The existence of n-flimsy numbers in a given base

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Abstract

Let the function s_g map a positive integer to the sum of its digits in the base g. A number k is called *n-flimsy in the base g* if $s_g(nk) < s_g(k)$. Clearly, given a base g, $g \geqslant 2$, if n is a power of g, then there does not exist an n-flimsy number in the base g. We give a constructive proof of the existence of an n-flimsy number in the base g for all the other values of n (such an existence follows from the results of Schmidt and Steiner, but the explicit construction is a novelty). Our algorithm for construction of such a number, say k, is very flexible in the sense that, by easy modifications, we can impose further requirements on k such as, for example, that k ends with a given sequence of digits, k begins with a given sequence of digits, k is divisible by a given number (or belongs to a certain congruence class modulo a given number) etc.

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1 Introduction

Research on the sum of digits of particular multiples of positive integers often leads to interesting results. See, for example, [7] (where an unexpected phenomenon, now called *Newman phenomenon*, was discovered), [4, 2] (some further investigation of the Newman phenomenon), [3, 1] (some results on the correlation between the sum of digits of a given number and the sum of digits of its multiples), [5, 6] (some results on the distribution of numbers that have a given remainder when divided by a given modulus such that their sum of digits also has a given remainder when divided by a given modulus), etc.

Roots of the present work can be found in a paper by Stolarsky [11], who called a number k *n-flimsy* if the sum of binary digits of the number nk is smaller than the sum of binary digits of the number k. A number k is called *flimsy* if it is n-flimsy for some $n \in \mathbb{N}$ (here and onward, by $\mathbb{N}$ we denote the set of positive integers, while the set of nonnegative integers is denoted by $\mathbb{N}_0$). The sequence of flimsy numbers is the sequence A005360 in [8].

Although the Stolarsky's definition considers only binary notation, there is a natural way to extend this definition to any base. For a given base $g \geqslant 2$, let the function s_g map a positive integer to the sum of its digits in the base g. We say that a number k is *n-flimsy in the base g* if $s_g(nk) < s_g(k)$. Given a base g and an integer $n \in \mathbb{N}$, if n is a power of g, then there obviously does not exist an n-flimsy integer in the base g (because in that case the equality $s_g(nk) = s_g(k)$ holds, since the only difference between the numbers nk and k is a string of zeros at the end of nk). The results of Schmidt [9] (for the case $g = 2$) and Steiner [10] (for the general case $g \geqslant 2$) imply the existence of n-flimsy numbers in the base g for the other values of n. Although these results even give some informations about the density of such numbers, their drawback is the fact that they are not constructive, that is, given a base g and an integer n, it is not clear how to find an n-flimsy number in the base g.

In this paper we give a constructive proof of the following result.

Theorem 1.1. *For every base $g \geqslant 2$ and every $n \in \mathbb{N}$ such that $\log_g n \notin \mathbb{N}_0$, there exists $k \in \mathbb{N}$ such that $s_g(nk) < s_g(k)$.*

Given g and n, we present an algorithm to construct such a number k. The presented algorithm is very flexible in the sense that, by easy modifications, we can impose further requirements on k such as, for example, that k ends with a given sequence of digits, k begins with a given sequence of digits, k is divisible by a given number (or belongs to a certain congruence class modulo a given number) etc.

2 Main section

Before we begin the proof of Theorem 1.1, we prove some necessary lemmas. From this point onward, we assume that a base g is given, $g \geqslant 2$, and n is a given number that is not a power of g. Whenever we speak about the digits of a number, we mean the digits in the base g.

Lemma 2.1. *For every a and j, where $2 \leqslant a \leqslant g-1$ and $j \in \mathbb{N}_0$, there exists $l \in \{g-1, g-2\}$ such that $al + j$ ends with a digit less than l.*

Proof. If $g-1$ is a suitable value for l, the proof is completed. Assume, therefore, that $g-1$ is not a suitable value of l, that is, $a(g-1)+j$ ends with $g-1$.

