Characterization of arithmetic functions that preserve the sum-of-squares operation

Bojan Bašić

Department of Mathematics and Informatics, University of Novi Sad,

Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia

`bojan.basic@dmi.uns.ac.rs`

Abstract

We characterize all functions $f : \mathbb{N} \to \mathbb{C}$ such that $f(m^2 + n^2) = f(m)^2 + f(n)^2$ for all $m, n \in \mathbb{N}$. It turns out that all such functions can be grouped into three families, namely $f \equiv 0$, $f(n) = \pm n$ (subject to some restrictions on when the choice of the sign is possible) and $f(n) = \pm \frac{1}{2}$ (again subject to some restrictions on when the choice of the sign is possible).

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1 Introduction

During the last two decades, there has been a lot of work on functions on positive integers satisfying some (more or less) Cauchy-like functional equation. In 1992, Spiro-Silverman [14] showed that the only multiplicative function $f : \mathbb{N} \to \mathbb{C}$ such that $f(p + q) = f(p) + f(q)$ for all primes p, q and that $f(p_0) \neq 0$ for some prime p_0 is the identity function $f(n) = n$. (By $\mathbb{N}$ we denote the set of positive integers. A function defined on $\mathbb{N}$ is called *multiplicative* if $f(mn) = f(m)f(n)$ for all coprime $m, n \in \mathbb{N}$, and is called *completely multiplicative* if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.) Very recently Fang [5] extended the same conclusion to the equation $f(p+q+r) = f(p)+f(q)+f(r)$, and Dubickas and Šarka [4] settled the general case $f(p_1 + p_2 + \cdots + p_k) = f(p_1) + f(p_2) + \cdots + f(p_k)$, where $k \geqslant 2$ is fixed. Phong [12] considered a similar equation $f(p+q+pq) = f(p)+f(q)+f(pq)$ (p and q are primes), and proved that the only completely multiplicative function f that satisfies this equation such that $f(p_0) \neq 0$ for some prime p_0 is the identity function. De Koninck, Kátai and Phong [9] proved that the only multiplicative function f that satisfies $f(1) = 1$ and $f(p+m^2) = f(p) + f(m^2)$ (p is prime, $m \in \mathbb{N}$) is the identity function. Chung [2] described all multiplicative and completely multiplicative functions f such that $f(m^2+n^2) = f(m^2)+f(n^2)$ ($m, n \in \mathbb{N}$). Some other related problems are treated in [1, 3, 7, 13].

We hereby consider a functional equation similar to those mentioned above, namely a modification of the functional equation treated by Chung. We prove the following theorem.

Theorem 1.1. *Let* $f : \mathbb{N} \to \mathbb{C}$ *satisfy*

$$f(m^2 + n^2) = f(m)^2 + f(n)^2 \tag{1.1}$$

for all $m, n \in \mathbb{N}$. *Then one of the following holds:*

1) $f \equiv 0$;

2)
   - $f(n) = \pm n$ *for each* n *such that either there exists a prime factor of* n *congruent to* 3 *modulo* 4 *that occurs in* n *to an odd exponent, or* n *is a perfect square that is not divisible by any prime congruent to* 1 *modulo* 4;
   - $f(n) = n$ *for all other* n

   (in the former case, the sign for each such n *can be chosen independently of the others);*

3)
   - $f(n) = \pm\frac{1}{2}$ *for each* n *such that either there exists a prime factor of* n *congruent to* 3 *modulo* 4 *that occurs in* n *to an odd exponent, or* n *is a perfect square that is not divisible by any prime congruent to* 1 *modulo* 4;
   - $f(n) = \frac{1}{2}$ *for all other* n

   (in the former case, the sign for each such n *can be chosen independently of the others).*

Note that there is no assumption on multiplicative properties of f. The proof, which is somewhat technical but elementary, is distributed through the following sections.

2 Evaluating $f(1)$

For the rest of the paper, let f denote an arithmetic function that satisfies the functional equation (1.1).

Lemma 2.1. $f(1) \in \{0, 1, -1, \frac{1}{2}, -\frac{1}{2}\}$.

