includes changes which remain effective at least for the life time of considered transaction execution while the temporary impact means short-term changes in the price caused by short time imbalances in supply and demand.

In order to express these costs formally we need a couple of technical details of the trading process itself. There are two opposite sides in trading: buyers and sellers. They are submitting their offers through trading system specifying the stock they want to buy/sell, volume and price as well as type of order - market or limit order. The prices are discrete and change according to the tick size rule. Tick size is the smallest possible change in price for a particular stock and it depends on the stock's properties like total market capitalization, liquidity, etc., and is determined by a stock exchange rules. At some exchanges tick size is always the smallest monetary unit (for example 1 cent at NYSE) but nevertheless the prices are still discrete. The prices submitted by buyers are called *bid* prices while the opposite are *ask* prices. Transaction is happening whenever there is an agreement in price at both sides - bid and ask. Different bid and ask prices are ordered by value so we have bid price level 1, level 2 etc., as well as ask price level 1, level 2 and so on. We will denote bid prices as $b_1, b_2, \dots$ and ask prices $a_1, a_2, \dots$. The mid price is defined as $p = (a_1 + b_1)/2$ while the *spread* is $s = a_1 - b_1$. Clearly all these values are time dependent but we omit the time symbol for reasons of simplicity. Order type plays an important role when it comes to impact costs. In general orders of large size are never executed as a single trade but decomposed into a subsequence of smaller orders that are executed within a given time window. This way the impact of trade is reduced. Choice of order type and execution strategy is quite important and nontrivial issue, for details see [11, 15, 16]. For the purpose of analysis in this paper we will assume that the execution strategy is optimal in the sense of [1, 2] and thus follow the impact model presented there. For completeness we will briefly describe the impact costs here. Let v_i denote the velocity of trade (number of traded shares in the unit time period, here one day) and ADV_i the average daily traded volume for each asset i, while s_i denotes the spread. Then, according to [1], the permanent impact is modeled by

$$g(v_i) = \gamma_i v_i^{\beta}$$

and the temporary impact by

$$h(v_i) = 0.5s_i + \eta_i v_i^{\beta},$$

where $\beta \in (0, 1]$. Parameters γ_i, s_i and η_i are specific values for the i-th asset, and according to the rule of thumb, [1] have the following values with the spread s_i expressed in £/share,

$$\begin{aligned}
\gamma_i &= \frac{s_i}{10\% ADV_i} \qquad \left(\frac{\text{£}}{\text{share}^2}\right), \\
\eta_i &= \frac{s_i}{1\% ADV_i} \qquad \left(\frac{\text{£}}{\text{share}} \cdot \frac{\text{time}}{\text{share}}\right).
\end{aligned}$$

Monetary unit clearly depends on the data we are considering and we used £ since our examples are from London Stock Exchange. When the impact is modeled by linear function, then $\beta = 1$. In our examples we used $\beta = 1$ and $\beta = 1/2$ for the nonlinear model of impact. Functions g and h are clearly concave functions.

As already stated, the fixed cost depend on transaction size. We assume that the fixed costs for a small investor are 1% of the transaction size, i.e. the cost of trading one share with the price p_i is $q_i = 0.01 \cdot p_i$ and for a large institutional investor fixed costs of trading one share of ith asset is $q_i = 3 \cdot 10^{-4} p_i$. Here we actually stated that the fixed costs are equal to 3 *basis points* of the transaction size. Basis points are commonly used unit and $1bp = 10^{-4}$. Therefore, with one day as the time unit, the transaction costs of trading z_i shares of ith asset, expressed in £, are

$$t_i(z_i) = z_i(g_i(z_i) + h_i(z_i) + q_i).$$

Let $\hat{n}_i$ be the number of shares of the ith asset in the existing portfolio $\hat{x}$, while n_i denotes the number of shares in the new optimal portfolio x. Since we assume that the whole budget will be invested, i.e. $\sum_{i=1}^{n} x_i = 1$, the total portfolio value will be $y = \sum_{i=1}^{n} n_i p_i$, we have to normalize the cost function so the total transaction cost is

$$\tilde{t}(\hat{x} - x) = \frac{\sum_{i=1}^{n} t_i(|\hat{n}_i - n_i|)}{y}.$$

As usual, we have that the weight of the portfolio represent the relative value of each asset, i.e. $x_i = n_i p_i / y$.

