On $d$-digit palindromes in different bases: The number of bases is unbounded

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Abstract

The following problem was posed in [E. H. Goins, Palindromes in Different Bases: A Conjecture of J. Ernest Wilkins, Integers 9 (2009), 725–734]: “What is the largest list of bases $b$ for which an integer $N \geq 10$ is a $d$-digit palindrome base $b$ for every base in the list?” We show that it is possible to construct such a list as large as we please. Furthermore, we show that it is possible to construct such arbitrarily large list for any given $d$.

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1 Introduction

We call a number a palindrome in base $b$ if for its expansion in base $b$, say $\langle c_{d-1}, c_{d-2}, \ldots, c_0 \rangle_b$ ($c_{d-1} \neq 0$), it holds that $c_j = c_{d-1-j}$ for every $0 \leq j \leq d - 1$. Among the properties of palindromes in different bases studied so far there are: compositeness and prime factors [1, 2], being a perfect power [7, 3], distribution of palindromes in arithmetic progressions [4], whether palindromes always appear in “reverse-then-add” sequences [6, 9, 8] (in base 10, it is a long-standing open problem whether a palindrome appears in “reverse-then-add” sequence starting with number 196), etc. In some of these works, e.g. [2] and [3], the results are obtained with respect to the number of digits of the considered palindromes.
Motivated by a question from Wilkins, Goins [5] proved that there are exactly 203 positive integers which are $d$-digit palindrome in base 10 and $d$-digit palindrome in another base (where $d$ is fixed, and $d \geq 2$ to avoid trivial cases), ranging from 22 to 9986831781362631871386899 ($d = 2$ to $d = 25$). He noted that a few of these palindromes are $d$-digit palindromes in as much as four different bases: $\langle 6,6 \rangle_{10} = \langle 3,3 \rangle_{21} = \langle 2,2 \rangle_{32} = \langle 1,1 \rangle_{65}$; $\langle 8,8 \rangle_{10} = \langle 4,4 \rangle_{21} = \langle 2,2 \rangle_{43} = \langle 1,1 \rangle_{87}$; $\langle 6,7,6 \rangle_{10} = \langle 5,6,5 \rangle_{11} = \langle 4,8,4 \rangle_{12} = \langle 1,2,1 \rangle_{25}$; $\langle 9,8,9 \rangle_{10} = \langle 3,7,3 \rangle_{17} = \langle 2,5,2 \rangle_{21} = \langle 1,12,1 \rangle_{26}$. This led him to ask whether it is possible find an even larger list of bases (clearly, base 10 will not be among them) such that there is a number which is $d$-digit palindrome simultaneously in all those bases; if possible, then what is the largest such list.

In this paper we prove that the largest such list does not exist, that is, it is possible to construct such a list as large as we please. Furthermore, we show that it is possible to construct such arbitrarily large list for any given $d$. Namely, our main results are the following two theorems.

**Theorem 1.1.** Given any $K \in \mathbb{N}$, there exists $d \geq 2$ and $n \in \mathbb{N}$ and a list of bases $\{b_1, b_2, \ldots, b_K\}$ such that, for each $1 \leq i \leq K$, $n$ is a $d$-digit palindrome in base $b_i$.

**Theorem 1.2.** Given any $K \in \mathbb{N}$ and $d \geq 2$, there exists $n \in \mathbb{N}$ and a list of bases $\{b_1, b_2, \ldots, b_K\}$ such that, for each $1 \leq i \leq K$, $n$ is a $d$-digit palindrome in base $b_i$.

We prove these theorems in the following section.

## 2 Proofs of the theorems

Clearly, Theorem 1.2 is a generalization of Theorem 1.1. Nevertheless, we prove them independently, since the proof of Theorem 1.1 nicely serve as a motivation for the proof of Theorem 1.2.

**Proof of Theorem 1.1.** Let $K \in \mathbb{N}$. Choose any $n \in \mathbb{N}$ such that $\tau(n) \geq 2K + 1$, where $\tau(n)$ denotes the number of divisors of $n$. Let $1 = a_1 < a_2 < \cdots < a_K < a_{K+1}$ be the smallest $K + 1$ divisors of $n$. Notice that $a_{K+1} \leq \lfloor \sqrt{n} \rfloor$, and therefore $a_i \leq \lfloor \sqrt{n} \rfloor - 1$ for each $1 \leq i \leq K$. Denote $b_i = \frac{n}{a_i} - 1$. We claim that $n = \langle a_i, a_i \rangle_{b_i}$ for each $1 \leq i \leq K$, that is: for each $1 \leq i \leq K$, $n$ is a 2-digit palindrome in base $b_i$. And indeed, since we have
\[ a_i, a_i \rangle_{b_i} = a_i + a_i b_i = a_i (b_i + 1) = n, \] it is enough to check whether \( b_i > a_i \).

It holds
\[ b_i = \frac{n}{a_i} - 1 \geq \frac{n}{\sqrt{n}} - 1 \geq (\lfloor \sqrt{n} \rfloor + 1) - 1 = \lfloor \sqrt{n} \rfloor > a_i, \]
and the proof is completed.