Proof. Let $f(1) = a$. We get $f(2) = f(1^2 + 1^2) = f(1)^2 + f(1)^2 = 2a^2$. Therefore,

$$f(2)^2 = 4a^4. \tag{2.1}$$

Using (2.1), we get $f(8) = f(2^2 + 2^2) = f(2)^2 + f(2)^2 = 8a^4$. Therefore,

$$f(8)^2 = 64a^8. \tag{2.2}$$

Using (2.1), we get $f(5) = f(2^2 + 1^2) = f(2)^2 + f(1)^2 = 4a^4 + a^2$. Therefore,

$$f(5)^2 = 16a^8 + 8a^6 + a^4. \tag{2.3}$$

Since $f(5)^2 + f(5)^2 = f(5^2 + 5^2) = f(50) = f(7^2 + 1^2) = f(7)^2 + f(1)^2$, by (2.3) we deduce

$$f(7)^2 = 2(16a^8 + 8a^6 + a^4) - a^2 = 32a^8 + 16a^6 + 2a^4 - a^2. \tag{2.4}$$

Since $f(8)^2 + f(1)^2 = f(8^2 + 1^2) = f(65) = f(7^2 + 4^2) = f(7)^2 + f(4)^2$, by (2.2) and (2.4) we deduce

$$f(4)^2 = 64a^8 + a^2 - (32a^8 + 16a^6 + 2a^4 - a^2) = 32a^8 - 16a^6 - 2a^4 + 2a^2. \tag{2.5}$$

Using (2.5), we get $f(32) = f(4^2 + 4^2) = f(4)^2 + f(4)^2 = 64a^8 - 32a^6 - 4a^4 + 4a^2$. Therefore,

$$f(32)^2 = 4096a^{16} - 4096a^{14} + 512a^{12} + 768a^{10} - 240a^8 - 32a^6 + 16a^4. \tag{2.6}$$

Using (2.5), we get $f(17) = f(4^2 + 1^2) = f(4)^2 + f(1)^2 = 32a^8 - 16a^6 - 2a^4 + 3a^2$. Therefore,

$$f(17)^2 = 1024a^{16} - 1024a^{14} + 128a^{12} + 256a^{10} - 92a^8 - 12a^6 + 9a^4. \tag{2.7}$$

Since $f(32)^2+f(7)^2 = f(32^2+7^2) = f(1073) = f(28^2+17^2) = f(28)^2+f(17)^2$, by (2.6), (2.4) and (2.7) we deduce

$$\begin{aligned} f(28)^2 &= 4096a^{16} - 4096a^{14} + 512a^{12} + 768a^{10} - 240a^8 - 32a^6 + 16a^4 \\ &\quad + 32a^8 + 16a^6 + 2a^4 - a^2 \\ &\quad - (1024a^{16} - 1024a^{14} + 128a^{12} + 256a^{10} - 92a^8 - 12a^6 + 9a^4) \\ &= 3072a^{16} - 3072a^{14} + 384a^{12} + 512a^{10} - 116a^8 - 4a^6 + 9a^4 - a^2. \end{aligned} \tag{2.8}$$

Since $f(17)^2 + f(4)^2 = f(17^2 + 4^2) = f(305) = f(16^2 + 7^2) = f(16)^2 + f(7)^2$, by (2.7), (2.5) and (2.4) we deduce

$$\begin{aligned} f(16)^2 &= 1024a^{16} - 1024a^{14} + 128a^{12} + 256a^{10} - 92a^8 - 12a^6 + 9a^4 \\ &\quad + 32a^8 - 16a^6 - 2a^4 + 2a^2 - (32a^8 + 16a^6 + 2a^4 - a^2) \\ &= 1024a^{16} - 1024a^{14} + 128a^{12} + 256a^{10} - 92a^8 - 44a^6 + 5a^4 + 3a^2. \end{aligned} \tag{2.9}$$

Since $f(17)^2+f(17)^2 = f(17^2+17^2) = f(578) = f(23^2+7^2) = f(23)^2+f(7)^2$, by (2.7) and (2.4) we deduce

$$\begin{aligned} f(23)^2 &= 2(1024a^{16} - 1024a^{14} + 128a^{12} + 256a^{10} - 92a^8 - 12a^6 + 9a^4) \\ &\quad - (32a^8 + 16a^6 + 2a^4 - a^2) \\ &= 2048a^{16} - 2048a^{14} + 256a^{12} + 512a^{10} - 216a^8 - 40a^6 + 16a^4 + a^2. \end{aligned} \tag{2.10}$$