Therefore the VaR optimal portfolio with transaction costs problem is

$$\begin{aligned}
\min_x \; & -F(m - \lfloor \alpha m \rfloor - 1, x) \\
\text{s.t.} \; & x^T \bar{\xi} - \tilde{t}(\hat{x} - x) \geq \mu \\
& \sum_{i=1}^{n} x_i = 1 \\
& x \geq 0.
\end{aligned} \tag{6}$$

One can easily notice that $\tilde{t}(\hat{x}-x)$ is a nonsmooth function as well. However that did not cause any problems for the numerical procedure as will be shown by results presented in the last Section.

The model (6) does not necessarily assume the self-financing portfolio as the problem of costs is equally important for portfolio optimization problems if one is increasing/decreasing the capital or maintaining the same capital. The self-financing condition can be expressed as

$$\sum_{i=1}^{n} \hat{n}_i p_i = (1+\tilde{t}(\hat{x}-x)) \sum_{i=1}^{n} n_i p_i.$$

This condition can replace the budget equation in (6). But the results would be slightly different than the ones presented here as the new budget equation would imply another feasible set. The procedure presented in the next Section is applicable to the self-financing portfolio problem as well.

3 SVaR optimal portfolio

The historical VaR function in general posses many local minima and maxima. According to [12] the number of local minima grows with the number of observations. Moreover the VaR function is nondifferentiable in every local minimizer. Therefore application of standard optimization tools is not a good option. On the other hand it was observed that VaR is composed of two components - the first one with global pattern, a unique global minimum and smooth behavior. The second component has a highly irregular pattern which produces all local minima and nondifferentiablity. In order to smooth out the local noisy component of VaR the smoothed VaR, SVaR, function is proposed and analyzed in [12]. VaR function defined by the portfolio weights x, the observations of return $\xi^1, \ldots, \xi^m$ and the probability level α is approximated by the family of smoothed quantile functions $F_\varepsilon(k,x)$ parameterized by smoothing parameter $\varepsilon > 0$:

$$F_\varepsilon(k,x) = \sum_{i=1}^{m} c_i^\varepsilon(x) f_i(x).$$

where $f_i(x) = x^T \xi^i$, $i = 1, 2, \ldots, m$ and c_i^ε are smoothing parameters determined by ε. The smoothing parameters can be determined in many ways but one particularly efficient way in terms of computational costs is described in [12] and we adopted that procedure in our experiments.

Let the $F_\varepsilon(k,x)$ with $\varepsilon > 0$ be the smoothed function for $F(k,x)$. It is shown in [12] that for any fixed x

$$\lim_{\varepsilon \to 0} F_\varepsilon(k,x) = F(k,x). \tag{7}$$

Using SVaR function and taking into account the transaction costs described above we obtain the following SVaR portfolio optimization problem.

$$
\begin{aligned}
\min_x \; & -F_\varepsilon(m - \lfloor \alpha m \rfloor - 1, x) \\
\text{s.t. } & x^T \bar{\xi} - \tilde{t}(\hat{x} - x) \geq \mu \\
& \sum_{i=1}^{n} x_i = 1 \\
& x \geq 0.
\end{aligned} \tag{8}
$$

This problem is solvable by standard optimization tools since the objective function is smooth enough. The first constraint is not smooth if any component of $\hat{x} - x$ is zero but that did not cause difficulties in numerical procedure so we did not apply any additional smoothing to the constraints. The solution of this problem is called *SVaR optimal portfolio.* However we are still interested in VaR optimal portfolio. One kind of postprocessing is suggested in [12], starting from SVaR minimizer a local minimization is performed taking VaR as the objective function.

We will pursue a different approach here. The standard idea in noonsmooth optimization is to considering a sequence of smoothed problem with a decreasing smoothing parameters. We will state the theoretical properties of such sequence for this particular case and then apply the idea in order to get the VaR optimal portfolio. Such procedure, in our experience, gives good results in a couple of smoothing steps only. A similar approach is adopted in [21] for different smoothing procedure. Also, it seems interesting from theoretical point of view to understand the relationship between VaR and SVaR optimal portfolios when $\varepsilon \to 0$.

Theorem 1 *Let $\{\varepsilon_\nu\}_{\nu \in \mathbf{N}}$ be a decreasing positive sequence such that*

$$\lim_{\nu \to \infty} \varepsilon_\nu = 0.$$

Let x^{VaR} be a global solution of VaR optimization problem (6) and x^ν a global solution of SVaR optimization problem (8) with parameter ε_ν. Then

$$\lim_{\nu \to \infty} F_{\varepsilon_\nu}(k, x^\nu) = F(k, x^{VaR}), \; k = m - \lfloor \alpha m \rfloor - 1.$$

Proof.

