The Heesch number for multiple prototiles is unbounded

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Abstract

In this paper we show that, for all integers $k \geqslant 3$ and $n \geqslant 1$, there exists a protoset consisting of k prototiles, whose Heesch number is n. This disproves a conjecture by Grünbaum and Shephard.

Mathematics Subject Classification (2010): 52C20, 05B45

Keywords: Heesch number, tiling, protoset, prototile, corona

1 Introduction

A *tiling* of the Euclidean plane $\mathbb{E}^2$ is a set $\mathscr{T}$ such that each $T \in \mathscr{T}$ is a closed topological disc, every two different $T', T'' \in \mathscr{T}$ have disjoint interiors, and $\bigcup \mathscr{T} = \mathbb{E}^2$. The elements of $\mathscr{T}$ are called *tiles*.

Given a tiling $\mathscr{T}$, its tiles can be partitioned into equivalence classes where two tiles are in the same class if and only if they are congruent (by any isometry). The set $\mathscr{P}$ obtained by choosing one representative from each such class is called the *protoset* of the tiling $\mathscr{T}$, while the elements of $\mathscr{P}$ are called *prototiles*. If $\mathscr{P}$ is the protoset of some tiling $\mathscr{T}$, we say that $\mathscr{P}$ *admits* the tiling $\mathscr{T}$. More generally, we shall call any set of pairwise noncongruent closed topological discs a *protoset* (even if it is not the protoset of some tiling), call its elements *prototiles*, and then it can be asked whether a given protoset admits *any* tiling.

For a long time, tilings were mainly studied within the scope of recreational mathematics, but by the 1980s it became clear not only that there are many unexpected connections between tilings and different branches of mathematics [14, 12, 2], but that the theory of tilings also has many real-world applications [11, 1, 7]. The book by Grünbaum and Shephard [5] is a very comprehensive treatment of the theoretical foundations of tilings, and it is a very valuable source of information even still today.

In this paper we investigate the so-called *Heesch number*. The Heesch number of a protoset measures, loosely speaking, how "close" to a tiling we can get with the given protoset (the larger Heesch number is, we can get "closer" to a tiling). The Heesch number of a protoset is either infinite (if the protoset admits some tiling) or a nonnegative integer. Grünbaum and Shephard conjectured that, among all protosets that contain a fixed number of prototiles and whose Heesch number is finite, there is an upper bound for the Heesch number (in other words, it takes only finitely many values). Most of the progress on this conjecture deals with protosets that consist of a single prototile, in which case some interesting prototiles were found, having larger and larger Heesch number (the current record-holder is a prototile whose Heesch number is 5).

In this paper we disprove the conjecture of Grünbaum and Shephard. In particular, we show that, for any given integer $k \geqslant 3$, the Heesch number of a protoset consisting of k prototiles can be arbitrarily large.

2 Heesch number

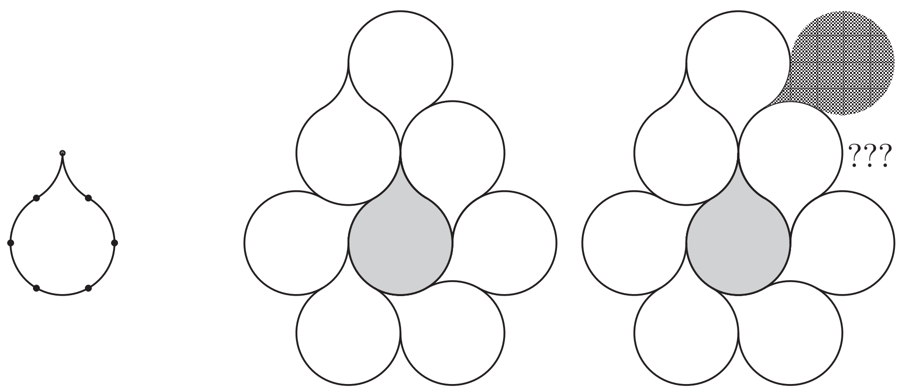
In this section we formally define the Heesch number and briefly review its history.