We claim that $g-2$ is then a suitable value for l. Indeed, if this were not the case, then either $a(g-2)+j \equiv g-1 \pmod{g}$ or $a(g-2)+j \equiv g-2 \pmod{g}$, which together with $a(g-1)+j \equiv g-1 \pmod{g}$ gives $a \equiv 0 \pmod{g}$, respectively $a \equiv 1 \pmod{g}$; a contradiction in both cases. Therefore, $g-2$ is indeed a suitable value for l. ■

Lemma 2.2. *For every b and j, where $1 \leqslant b \leqslant g-1$ and $j \in \mathbb{N}_0$, there exists $l \in \{g-1, g-2\}$ such that $bl + j$ ends with a digit different from 0.*

Proof. If $b(g-1)+j$ ends with a digit different from 0, we can take $l = g-1$. Otherwise, because of $b(g-2)+j = b(g-1)+j-b$ and $1 \leqslant b \leqslant g-1$, it follows that $b(g-2)+j$ ends with a digit different from 0; therefore, we can take $l = g-2$. ■

We shall also need the following two properties of the floor function.

Lemma 2.3. *For any $m, m' \in \mathbb{N}_0$ and $i \in \mathbb{N}$, if $m \equiv m' \pmod{g^i}$, then $\left\lfloor \frac{m}{g} \right\rfloor \equiv \left\lfloor \frac{m'}{g} \right\rfloor \pmod{g^{i-1}}$.*

Proof. From $m \equiv m' \pmod{g^i}$ we obtain $m \bmod g = m' \bmod g$, and therefore

$$\begin{aligned} m - m' &= \left(g \left\lfloor \frac{m}{g} \right\rfloor + m \bmod g \right) - \left(g \left\lfloor \frac{m'}{g} \right\rfloor + m' \bmod g \right) \\ &= g \left(\left\lfloor \frac{m}{g} \right\rfloor - \left\lfloor \frac{m'}{g} \right\rfloor \right). \end{aligned}$$

Since $g^i \mid m - m'$, we get $g^{i-1} \mid \left\lfloor \frac{m}{g} \right\rfloor - \left\lfloor \frac{m'}{g} \right\rfloor$, which was to be proved. ■

Lemma 2.4. *For any $m, y, y' \in \mathbb{N}$ we have*

$$\left\lfloor \frac{\left\lfloor \frac{m}{y} \right\rfloor}{y'} \right\rfloor = \left\lfloor \frac{m}{yy'} \right\rfloor. \tag{1}$$

Proof. Let $m = wyy' + p$, where $0 \leqslant p < yy'$. Then

$$\left\lfloor \frac{m}{yy'} \right\rfloor = \left\lfloor w + \frac{p}{yy'} \right\rfloor = w + \left\lfloor \frac{p}{yy'} \right\rfloor = w$$

and

$$\left\lfloor \frac{\left\lfloor \frac{m}{y} \right\rfloor}{y'} \right\rfloor = \left\lfloor \frac{wy' + \left\lfloor \frac{p}{y} \right\rfloor}{y'} \right\rfloor = w + \left\lfloor \frac{\left\lfloor \frac{p}{y} \right\rfloor}{y'} \right\rfloor.$$

The assertion now follows because $0 \leqslant \frac{\lfloor \frac{p}{y} \rfloor}{y'} \leqslant \frac{\frac{p}{y}}{y'} = \frac{p}{yy'} < 1$, that is, $\left\lfloor \frac{\lfloor \frac{p}{y} \rfloor}{y'} \right\rfloor = 0$. ■

Proof of Theorem 1.1. Clearly, we may assume $g \nmid n$, that is, n ends with a nonzero digit (because the ending sequence of zeros makes no difference to $s_g(n)$ nor to $s_g(nk)$ for any integer k, and thus we may simply erase such a sequence of zeros from a given n and work with what remains). Let n end with a digit a. Since it will turn out that the proof is much more complicated when $a = 1$ in comparison to any other value of a, we shall distinguish these two cases.