Let us suppose the contrary, that there exists $\delta > 0$ such that for any $\bar{\nu}$ large enough we have

$$|F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) - F(k, x^{VaR})| \geq \delta. \tag{9}$$

As (7) holds for any fixed x, we have

$$\lim_{\varepsilon_\nu \to 0} F_{\varepsilon_\nu}(k, x^{VaR}) = F(k, x^{VaR})$$

and

$$\lim_{\varepsilon_\nu \to 0} F_{\varepsilon_\nu}(k, x^{\bar{\nu}}) = F(k, x^{\bar{\nu}}).$$

So for every ν large enough

$$|F_{\varepsilon_\nu}(k, x^{VaR}) - F(k, x^{VaR})| < \delta \tag{10}$$

and

$$|F_{\varepsilon_\nu}(k, x^{\bar{\nu}}) - F(k, x^{\bar{\nu}})| < \delta \ . \tag{11}$$

Clearly (9) can be separated into the following two cases:

1^o $F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) - F(k, x^{VaR}) \geq \delta$
Since x^{VaR} is a global solution of VaR optimization problem then

$$F(k, x^{VaR}) - F(k, x^{\bar{\nu}}) \geq 0.$$

Using inequality (11) we obtain the following contradiction

$$\delta > F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) - F(k, x^{\bar{\nu}}) =$$

$$= F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) - F(k, x^{VaR}) + F(k, x^{VaR}) - F(k, x^{\bar{\nu}}) \geq$$
$$\geq F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) - F(k, x^{VaR}) \geq \delta,$$

2^o $F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) - F(k, x^{VaR}) \leq -\delta$
Since $x^{\bar{\nu}}$ is a global solution of SVaR optimization problem with parameter $\varepsilon_{\bar{\nu}}$ then

$$F_{\varepsilon_{\bar{\nu}}}(k, x^{VaR}) - F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) \leq 0.$$

Inequality (10) now implies

$$-\delta < F_{\varepsilon_{\bar{\nu}}}(k, x^{VaR}) - F(k, x^{VaR}) =$$

$$= F_{\varepsilon_{\bar{\nu}}}(k, x^{VaR}) - F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) + F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) - F(k, x^{VaR}) \leq$$
$$\leq F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) - F(k, x^{VaR}) \leq -\delta.$$

and therefore we can conclude that

$$|F_{\varepsilon_\nu}(k, x^{\nu}) - F(k, x^{VaR})| < \delta$$

holds for any $\delta > 0$ for ν large enough and the statement is proved.

The following theorem states that if the sequence of SVaR optimal portfolios converges when smoothing parameters tend to zero, then the sequence of optimal SVaR values tends to the value of VaR function.

Theorem 2 *Let* $\{\varepsilon_\nu\}_{\nu\in\mathbf{N}}$ *be a decreasing positive sequence*

$$\lim_{\nu\to\infty} \varepsilon_\nu = 0.$$

Let x^ν *be a global solution of SVaR optimization problem (8) with parameter* ε_ν*. If*

$$\lim_{\nu\to\infty} x^\nu = x^*,$$

then

$$\lim_{\nu\to\infty} F_{\varepsilon_\nu}(k, x^\nu) = F(k, x^*),\ k = m - \lfloor \alpha m \rfloor - 1.$$

Proof. Suppose by contradiction that there exists $\delta > 0$ such that for every $\nu_0 \in \mathbf{N}$ exists $\bar{\nu} \geq \nu_0$ for which

$$|F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) - F(k, x^*)| \geq \delta \tag{12}$$

holds. According to (7) for fixed $x^{\bar{\nu}}$

$$\lim_{\varepsilon_\nu\to 0} F_{\varepsilon_\nu}(k, x^{\bar{\nu}}) = F(k, x^{\bar{\nu}}).$$

This limit implies that for ν large enough

$$|F_{\varepsilon_\nu}(k, x^{\bar{\nu}}) - F(k, x^{\bar{\nu}})| < \frac{\delta}{3}. \tag{13}$$

From inequality (13) for $x^{\bar{\nu}}$ and continuousness of function F it follows that

$$-\frac{\delta}{3} < F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) - F(k, x^{\bar{\nu}}) < \frac{\delta}{3}$$

and

$$-\frac{\delta}{3} < F(k, x^{\bar{\nu}}) - F(k, x^*) < \frac{\delta}{3},$$

for $\bar{\nu}$ large enough. Consequently

$$\frac{-2\delta}{3} < F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) - F(k, x^*) \leq \frac{2\delta}{3},$$

i.e.

$$|F_{\varepsilon_{\bar{\nu}}}(k, x^{\bar{\nu}}) - F(k, x^*)| \leq \frac{2\delta}{3},$$

what is a contradiction with (12).