Since $f(23)^2+f(16)^2 = f(23^2+16^2) = f(785) = f(28^2+1^2) = f(28)^2+f(1)^2$, by (2.10), (2.9) and (2.8) we deduce

$$\begin{aligned} f(1)^2 &= 2048a^{16} - 2048a^{14} + 256a^{12} + 512a^{10} - 216a^8 - 40a^6 + 16a^4 + a^2 \\ &\quad + 1024a^{16} - 1024a^{14} + 128a^{12} + 256a^{10} - 92a^8 - 44a^6 + 5a^4 + 3a^2 \\ &\quad - (3072a^{16} - 3072a^{14} + 384a^{12} + 512a^{10} - 116a^8 - 4a^6 + 9a^4 - a^2) \\ &= 256a^{10} - 192a^8 - 80a^6 + 12a^4 + 5a^2. \end{aligned}$$

On the other hand, since $f(1)^2 = a^2$, the difference of the right-hand sides of these two expressions must equal 0, that is,

$$256a^{10} - 192a^8 - 80a^6 + 12a^4 + 4a^2 = 0.$$

The polynomial on the left-hand side can be factored as

$$4a^2(a-1)(a+1)(2a-1)(2a+1)(4a^2+1)^2 = 0.$$

Therefore, we see that its zeros are $a_1 = a_2 = 0$, $a_3 = 1$, $a_4 = -1$, $a_5 = \frac{1}{2}$, $a_6 = -\frac{1}{2}$, $a_7 = a_8 = \frac{i}{2}$, $a_9 = a_{10} = -\frac{i}{2}$. Thus, in order to finish the proof we need to show that $f(1) = \pm\frac{i}{2}$ is impossible, that is, that $a^2 = -\frac{1}{4}$ is impossible.

Using (2.3) and (2.5), we get $f(41) = f(5^2+4^2) = f(5)^2 + f(4)^2 = 48a^8 - 8a^6 - a^4 + 2a^2$. Therefore,

$$f(41)^2 = 2304a^{16} - 768a^{14} - 32a^{12} + 208a^{10} - 31a^8 - 4a^6 + 4a^4. \tag{2.11}$$

On the other hand, using (2.3) and (2.1), we get $f(29) = f(5^2+2^2) = f(5)^2 + f(2)^2 = 16a^8 + 8a^6 + 5a^4$ and therefore

$$f(29)^2 = 256a^{16} + 256a^{14} + 224a^{12} + 80a^{10} + 25a^8, \tag{2.12}$$

which, since $f(29)^2 + f(29)^2 = f(29^2+29^2) = f(1682) = f(41^2+1^2) = f(41)^2 + f(1)^2$, leads to

$$\begin{aligned} f(41)^2 &= 2(256a^{16} + 256a^{14} + 224a^{12} + 80a^{10} + 25a^8) - a^2 \\ &= 512a^{16} + 512a^{14} + 448a^{12} + 160a^{10} + 50a^8 - a^2. \end{aligned} \tag{2.13}$$

Therefore, the difference of the right-hand sides of (2.11) and (2.13) must equal 0, that is,

$$1792a^{16} - 1280a^{14} - 480a^{12} + 48a^{10} - 81a^8 - 4a^6 + 4a^4 + a^2 = 0.$$

We calculate that for $a^2 = -\frac{1}{4}$ the expression on the left equals $-\frac{5}{16}$. This shows that $f(1) = \pm\frac{i}{2}$ is indeed impossible, and the proof is thus completed. ∎

3 Evaluating $f(2), f(3), \ldots, f(10)$ up to a sign

The equations obtained in the proof Lemma 2.1 will still be useful, and we need some more.

Using (2.5) and (2.1), we get $f(20) = f(4^2+2^2) = f(4)^2 + f(2)^2 = 32a^8 - 16a^6 + 2a^4 + 2a^2$. Therefore,

$$f(20)^2 = 1024a^{16} - 1024a^{14} + 384a^{12} + 64a^{10} - 60a^8 + 8a^6 + 4a^4. \quad (3.1)$$

Since $f(17)^2+f(20)^2 = f(17^2+20^2) = f(689) = f(25^2+8^2) = f(25)^2+f(8)^2$, by (2.7), (3.1) and (2.2) we deduce

$$\begin{aligned} f(25)^2 &= 1024a^{16} - 1024a^{14} + 128a^{12} + 256a^{10} - 92a^8 - 12a^6 + 9a^4 \\ &\quad + 1024a^{16} - 1024a^{14} + 384a^{12} + 64a^{10} - 60a^8 + 8a^6 + 4a^4 \\ &\quad - 64a^8 \\ &= 2048a^{16} - 2048a^{14} + 512a^{12} + 320a^{10} - 216a^8 - 4a^6 + 13a^4. \end{aligned}$$