The case $a \neq 1$**.** Let the function $f(a, j)$ map the pair (a, j) (where $j \in \mathbb{N}_0$) to a value $l \in \{g-1, g-2\}$ such that $al + j$ ends with a digit less than l (such a value exists by Lemma 2.1); in case of multiple possibilities for l, choose arbitrarily. Define $d_0 = f(a, 0)$, $k_0 = d_0$, and for $u = 1, 2, \ldots$ recursively define

$$\begin{aligned} d_u &= f\Big(a, \Big\lfloor \frac{nk_{u-1}}{g^u} \Big\rfloor\Big), \\ k_u &= g^u d_u + k_{u-1} = \overline{d_u \ldots d_1 d_0}_{(g)}. \end{aligned}$$

Let $u \in \mathbb{N}_0$ be fixed. For each v, $0 \leqslant v \leqslant u$, the $(v+1)^{\text{st}}$ digit from the right in the product nk_u equals

$$\begin{aligned}
\left\lfloor \frac{nk_u}{g^v} \right\rfloor \bmod g &= \left\lfloor \frac{n(g^v \cdot \overline{d_u d_{u-1} \dots d_v}_{(g)} + \overline{d_{v-1} \dots d_1 d_0}_{(g)})}{g^v} \right\rfloor \bmod g \\
&= \left\lfloor n \cdot \overline{d_u d_{u-1} \dots d_v}_{(g)} + \frac{n \cdot \overline{d_{v-1} \dots d_1 d_0}_{(g)}}{g^v} \right\rfloor \bmod g \\
&= \left(n \cdot \overline{d_u d_{u-1} \dots d_v} + \left\lfloor \frac{nk_{v-1}}{g^v} \right\rfloor \right) \bmod g \\
&= \left(ad_v + \left\lfloor \frac{nk_{v-1}}{g^v} \right\rfloor \right) \bmod g \\
&= \left(a \cdot f\Big(a, \left\lfloor \frac{nk_{v-1}}{g^v} \right\rfloor\Big) + \left\lfloor \frac{nk_{v-1}}{g^v} \right\rfloor \right) \bmod g \\
&\leqslant f\Big(a, \left\lfloor \frac{nk_{v-1}}{g^v} \right\rfloor\Big) - 1 = d_v - 1
\end{aligned}$$

(the inequality at the end follows by the definition of f). Since $nk_u < ng^{u+1}$, we have that nk_u has at most $\lfloor \log_g n \rfloor + u + 2$ digits, and by the previous and this observation it follows

$$\begin{aligned}
s_g(nk_u) &\leqslant (g-1)(\lfloor \log_g n \rfloor + 1) + \sum_{z=0}^{u} (d_z - 1) \\
&= \sum_{z=0}^{u} d_z + (g-1)(\lfloor \log_g n \rfloor + 1) - u - 1 \\
&= s_g(k_u) + (g-1)(\lfloor \log_g n \rfloor + 1) - u - 1.
\end{aligned}$$

Therefore, for a large enough u we have $s_g(nk_u) < s_g(k_u)$, which was to be proved.

The case $a = 1$. Let b be the first nonzero digit from the right after the final 1 (which exists since n is not a power of g), and let t, $t \geqslant 0$, be the number of zeros between the final 1 and b, that is,

$$n = g^{t+2}x + g^{t+1}b + 1.$$

Having Lemma 2.2 in mind, let the function $h(b, j)$ map the pair (b, j) to a value $l \in \{g-1, g-2\}$ such that $bl + j$ ends with a digit different from 0;

in case of multiple possibilities for l, choose arbitrarily. We define two more auxiliary functions, namely

$$\iota(m) = \begin{cases} 0, & \text{if } m \bmod g = 0; \\ 1, & \text{otherwise} \end{cases}$$

and

$$\mu(m) = \left\lfloor \frac{m}{g} \right\rfloor + \iota(m). \tag{2}$$

The required value k shall now be constructed in a way that is somewhat similar to the one seen in the case $a \neq 1$, but more technically challenging. Define $d_0 = h(b, 0)$, $k_0 = d_0$, and for $u = 1, 2, \ldots$ recursively define

