Some Applications of Higher Commutators to Mal'cev Algebras

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Novi Sad, August 2008

Some Notions

Mal'cev algebras: **A** has a Mal'cev term m

$$m(x, y, y) = x$$

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for all $x, y \in A$.

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Expanded groups: $\mathbf{V} = \langle V, +, f_1, \ldots, f_n \rangle$
$f_1, \ldots, f_n$ are operations on A and $+$ is a group operation

Centralizers

Definition. (Hobby,McKenzie $C(\alpha_1, \alpha_2; \eta)$) Let $\mathbf{A}$ be an algebra, $\alpha_1, \alpha_2, \eta \in \operatorname{Con} \mathbf{A}$. Then we say that α_1 centralizes α_2 modulo η if for all polynomials $f(\mathbf{x}_1, \mathbf{x}_2)$ and vectors $\mathbf{a}_1, \mathbf{b}_1, \mathbf{u}, \mathbf{v}$ from $\mathbf{A}$ satisfying $\mathbf{a}_1 \equiv \mathbf{b}_1 \pmod{\alpha_1}$, $\mathbf{u} \equiv \mathbf{v} \pmod{\alpha_2}$ and

$$f(\mathbf{a}_1, \mathbf{u}) \equiv f(\mathbf{a}_1, \mathbf{v}) \pmod{\eta},$$

we have

$$f(\mathbf{b}_1, \mathbf{u}) \equiv f(\mathbf{b}_1, \mathbf{v}) \pmod{\eta}.$$

Comutators and Nilpotent Property

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Definition. (Hobby, McKenzie) Let **A** be an algebra from a congruence modular variety. **A** is nilpotent (of class n, $n \in \mathbb{N}$) if

$$\underbrace{[1 \ldots [1,}_{n} 1]] = 0$$

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Theorem. (Hunt, Stearns 1990, Burris, Lawrence 1993)
For a finite nilpotent ring, term equivalence problem can be decided in polynomial time.

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Is there a wider class of algebras where affine completeness is a decidable property?

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Theorem. (A. Bulatov) There are countably many clones on $\mathbb{Z}_p \times \mathbb{Z}_p$ that contain $f(x, y, z) = x - y + z$ and all constant operations.

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Is there a finite set A such that there are uncountably many clones on A that contain a Mal'cev operation?

Given a finite algebra **A** with a Mal'cev operation, is there an $n \in \mathbb{N}$ such that the following is true: if a function f preserves all n-ary relations that are invariant under all polynomial functions, then f is a polynomial function?

Higher Centralizers

Definition. (Bulatov $C(\alpha_1, \dots, \alpha_n; \eta)$) Let $\mathbf{A}$ be an algebra, $\alpha_1, \dots, \alpha_n, \eta \in \operatorname{Con} \mathbf{A}$. Then we say that $\alpha_1, \dots, \alpha_{n-1}$ centralize α_n modulo η if for all polynomials $f(\mathbf{x}_1, \dots, \mathbf{x}_n)$ and vectors $\mathbf{a}_1, \dots, \mathbf{a}_{n-1}, \mathbf{b}_1 \dots, \mathbf{b}_{n-1}, \mathbf{u}, \mathbf{v}$ from $\mathbf{A}$ satisfying $\mathbf{a}_i \equiv \mathbf{b}_i \pmod{\alpha_i}$, $1 \leq i \leq n$, $\mathbf{u} \equiv \mathbf{v} \pmod{\alpha_n}$ and

$$f(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}, \mathbf{u}) \equiv f(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}, \mathbf{v}) \pmod{\eta},$$

for all $(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) \in \{\mathbf{a}_1, \mathbf{b}_1\} \times \dots \times \{\mathbf{a}_{n-1}, \mathbf{b}_{n-1}\}$ and $(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) \neq (\mathbf{b}_1, \dots, \mathbf{b}_{n-1})$, we have

$$f(\mathbf{b}_1, \dots, \mathbf{b}_{n-1}, \mathbf{u}) \equiv f(\mathbf{b}_1, \dots, \mathbf{b}_{n-1}, \mathbf{v}) \pmod{\eta}.$$

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Example:

$$[1_V, [1_V, 1_V]] \neq [1_V, 1_V, 1_V] \text{ for } \mathbf{V} = \langle \mathbb{Z}_4, +, 2xyz \rangle$$

Bulatov's Properties

Proposition. $\mathbf{A}$ an arbitrary algebra and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathrm{Con}\,\mathbf{A}$

- $[\alpha_1, \ldots, \alpha_n] \leq \bigwedge_{i=1}^{n} \alpha_i$

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Claim. If **A** is in a congruence modular variety and π is any permutation of $\{1, \ldots, n\}$ then

$$[\alpha_1, \ldots, \alpha_n] = [\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)}].$$

Some Properties of Higher Commutators in Mal'cev Algebras

Proposition.

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- $\bigvee_{i \in I} [\alpha_0, \dots, \alpha_{j-1}, \rho_i, \alpha_{j+1}, \dots, \alpha_k] = [\alpha_0, \dots, \alpha_{j-1}, \bigvee_{i \in I} \rho_i, \alpha_{j+1}, \dots, \alpha_k]$.

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- $[\alpha_0, \dots, \alpha_j, [\alpha_{j+1}, \dots, \alpha_k]] \leq [\alpha_0, \alpha_1, \dots, \alpha_k]$.

Supernilpotent Algebras

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Abelian Algebras $\subseteq$ Supernilpotent Algebras $\subseteq$ Nilpotent Algebras

How Does the Supernilpotency Help?

Proposition. Let $\mathbf{A}$ be a finite nilpotent algebra of finite type that generates a congruence modular variety. If $\mathbf{A}$ factors as a direct product of algebras of prime power cardinality then $\mathbf{A}$ is a supernilpotent Mal'cev algebra.

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Proposition. Let **A** be an n-supernilpotent Mal'cev algebra. Then the polynomial clone of **A** is generated by all polynomials of arity at most $n-1$ and the Mal'cev term.

The Polynomial Equivalence Problem

Theorem. The polynomial equivalence problem for a finite nilpotent algebra **A** of finite type that is a product of algebras of prime power order and generates a congruence modular variety has polynomial time complexity in the length of the input terms.

Affine Completeness

Theorem. There is an algorithm that decides whether a finite nilpotent algebra of finite type that is a product of algebras of prime power order and generates a congruence modular variety is affine complete.

Mal'cev Clones

Theorem. Let **A** be a finite Mal'cev algebra with congruence lattice of height two. Then there is an $n \in \mathbb{N}$ such that: if a function f preserves all n-ary relations that are invariant under all polynomial functions, then f is a polynomial function.