Therefore for every $\delta > 0$ and ν large enough

$$|F_{\varepsilon_\nu}(k, x^\nu) - F(k, x^*)| < \delta,$$

what proves the statement.

These two theorems allow us to formulate the following Corollary which serves as a base for the iterative application of SVaR procedure presented below as Algorithm SVaR.

Corollary 1 *Let* $\{\varepsilon_\nu\}_{\nu\in\mathbf{N}}$ *be a positive decreasing sequence such that*

$$\lim_{\nu\to\infty} \varepsilon_\nu = 0.$$

Let x^{VaR} *be a global solution of the VaR optimization problem (6), and* x^ν *a global solution of the SVaR optimization problem (8) with parameter* ε_ν*. If*

$$\lim_{\nu\to\infty} x^\nu = x^*,$$

then

$$F(k, x^*) = F(k, x^{VaR}),\ k = m - \lfloor \alpha m \rfloor - 1.$$

Consequently, the VaR optimal portfolio can be found by solving a sequence of SVaR optimization problems (8) with the sequence of smoothing parameters that tends to zero. Below we present the general algorithm used in Section 4. Let Ω be the feasible set of (8).

Algorithm SVaR

Select the initial point $x^0 \in \Omega$, smoothing parameter $\varepsilon > 0$, decrease factor $\rho \in (0,1)$ and tolerance tol. Let $k = 0$.

Step 1 Determine $F_\varepsilon(m - \lfloor \alpha m \rfloor - 1, x)$.

Step 2 Find x^{k+1} such that

$$x^{k+1} = arg\min_x -F_\varepsilon(m - \lfloor \alpha m \rfloor - 1, x)$$

$$\text{s.t. } x^T\bar{\xi} - \tilde{t}(\hat{x} - x) \geq \mu, \quad \sum_{i=1}^{n} x_i = 1, \quad x \geq 0.$$

Step 3 If $||x^{k+1} - x^k|| \leq tol$ stop and take $x^* = x^{k+1}$.

Step 4 Set $k = k + 1$, $\varepsilon = \rho\varepsilon$, and go to step 1.

4 Numerical results

In this section we present the numerical examples obtained by SVaR algorithm. Two toy portfolios were constructed with the purpose of demonstrating the influence of transaction costs and emphasizing the difference between costs for small and large investor. No additional considerations like sector constraints, company size, mutual correlation of portfolio components nor any kind of investment analysis are taken into account with these portfolios. We used 7 stocks from London Stock Exchange (LSE), Astrazeneca (AZN), Bae Systems (BA), Lonmin (LMI), Rio Tinto (RIO), Smith & Nephew (SN), Tesco (TSCO) and Vodafone (VOD). All data are taken from www.advfn.com and www.finance.yahoo.com sites. VaR is computed on the basis of 500 daily returns from February 13, 2006 to February 12, 2008 with confidence level 0.95. We considered four types of problems:

P1 - there are no transaction costs,

P2 - the models of permanent and temporary impact are linear,

P3 - the model of permanent impact is linear and the model of temporary impact is nonlinear,

P4 - the model of permanent impact is nonlinear and the model of temporary impact is linear.

All cases are observed for a large institutional investor and a small investor. The value of portfolio for the large investor was hundred million pounds and in the case of small investor it was hundred thousand pounds. For the initial portfolio we used equal weights, i.e.

$$\hat{x} = \left(\frac{1}{7}, \ldots, \frac{1}{7}\right).$$

Initial smoothing parameter was set to $\varepsilon = 10^{-3}$. Step 2 of the algorithm was performed using Matlab built-in function *fmincon*. In the vast majority of cases the problem of SVaR optimal portfolio from Step 2 was successfully solved. The smoothing parameter reduction was done by $\rho = 0.25$, while $tol = 10^{-5}$ was taken.