$$d_u = \begin{cases} h\Big(b, \mu^{t+1}\big(\lfloor \frac{nk_{u-1}}{g^u} \rfloor\big)\Big), & \text{if } \lfloor \frac{nk_{u-1}}{g^u} \rfloor \bmod g = 0; \\ \\ g - 1, & \text{otherwise}, \end{cases} \tag{3}$$

$$k_u = g^u d_u + k_{u-1} = \overline{d_u \ldots d_1 d_0}_{(g)}.$$

(By μ^{t+1} we mean the function μ iterated $t+1$ times.)

We proceed by proving a few important claims about the behavior of the construction. Claims 2.5 and 2.6 present some technical properties of the function μ. Claim 2.7 provides a tie between the function μ and the digit d_u returned by the procedure at the u^{th} step. Claim 2.8 is an intermediate step toward Claim 2.9, and Claim 2.9 toward Claim 2.10, which then provides a crucial relationship between the numbers k_{u-1} and k_{u+t} (of course, the function μ is of a key importance here). Finally, Claims 2.11 and 2.12 are the final ingredients for this theorem: we first show that the digits of nk_u show some kind of tendency to be smaller than the digits of k_u, and then we use this behavior to show that the sum of digits of nk_u is indeed smaller than the sum of digits of k_u for u large enough.

Claim 2.5. *For any $m, m' \in \mathbb{N}_0$ we have*

$$\mu(gm + m') = m + \mu(m'). \tag{4}$$

Proof. Indeed, doing some simple transformations and having in mind that ι depends only on the remainder modulo g of the argument, we obtain

$$\mu(gm + m') = \left\lfloor \frac{gm + m'}{g} \right\rfloor + \iota(gm + m') = m + \left\lfloor \frac{m'}{g} \right\rfloor + \iota(m') = m + \mu(m'),$$

which proves the claim.

Claim 2.6. *For any $m, m' \in \mathbb{N}_0$ and $i \in \mathbb{N}$, if $m \equiv m' \pmod{g^i}$, then $\mu(m) \equiv \mu(m') \pmod{g^{i-1}}$.*

Proof. If $m \equiv m' \pmod{g^i}$, then $m \equiv m' \pmod{g}$, and thus $\iota(m) = \iota(m')$. Further, from $m \equiv m' \pmod{g^i}$, by Lemma 2.3 we also obtain $\lfloor \frac{m}{g} \rfloor \equiv \lfloor \frac{m'}{g} \rfloor \pmod{g^{i-1}}$. This gives $\mu(m) \equiv \mu(m') \pmod{g^{i-1}}$, which proves the claim.

Claim 2.7. *For each $u \in \mathbb{N}$, we have*

$$\mu\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right) = \left\lfloor \frac{d_u + \lfloor \frac{nk_{u-1}}{g^u} \rfloor}{g} \right\rfloor = \left\lfloor \frac{d_u + \frac{nk_{u-1}}{g^u}}{g} \right\rfloor. \tag{5}$$

Proof. We first note

$$\begin{aligned}\left\lfloor \frac{d_u + \lfloor \frac{nk_{u-1}}{g^u} \rfloor}{g} \right\rfloor &= \left\lfloor \frac{d_u + g\left\lfloor \frac{\lfloor \frac{nk_{u-1}}{g^u} \rfloor}{g} \right\rfloor + \lfloor \frac{nk_{u-1}}{g^u} \rfloor \bmod g}{g} \right\rfloor \\ &= \left\lfloor \frac{\lfloor \frac{nk_{u-1}}{g^u} \rfloor}{g} \right\rfloor + \left\lfloor \frac{d_u + \lfloor \frac{nk_{u-1}}{g^u} \rfloor \bmod g}{g} \right\rfloor.\end{aligned} \tag{6}$$