Since $f(25)^2+f(2)^2 = f(25^2+2^2) = f(629) = f(23^2+10^2) = f(23)^2+f(10)^2$, by the previous equation and (2.1) and (2.10) we deduce

$$\begin{aligned} f(10)^2 &= 2048a^{16} - 2048a^{14} + 512a^{12} + 320a^{10} - 216a^8 - 4a^6 + 13a^4 \\ &\quad + 4a^4 \\ &\quad - (2048a^{16} - 2048a^{14} + 256a^{12} + 512a^{10} - 216a^8 \\ &\qquad\qquad -40a^6 + 16a^4 + a^2) \\ &= 256a^{12} - 192a^{10} + 36a^6 + a^4 - a^2. \end{aligned} \quad (3.2)$$

This makes enough prerequisites for this section.

Lemma 3.1. *Let $f(1) = 0$. Then $f(n) = 0$ for all n such that $1 \leqslant n \leqslant 10$.*

Proof. Putting $a = 0$ in the equations (2.1), (2.5), (2.3), (2.4), (2.2) and (3.2) gives $f(n) = 0$ for all n such that $1 \leqslant n \leqslant 10$ apart from $n = 3, 6, 9$.

We further have $f(26) = f(5^2+1^2) = f(5)^2 + f(1)^2 = 0$. Putting $a = 0$ in the equation (2.12) gives $f(29) = 0$. Since $f(29)^2 + f(2)^2 = f(29^2+2^2) = f(845) = f(26^2+13^2) = f(26)^2+f(13)^2$, we deduce $f(13)^2 = f(29)^2+f(2)^2 - f(26)^2 = 0$, that is, $f(13) = 0$. Since $f(13) = f(3^2+2^2) = f(3)^2 + f(2)^2$, we deduce $f(3)^2 = f(13) - f(2)^2 = 0$, that is, $f(3) = 0$.

Since $f(10)^2 + f(10)^2 = f(10^2+10^2) = f(200) = f(14^2+2^2) = f(14)^2 + f(2)^2$, we deduce $f(14)^2 = 2f(10)^2 - f(2)^2 = 0$. Since $f(14)^2 + f(3)^2 =$

$f(14^2+3^2) = f(205) = f(13^2+6^2) = f(13)^2 + f(6)^2$, we deduce $f(6)^2 = f(14)^2 + f(3)^2 - f(13)^2 = 0$ (recall that $f(13) = 0$ is obtained in the previous paragraph), that is, $f(6) = 0$.

Finally, since $f(6)^2 + f(7)^2 = f(6^2+7^2) = f(85) = f(9^2+2^2) = f(9)^2 + f(2)^2$, we deduce $f(9)^2 = f(6)^2 + f(7)^2 - f(2)^2 = 0$, that is, $f(9) = 0$, which completes the proof. ∎

Lemma 3.2. *Let $f(1) = \pm 1$. Then $f(n) = \pm n$ for all n such that $1 \leqslant n \leqslant 10$.*

Proof. Putting $a^2 = 1$ in the equations (2.1), (2.5), (2.3), (2.4), (2.2) and (3.2) gives $f(n)^2 = n^2$, that is, $f(n) = \pm n$, for all n such that $1 \leqslant n \leqslant 10$ apart from $n = 3, 6, 9$.

Let $f(3) = b$. We have $f(13) = f(3^2+2^2) = f(3)^2 + f(2)^2 = b^2 + 4$ and $f(25) = f(4^2+3^2) = f(4)^2 + f(3)^2 = b^2 + 16$; therefore, $f(13)^2 = b^4 + 8b^2 + 16$ and $f(25)^2 = b^4 + 32b^2 + 256$. Putting $a^2 = 1$ in the equation (2.13) gives $f(41)^2 = 1681$. Since $f(41)^2 + f(13)^2 = f(41^2+13^2) = f(1850) = f(35^2 + 25^2) = f(35)^2 + f(25)^2$, we deduce

$$f(35)^2 = 1681 + b^4 + 8b^2 + 16 - (b^4 + 32b^2 + 256) = 1441 - 24b^2.$$

On the other hand, putting $a^2 = 1$ in the equations (2.12) and (3.1) gives $f(29)^2 = 841$ and $f(20)^2 = 400$, and since $f(29)^2 + f(20)^2 = f(29^2+20^2) = f(1241) = f(35^2+4^2) = f(35)^2 + f(4)^2$, we deduce

$$f(35)^2 = 841 + 400 - 16 = 1225.$$

Therefore, the difference of the right-hand sides of the last two equations must equal 0, that is,

$$216 - 24b^2 = 0.$$

We thus find $b^2 = 9$, that is, $f(3) = \pm 3$.