**Proof of Theorem 1.2.** Let \( K \in \mathbb{N} \) and \( d \geq 2 \). Choose any \( m \in \mathbb{N} \) such that \( \tau(m) \geq 2K + 1 \). Let \( 1 = a'_1 < a'_2 < \cdots < a'_K \) be the smallest \( K \) divisors of \( m \), and let \( a_i = (a'_i)^{d-1} \). As in the previous proof, we have \( a'_i \leq \lfloor \sqrt{m} \rfloor - 1 \), and it follows that \( a_i \leq \lfloor \sqrt{m} \rfloor^{d-1} - 1 \). Let
\[ n = \left( \frac{d-1}{\lfloor \frac{d-1}{2} \rfloor} \right)^{d-1} m^{(d-1)^2}. \]

Define
\[ b_i = \sqrt[d-1]{\frac{n}{a_i}} - 1 = \sqrt[d-1]{\frac{(d-1)^{d-1} m^{(d-1)^2}}{(a'_i)^{d-1}}} - 1 = \left( \frac{d-1}{\lfloor \frac{d-1}{2} \rfloor} \right) \frac{m^{d-1}}{a'_i} - 1. \]

We claim that
\[ n = \left\langle \left( \frac{d-1}{d-1} \right) a_i, \left( \frac{d-1}{d-2} \right) a_i, \left( \frac{d-1}{d-3} \right) a_i, \ldots, \left( \frac{d-1}{1} \right) a_i, \left( \frac{d-1}{0} \right) a_i \right\rangle_{b_i} \quad (1) \]
for each \( 1 \leq i \leq K \), that is: for each \( 1 \leq i \leq K \), \( n \) is a \( d \)-digit palindrome in base \( b_i \) (because \( \binom{d-1}{j} = \binom{d-1}{d-1-j} \) for each \( 0 \leq j \leq d-1 \)). And indeed, since we have
\[ \left\langle \left( \frac{d-1}{d-1} \right) a_i, \left( \frac{d-1}{d-2} \right) a_i, \left( \frac{d-1}{d-3} \right) a_i, \ldots, \left( \frac{d-1}{1} \right) a_i, \left( \frac{d-1}{0} \right) a_i \right\rangle_{b_i} = \sum_{j=0}^{d-1} \binom{d-1}{j} a_i b'_i = a_i \sum_{j=0}^{d-1} \binom{d-1}{j} b'_i = a_i (b_i + 1)^{d-1} = n, \]

it is enough to check whether \( b_i > \left( \frac{d-1}{\lfloor \frac{d-1}{2} \rfloor} \right) a_i \) (since the right-hand side is the
largest digit in (1)). It holds

\[ b_i = \left( \frac{d-1}{\left\lfloor \frac{d-1}{2} \right\rfloor} \right) \frac{m^{d-1}}{a'_i} - 1 \geq \left( \frac{d-1}{\left\lfloor \frac{d-1}{2} \right\rfloor} \right) \frac{m^{d-1}}{\sqrt{m}} - 1 \]

\[ \geq \left( \frac{d-1}{\left\lfloor \frac{d-1}{2} \right\rfloor} \right) \frac{m^{d-1}}{\sqrt{m}} - 1 \geq \left( \frac{d-1}{\left\lfloor \frac{d-1}{2} \right\rfloor} \right) \left( \frac{m^{d-1}}{\sqrt{m}} + 1 \right) - 1 \]

\[ > \left( \frac{d-1}{\left\lfloor \frac{d-1}{2} \right\rfloor} \right) (a_i + 1) - 1 = \left( \frac{d-1}{\left\lfloor \frac{d-1}{2} \right\rfloor} \right) a_i + \left( \frac{d-1}{\left\lfloor \frac{d-1}{2} \right\rfloor} \right) - 1 \geq \left( \frac{d-1}{\left\lfloor \frac{d-1}{2} \right\rfloor} \right) a_i, \]

and the proof is completed.

**3 Future directions**

It would be interesting to see which palindromic sequences \( \langle c_{d-1}, c_{d-2}, \ldots, c_0 \rangle \), \( c_{d-1} \neq 0 \), have the property that for any \( K \in \mathbb{N} \) there exists \( n \in \mathbb{N} \) and a list of bases \( \{b_1, b_2, \ldots, b_K\} \) such that, for each \( 1 \leq i \leq K \), \( n \) is a \( d \)-digit palindrome in base \( b_i \), and that \( n = \langle c_{d-1}, c_{d-2}, \ldots, c_0 \rangle_{b_{i_0}} \) for some \( 1 \leq i_0 \leq K \). From our result it follows that all the sequences \( \langle (d-1), (d-1), (d-1), \ldots, (d-1), (d-1) \rangle \) where \( d \geq 2 \), as well as their multiples by a factor of form \( t^{d-1} \), have such property. Do sequences \( \langle 1, 1, 1 \rangle \) and \( \langle 1, 0, 1 \rangle \) and, more generally, \( \langle 1, 1, \ldots, 1 \rangle \) and \( \langle 1, 0, 0, \ldots, 0, 1 \rangle \) have this property? Is it perhaps true that all palindromic sequences have this property? If not, could the ones having this property be characterized? These questions are open for future research.

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**References**