Now, observe the way d_u is defined. If $\lfloor \frac{nk_{u-1}}{g^u} \rfloor \bmod g = 0$, then whatever d_u is, we have the inequality $d_u + \lfloor \frac{nk_{u-1}}{g^u} \rfloor \bmod g = d_u < g$; on the other hand, if $\lfloor \frac{nk_{u-1}}{g^u} \rfloor \bmod g \neq 0$, then $d_u = g-1$, and we now have $d_u + \lfloor \frac{nk_{u-1}}{g^u} \rfloor \bmod g \geqslant g$. In other words, the last summand at (2.6) equals 0 if and only if $\lfloor \frac{nk_{u-1}}{g^u} \rfloor \bmod g = 0$, and equals 1 otherwise (since then $g \leqslant d_u + \lfloor \frac{nk_{u-1}}{g^u} \rfloor \bmod g < 2g$); in fact, it equals $\iota\big(\lfloor \frac{nk_{u-1}}{g^u} \rfloor\big)$ (directly by the definition of ι). Therefore, we have

$$\left\lfloor \frac{d_u + \lfloor \frac{nk_{u-1}}{g^u} \rfloor}{g} \right\rfloor = \left\lfloor \frac{\lfloor \frac{nk_{u-1}}{g^u} \rfloor}{g} \right\rfloor + \iota\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right) = \mu\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right),$$

which proves the first equality in (2.5).

Further, we have

$$\left\lfloor \frac{d_u + \lfloor \frac{nk_{u-1}}{g^u} \rfloor}{g} \right\rfloor = \left\lfloor \frac{\lfloor \frac{d_u g^u + nk_{u-1}}{g^u} \rfloor}{g} \right\rfloor \overset{(2.1)}{=} \left\lfloor \frac{d_u g^u + nk_{u-1}}{g^{u+1}} \right\rfloor = \left\lfloor \frac{d_u + \frac{nk_{u-1}}{g^u}}{g} \right\rfloor,$$

which completes the proof of the claim. (Here and onward, by $\stackrel{(2.1)}{=}$ we imply that the equality holds because of the formula (2.1).)

Claim 2.8. *For each $u \in \mathbb{N}$, we have*

$$\left\lfloor \frac{nk_u}{g^{u+1}} \right\rfloor \equiv \mu\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right) \pmod{g^t}.$$

Proof. The proof is a straightforward calculation:

$$\begin{aligned}
\left\lfloor \frac{nk_u}{g^{u+1}} \right\rfloor &= \left\lfloor \frac{n(d_u g^u + k_{u-1})}{g^{u+1}} \right\rfloor = \left\lfloor \frac{nd_u + \frac{nk_{u-1}}{g^u}}{g} \right\rfloor \\
&= \left\lfloor \frac{(g^{t+2}x + g^{t+1}b + 1)d_u + \frac{nk_{u-1}}{g^u}}{g} \right\rfloor \\
&= g^{t+1}xd_u + g^t bd_u + \left\lfloor \frac{d_u + \frac{nk_{u-1}}{g^u}}{g} \right\rfloor \\
&\equiv \left\lfloor \frac{d_u + \frac{nk_{u-1}}{g^u}}{g} \right\rfloor \stackrel{(2.5)}{=} \mu\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right) \pmod{g^t}.
\end{aligned}$$

Claim 2.9. *For each $u \in \mathbb{N}$, we have*

$$\left\lfloor \frac{nk_{u+t}}{g^{u+t+1}} \right\rfloor \equiv \mu^t\left(\left\lfloor \frac{nk_u}{g^{u+1}} \right\rfloor\right) \pmod{g}.$$

Proof. By induction on v, we shall prove that for each v, $0 \leqslant v \leqslant t$, we have

$$\left\lfloor \frac{nk_{u+t}}{g^{u+t+1}} \right\rfloor \equiv \mu^v\left(\left\lfloor \frac{nk_{u+t-v}}{g^{u+t-v+1}} \right\rfloor\right) \pmod{g^{t-v+1}}. \tag{7}$$