Since $f(10)^2 + f(10)^2 = f(10^2+10^2) = f(200) = f(14^2+2^2) = f(14)^2 + f(2)^2$, we deduce $f(14)^2 = 2 \cdot 100 - 4 = 196$. Putting $b^2 = 9$ in the expression for $f(13)^2$ obtained in the previous paragraph gives $f(13)^2 = 169$. Since $f(14)^2 + f(3)^2 = f(14^2+3^2) = f(205) = f(13^2+6^2) = f(13)^2 + f(6)^2$, we deduce $f(6)^2 = 196 + 9 - 169 = 36$, that is, $f(6) = \pm 6$.

Finally, since $f(6)^2 + f(7)^2 = f(6^2+7^2) = f(85) = f(9^2+2^2) = f(9)^2 + f(2)^2$, we deduce $f(9)^2 = 36 + 49 - 4 = 81$, that is, $f(9) = \pm 9$, which completes the proof. ∎

Lemma 3.3. *Let $f(1) = \pm\frac{1}{2}$. Then $f(n) = \pm\frac{1}{2}$ for all n such that $1 \leqslant n \leqslant 10$.*

Proof. Putting $a^2 = \frac{1}{4}$ in the equations (2.1), (2.5), (2.3), (2.4), (2.2) and (3.2) gives $f(n)^2 = \frac{1}{4}$, that is, $f(n) = \pm\frac{1}{2}$, for all n such that $1 \leqslant n \leqslant 10$ apart from $n = 3, 6, 9$.

Let $f(3) = b$. We have $f(18) = f(3^2 + 3^2) = f(3)^2 + f(3)^2 = 2b^2$ and $f(34) = f(5^2 + 3^2) = f(5)^2 + f(3)^2 = b^2 + \frac{1}{4}$; therefore, $f(18)^2 = 4b^4$ and $f(34)^2 = b^4 + \frac{b^2}{2} + \frac{1}{16}$. Putting $a^2 = \frac{1}{4}$ in the equation (2.12) gives $f(29)^2 = \frac{1}{4}$. Since $f(29)^2+f(18)^2 = f(29^2+18^2) = f(1165) = f(34^2+3^2) = f(34)^2+f(3)^2$, we deduce

$$f(3)^2 = \frac{1}{4} + 4b^4 - \left(b^4 + \frac{b^2}{2} + \frac{1}{16}\right) = 3b^4 - \frac{b^2}{2} + \frac{3}{16}.$$

On the other hand, since $f(3)^2 = b^2$, the difference of the right-hand sides of these two expressions must equal 0, that is,

$$3b^4 - \frac{3b^2}{2} + \frac{3}{16} = 0.$$

The polynomial on the left-hand side can be factored as

$$\frac{3}{16}(2b-1)^2(2b+1)^2 = 0.$$

Therefore, we see that its zeros are $b_1 = b_2 = \frac{1}{2}$, $b_3 = b_4 = -\frac{1}{2}$. Thus, $f(3) = \pm\frac{1}{2}$.

We have $f(13) = f(3^2+2^2) = f(3)^2 + f(2)^2 = b^2 + \frac{1}{4}$; therefore, $f(13)^2 = b^4 + \frac{b^2}{2} + \frac{1}{16}$. Since $f(10)^2 + f(10)^2 = f(10^2 + 10^2) = f(200) = f(14^2 + 2^2) = f(14)^2 + f(2)^2$, we deduce $f(14)^2 = 2 \cdot \frac{1}{4} - \frac{1}{4} = \frac{1}{4}$. Since $f(14)^2 + f(3)^2 = f(14^2 + 3^2) = f(205) = f(13^2 + 6^2) = f(13)^2 + f(6)^2$, we deduce $f(6)^2 = \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = \frac{1}{4}$, that is, $f(6) = \pm\frac{1}{2}$.

Finally, since $f(6)^2 + f(7)^2 = f(6^2 + 7^2) = f(85) = f(9^2 + 2^2) = f(9)^2 + f(2)^2$, we deduce $f(9)^2 = \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = \frac{1}{4}$, that is, $f(9) = \pm\frac{1}{2}$, which completes the proof. ■

4 Completing the proof

We are now ready for the final steps.