Figures 1a and 1b show the efficient frontiers for large and small institutional investors, respectively. The horizontal axis at these figures shows the VaR values expressed in percents relative to the portfolio value. The vertical axis shows the minimal expected return expressed in basis points. The black curve depicts the results without transaction costs. The red curve shows the results for P2. The efficient frontier for P3 is presented by the green curve and the blue curve is used for the results for P4. As we can see, the transaction costs decrease the value of VaR i.e. increase the lowest bound for portfolio return underperformance. Notice that the values of optimal VaR with transaction costs for the large investor are between 1.6% and 1.85% of portfolio value, but for the small investor they are between 1.7% and 1.85%. That means that the values of optimal VaR are larger for smaller portfolio, that is the VaR optimal portfolio is more risky for small investors than for large investors. One can also observe at Figure 1b that the choice of impact model has no influence for small investors, but that is not the case for large investors. Hence the fixed costs are indeed dominant for small portfolio while the impact costs prevail in the case of large investor.

Remaining figures show the VaR optimal portfolio for different minimal expected returns μ. The horizontal axis on these figures shows the minimal expected return expressed in basis points. The vertical axis shows the weights of all assets relative to the portfolio value. Figure 2 gives results for the problems without transaction costs. These results are the same for both large and small investors. The VaR optimal portfolios for large investor with transaction costs where the model of market impact is linear are presented at figure 3a. The results for linear permanent impact and nonlinear temporary impact and the results for nonlinear permanent impact and linear temporary impact are given at figures 4a and 5a, respectively. Figures 3b, 4b and 5b show the VaR optimal portfolios for small investor with transaction costs using the linear permanent and temporary impact model, linear permanent and nonlinear temporary impact model and nonlinear permanent and linear temporary impact model, respectively. We can see that the transaction costs significantly influence VaR optimal portfolio and that for higher values of μ the difference between the initial portfolio and VaR optimal portfolio decreases. Occasional jumps that are obvious are due to the termination of the algorithm at local minima.