For $v = 0$, the assertion is trivially true (the left-hand side and the right-hand side are equal). For the induction step, assume that (2.7) holds for a given v, $0 \leqslant v \leqslant t - 1$, and let us prove

$$\left\lfloor \frac{nk_{u+t}}{g^{u+t+1}} \right\rfloor \equiv \mu^{v+1}\left(\left\lfloor \frac{nk_{u+t-v-1}}{g^{u+t-v}} \right\rfloor\right) \pmod{g^{t-v}}. \tag{8}$$

By the previous claim, we have

$$\left\lfloor \frac{nk_{u+t-v}}{g^{u+t-v+1}} \right\rfloor \equiv \mu\left(\left\lfloor \frac{nk_{u+t-v-1}}{g^{u+t-v}} \right\rfloor\right) \pmod{g^t}.$$

Applying Claim 2.6 a total of v times, we obtain

$$\mu^v\left(\left\lfloor \frac{nk_{u+t-v}}{g^{u+t-v+1}} \right\rfloor\right) \equiv \mu^{v+1}\left(\left\lfloor \frac{nk_{u+t-v-1}}{g^{u+t-v}} \right\rfloor\right) \pmod{g^{t-v}}.$$

Together with the assumption (2.7), we obtain (2.8). This completes the inductive step. Putting $v = t$ in (2.7) gives the statement of the claim. The proof is completed.

Claim 2.10. *For each $u \in \mathbb{N}$, we have*

$$\left\lfloor \frac{nk_{u+t}}{g^{u+t+1}} \right\rfloor \equiv bd_u + \mu^{t+1}\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right) \pmod{g}.$$

Proof. Starting from the congruence from the previous claim, we obtain

$$\begin{aligned}
\left\lfloor \frac{nk_{u+t}}{g^{u+t+1}} \right\rfloor &\equiv \mu^t\left(\left\lfloor \frac{nk_u}{g^{u+1}} \right\rfloor\right) = \mu^t\left(\left\lfloor \frac{n(d_u g^u + k_{u-1})}{g^{u+1}} \right\rfloor\right) \pmod{q}\\
&= \mu^t\left(\left\lfloor \frac{(g^{t+2}x + g^{t+1}b + 1)d_u + \frac{nk_{u-1}}{g^u}}{g} \right\rfloor\right)\\
&= \mu^t\left(g^{t+1}xd_u + g^t bd_u + \left\lfloor \frac{d_u + \frac{nk_{u-1}}{g^u}}{g} \right\rfloor\right)\\
&\overset{(2.5)}{=} \mu^t\left(g^{t+1}xd_u + g^t bd_u + \mu\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right)\right)\\
&\overset{(2.4)}{=} \mu^{t-1}\left(g^t xd_u + g^{t-1}bd_u + \mu^2\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right)\right)\\
&\overset{(2.4)}{=} \cdots \overset{(2.4)}{=} \mu\left(g^2xd_u + gbd_u + \mu^t\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right)\right)\\
&\overset{(2.4)}{=} gxd_u + bd_u + \mu^{t+1}\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right)\\
&\equiv bd_u + \mu^{t+1}\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right) \pmod{g}.
\end{aligned}$$

This completes the proof of the claim.

Claim 2.11. *For each $u \in \mathbb{N}_0$, the following relations hold between the $(u+1)^{st}$ digit from the right in the product nk_u and the digit d_u (that is, the $(u+1)^{st}$ digit from the right in k_u):*

- $\lfloor \frac{nk_u}{g^u} \rfloor \bmod g - d_u = 0$ *if and only if* $\lfloor \frac{nk_{u-1}}{g^u} \rfloor \bmod g = 0$*; furthermore, in that case we have* $\lfloor \frac{nk_{u+t}}{g^{u+t+1}} \rfloor \bmod g \neq 0$*;*
- $\lfloor \frac{nk_u}{g^u} \rfloor \bmod g - d_u \leqslant -1$ *otherwise.*