Proof of Theorem 1.1. Recall the identity

$$(ab+cd)^2 + (ad-bc)^2 = (ab-cd)^2 + (ad+bc)^2.$$

Applying f to the both sides of this equation and using (1.1) leads to

$$f(ab+cd)^2 + f(ad-bc)^2 = f(ab-cd)^2 + f(ad+bc)^2 \tag{4.1}$$

whenever all four arguments are positive.

We shall now prove, by induction on n, that if f satisfies one of the statement from Lemmas 3.1, 3.2 or 3.3 up to $n = 10$, it then satisfies that statement on the whole domain. Let $n > 10$ be an odd number, say $n = 2k+1$. Putting $a = k$, $b = 2$, $c = 1$ and $d = 1$ in (4.1) gives

$$f(n)^2 = f(2k-1)^2 + f(k+2)^2 - f(k-2)^2.$$

Since all the arguments on the right-hand side are positive and smaller than n, by the inductive assumption we have $f(2k-1)^2 = f(k+2)^2 = f(k-2)^2 = 0$, respectively $f(2k-1)^2 = (2k-1)^2$, $f(k+2)^2 = (k+2)^2$ and $f(k-2)^2 = (k-2)^2$, respectively $f(2k-1)^2 = f(k+2)^2 = f(k-2)^2 = \frac{1}{4}$. In the first and the third case we immediately get $f(n)^2 = 0$, respectively $f(n)^2 = \frac{1}{4}$, and in the second case we calculate

$$f(n)^2 = 4k^2 - 4k + 1 + k^2 + 4k + 4 - (k^2 - 4k + 4) = 4k^2 + 4k + 1 = n^2,$$

as needed. Let now $n > 10$ be an even number, say $n = 2k$ ($k \geqslant 6$). Putting $a = k-1$, $b = 2$, $c = 2$ and $d = 1$ in (4.1) gives

$$f(n)^2 = f(2k-4)^2 + f(k+3)^2 - f(k-5)^2.$$

Since all the arguments on the right-hand side are positive (because $k \geqslant 6$) and smaller than n, we similarly as in the previous case get $f(n)^2 = 0$, respectively $f(n)^2 = \frac{1}{4}$, respectively

$$f(n)^2 = 4k^2 - 16k + 16 + k^2 + 6k + 9 - (k^2 - 10k + 25) = 4k^2 = n^2,$$

as needed.

Therefore, we have that for all n either $f(n) = 0$, $f(n) = \pm n$ or $f(n) = \pm\frac{1}{2}$. In order to finish the proof we need to check for which n the sign is actually fixed (for the latter two families of functions). We shall show that the sign is fixed (in particular, positive) if and only if n is a sum of two positive squares. Since it is known that n cannot be represented as a sum of two positive squares if and only if either there exists a prime factor of n congruent to 3 modulo 4 that occurs in n to an odd exponent, or n is a perfect square that is not divisible by any prime congruent to 1 modulo 4 (see, e.g., [11, Problem 5 in Section 3.6 and Theorem 3.22] or [8, Theorem 2.11]), this is enough to complete the proof. (Note that some different claims can be found in the literature, such as in [6] that n is a sum of two positive squares if and only if n is of the form $4^a n_1 n_2^2$ where $a \geqslant 0$, n_1 is a product of primes congruent to 1 modulo 4 and $n_1 > 1$, n_2 is a product of primes congruent to 3 modulo 4, and a similar claim in [10] but with 2^e, $e \geqslant 0$, instead of 4^a. However, both these claims miss the case, e.g., $n = 18$, which is a sum of two positive squares, $9 + 9$, but does not have the described form.)

Let n be a sum of two positive squares, say $n = s^2 + t^2$ $(s, t \neq 0)$. Then $f(n) = f(s^2 + t^2) = f(s)^2 + f(t)^2$, and since the right-hand side is positive (it equals either $s^2 + t^2$ or $\frac{1}{4}$), it follows that the positive sign has to be chosen for $f(n)$ whenever n is a sum of two positive squares. For the other direction, let $f(n) = \pm n$ with a freely chosen sign for each n that is not a sum of two positive squares, and $f(n) = n$ for all other n. (The case with $\pm\frac{1}{2}$ as the image is analogous.) Then for each $m, n \in \mathbb{N}$ we have $f(m^2 + n^2) = m^2 + n^2$ and $f(m)^2 + f(n)^2 = (\pm m)^2 + (\pm n)^2 = m^2 + n^2$, which completes the proof. ■

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