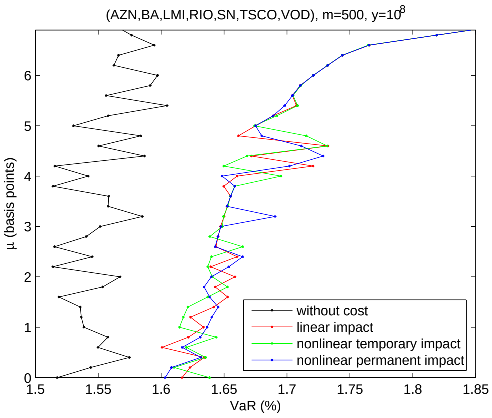

Figure 1a: P1 Efficient frontier - large investor

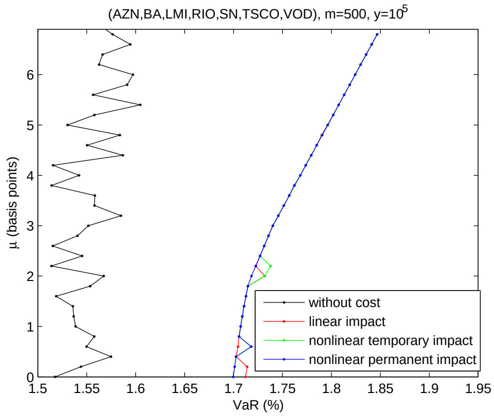

Figure 1b: P2 Efficient frontier - small investor

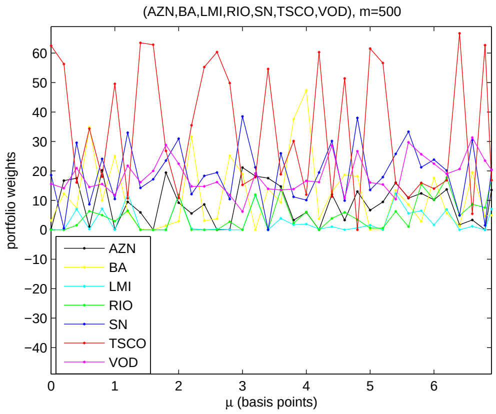

Figure 2: P1 VaR optimal portfolio weights

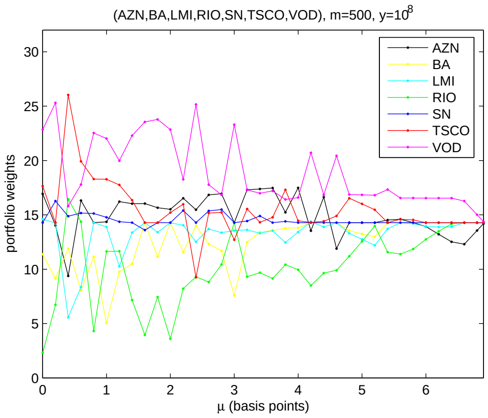

Figure 3a: P2 VaR optimal portfolio weights - large investor

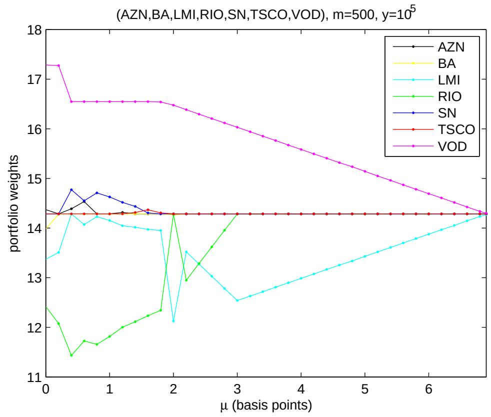

Figure 3b: P2 VaR optimal portfolio weights - small investor

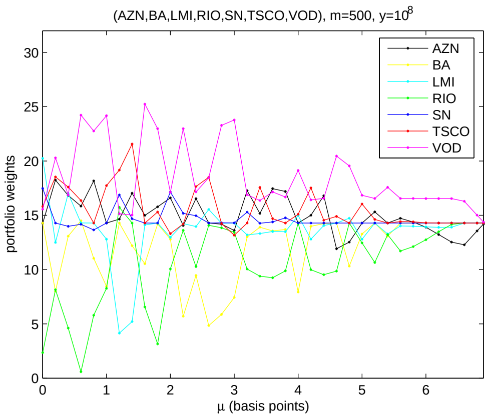

Figure 4a: P3 VaR optimal portfolio weights - large investor

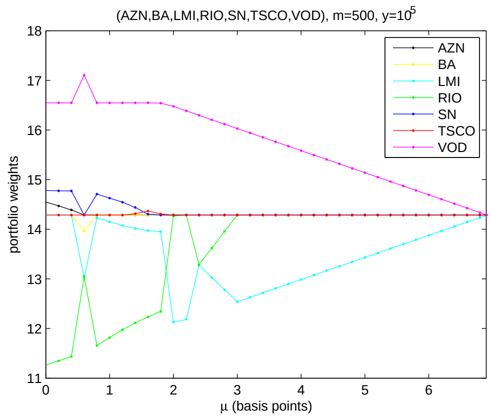

Figure 4b: P3 VaR optimal portfolio weights - small investor

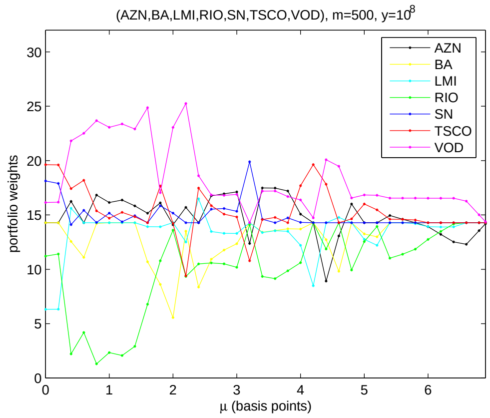

Figure 5a: P4 VaR optimal portfolio weights - large investor

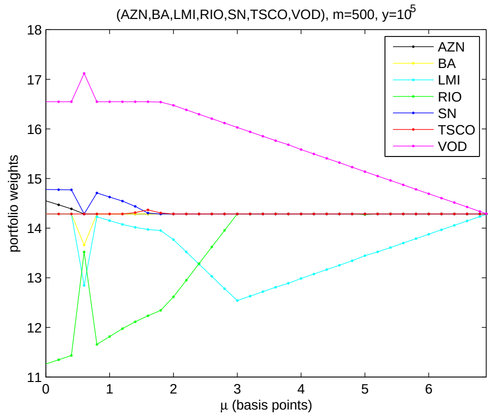

Figure 5b: P4 VaR optimal portfolio weights - small investor

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