Proof. We first note

$$\begin{aligned} \left\lfloor \frac{nk_u}{g^u} \right\rfloor - d_u = \left\lfloor \frac{n(g^u d_u + k_{u-1})}{g^u} \right\rfloor - d_u = nd_u + \left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor - d_u \\ \equiv d_u + \left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor - d_u = \left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor \pmod g. \end{aligned} \tag{9}$$

(Between the first and the second row we used the fact $n \equiv 1 \pmod g$.) Since $g-2 \leqslant d_u \leqslant g-1$, it follows that

$$-(g-1) \leqslant \left\lfloor \frac{nk_u}{g^u} \right\rfloor \bmod g - d_u \leqslant (g-1)-(g-2) = 1. \tag{10}$$

Together with (2.9), this immediately gives the first half of the first assertion from the statement. Further, if $\lfloor \frac{nk_{u-1}}{g^u} \rfloor \bmod g \neq 0$, then our construction gives $d_u = g-1$, and in that case the upper bound from (2.10) is actually $(g-1)-(g-1) = 0$, which gives the second assertion.

That leaves to prove the second half of the first assertion. If $\lfloor \frac{nk_{u-1}}{g^u} \rfloor \bmod g = 0$, then $d_u = h\Big(b, \mu^{t+1}\big(\lfloor \frac{nk_{u-1}}{g^u} \rfloor\big)\Big)$. By Claim 2.10, we have

$$\begin{aligned} \left\lfloor \frac{nk_{u+t}}{g^{u+t+1}} \right\rfloor &\equiv bd_u + \mu^{t+1}\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right) \\ &= b \cdot h\left(b, \mu^{t+1}\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right)\right) + \mu^{t+1}\left(\left\lfloor \frac{nk_{u-1}}{g^u} \right\rfloor\right) \pmod g, \end{aligned} \tag{11}$$

and directly by the definition of h we obtain that the above expression ends with a digit different from 0. This completes the proof of the claim.

Claim 2.12. *For each $u \in \mathbb{N}_0$, there exists a collection $\mathscr{S}$ of pairwise disjoint subsets of the set $\{0, 1, \ldots, u\}$ such that:*

- *$|S| \leqslant 2$ for each $S \in \mathscr{S}$;*
- *for each $S \in \mathscr{S}$, the sum $\sum_{v \in S} d_v$ is greater than the sum of digits at the corresponding positions in the product nk_u;*
- *$|\bigcup \mathscr{S}| \geqslant u - t$.*

Proof. Let $u \in \mathbb{N}_0$ be given. Let $nk_u = \overline{e_q e_{q-1} \dots e_2 e_1 e_0}_{(g)}$. In other words, $e_v = \lfloor \frac{nk_u}{g^v} \rfloor \bmod g$ for $0 \leqslant v \leqslant u$. Note that we also have

$$\begin{aligned} e_v = \left\lfloor \frac{nk_u}{g^v} \right\rfloor \bmod g &= \left\lfloor \frac{n(g^{v+1} \cdot \overline{d_u \dots d_{v+2} d_{v+1}}_{(g)} + k_v)}{g^v} \right\rfloor \bmod g \\ &= \left(ng \cdot \overline{d_u \dots d_{v+2} d_{v+1}}_{(g)} + \left\lfloor \frac{nk_v}{g^v} \right\rfloor \right) \bmod g = \left\lfloor \frac{nk_v}{g^v} \right\rfloor \bmod g. \end{aligned}$$

The collection $\mathscr{S}$ will be formed by the following algorithm. Initially set the collection $\mathscr{S}$ to be empty. For $v = 0, 1, 2 \dots$ do the following:

1. If v already belongs to some member of $\mathscr{S}$, proceed to the next value of v; otherwise, proceed to the next step.
2. If $d_v > e_v$, add $\{v\}$ to the collection and proceed to the next value of v; otherwise, proceed to the next step.
3. If $d_v = e_v$, note that in that case Claim 2.11 (the first part) gives $\lfloor \frac{nk_{v+t}}{g^{v+t+1}} \rfloor \bmod g \neq 0$; because of this observation, by Claim 2.11 again (the second part), we obtain $e_{v+t+1} - d_{v+t+1} \leqslant -1$, that is, $d_{v+t+1} > e_{v+t+1}$. We add $\{v, v+t+1\}$ to the collection and proceed to the next value of v (note that, since $d_v + d_{v+t+1} > e_v + e_{v+t+1}$, the set added to the collection fulfills the requirements).

The last value of v for which we apply the algorithm is $u - t - 1$ (because we need the inequality $v + t + 1 \leqslant u$, in order for all the indices we were mentioning to be defined). By the way the algorithm works, all the numbers from $\{0, 1, \dots, u-t-1\}$ will be added to $\bigcup \mathscr{S}$. This gives $|\bigcup \mathscr{S}| \geqslant u - t$, which completes the proof of the claim.

We are finally ready to complete the proof of Theorem 1.1. Fix $u \in \mathbb{N}_0$. Let $nk_u = \overline{e_q e_{q-1} \dots e_2 e_1 e_0}_{(g)}$. Since $k_u < g^{u+1}$ and $n < g^{\lfloor \log_g n \rfloor + 1}$, we have $nk_u < g^{\lfloor \log_g n \rfloor + u + 2}$; therefore, $q \leqslant \lfloor \log_g n \rfloor + u + 1$.

Let $\mathscr{S}$ be the collection from Claim 2.12. For each $S \in \mathscr{S}$ we have $\sum_{v\in S} e_v < \sum_{v\in S} d_v$, that is, $\sum_{v\in S} e_v \leqslant \sum_{v\in S} d_v - 1$. Since $|\bigcup \mathscr{S}| \geqslant u - t$ and each member of $\mathscr{S}$ is of cardinality at most 2, we conclude that the collection $\mathscr{S}$ contains at least $\frac{u-t}{2}$ members. We now have

$$\begin{aligned}\sum_{v\in\bigcup\mathscr{S}} e_v &= \sum_{S\in\mathscr{S}}\sum_{v\in S} e_v \leqslant \sum_{S\in\mathscr{S}}\left(\sum_{v\in S} d_v - 1\right) = \sum_{S\in\mathscr{S}}\sum_{v\in S} d_v - |\mathscr{S}| \\ &\leqslant \sum_{S\in\mathscr{S}}\sum_{v\in S} d_v - \frac{u-t}{2} = \sum_{v\in\bigcup\mathscr{S}} d_v - \frac{u-t}{2}.\end{aligned}$$

For each v, $0 \leqslant v \leqslant q$, such that $v \notin \bigcup\mathscr{S}$ we trivially have $e_v \leqslant g-1$, and there are at most

$$\begin{aligned}(q+1)-(u-t) &= q-u+t+1 \\ &\leqslant (\lfloor\log_g n\rfloor + u + 1) - u + t + 1 = \lfloor\log_g n\rfloor + t + 2\end{aligned}$$

such values of v. Finally, we calculate

$$\begin{aligned}s_g(nk_u) &= \sum_{v=0}^{q} e_v = \sum_{v\in\bigcup\mathscr{S}} e_v + \sum_{v\notin\bigcup\mathscr{S}} e_v \\ &\leqslant \left(\sum_{v\in\bigcup\mathscr{S}} d_v - \frac{u-t}{2}\right) + (g-1)(\lfloor\log_g n\rfloor + t + 2) \\ &\leqslant \sum_{v=0}^{u} d_v - \frac{u-t}{2} + (g-1)(\lfloor\log_g n\rfloor + t + 2) \\ &= s_g(k_u) - \frac{u}{2} + \left(\frac{t}{2} + (g-1)(\lfloor\log_g n\rfloor + t + 2)\right).\end{aligned}$$

The expression in the last brackets is constant with respect to u. Therefore, for u large enough we have $s_g(nk_u) < s_g(k_u)$, that is, we may take $k = k_u$ for u large enough. This completes the proof. ■

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