Definition 2.1. Let $\mathscr{P}$ be a protoset. We say that a tile T in the plane, congruent to a prototile from $\mathscr{P}$, can be *surrounded n times* if and only if there exist finite collections of tiles $\mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_n$ in the plane such that:

- for each i, $1 \leqslant i \leqslant n$, each tile from $\mathscr{C}_i$ is congruent to some prototile from $\mathscr{P}$;
- every two different tiles from $\{T\} \cup \bigcup_{i=1}^{n} \mathscr{C}_i$ have disjoint interiors;
- for each i, $1 \leqslant i \leqslant n$, each tile from $\mathscr{C}_i$ has a common boundary point with some tile from $\mathscr{C}_{i-1}$ (where, by convention, we let $\mathscr{C}_0 = \{T\}$);

- for each i, $1 \leqslant i \leqslant n$, $\bigcup\left(\bigcup_{j=0}^{i} \mathscr{C}_j\right)$ is a closed topological disc such that $\bigcup\left(\bigcup_{j=0}^{i-1} \mathscr{C}_j\right)$ is completely contained in its interior.

The collection $\mathscr{C}_i$ is called the i^{th} *corona*.



(a) A tile (its border is composed of seven congruent circle arcs).

(b) The tile surrounded by the first corona.

(c) The second corona cannot be formed.

Figure 1: Surrounding a tile.

See Figure 1 for an example. Part (b) shows a tile that is surrounded once (the gray tile is the tile T from the preceding definition, and the other tiles form the first corona). Part (c) shows that, at least after such a first corona, the second one cannot be formed: indeed, if we start forming the second corona, the tile with a checkerboard pattern must be placed as shown, and then it is obvious that we cannot continue because the space marked by "???" cannot be filled.

Definition 2.2. The *Heesch number* of a given protoset $\mathscr{P}$ is the maximal nonnegative integer n such that each prototile from $\mathscr{P}$ can be surrounded n times. If such a maximum does not exist, then we define the Heesch number as being infinite.

For a prototile T, by the *Heesch number of* T we mean the Heesch number of a protoset consisting of only one tile congruent to T.

The Extension Theorem [5, Theorem 3.8.1] states that, if for some protoset $\mathscr{P}$ the maximum from the preceding definition does not exist, then the plane can be tiled by tiles congruent to prototiles from $\mathscr{P}$. (This is, however, not really trivial as it might seem at the first glance.)

Heesch [6] asked for which positive integers n there exists a prototile T such that T can be surrounded n times by congruent copies of itself, but cannot be surrounded $n+1$ times (in other words: which positive integers can be the Heesch number of some prototile). The first prototile whose Heesch number is finite and positive had been discovered by Lietzmann [8], forty years before Heesch asked his question. This is the prototile shown at Figure 1; its Heesch number is 1 (that is, it can be proved that the second corona cannot be formed not only after the first corona from Figure 1, but after any first corona). Heesch himself provided another example of a prototile whose Heesch number is 1, which has the shape of a convex pentagon. The first example of a prototile whose Heesch number is finite and greater than 1 was discovered only in 1991 [3] (in particular, a family of tiles was constructed, each having the Heesch number 2). The largest currently known finite Heesch number is 5 [9], achieved for the prototile shown at Figure 2, where n is any positive integer greater than 3.

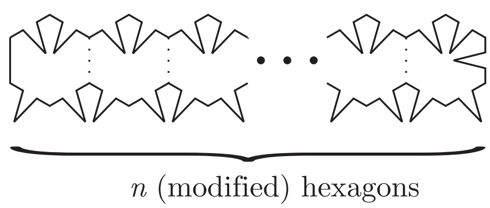


Figure 2: The so-called *n-hexapillar*.

Not much is known for protosets consisting of more than one prototile. In [5, Section 3.8], Grünbaum and Shephard conjectured that, in our terminology, for any positive integer k the set of all finite Heesch numbers of protosets consisting of k prototiles is bounded from above. Mann [10] recalled this conjecture and expressed his belief that it might actually be false, but apart from a few results in the case $k = 1$, not much progress has been made. In this paper we disprove the conjecture by showing that, for each $k \geqslant 3$, the Heesch number is unbounded.

Most of the recent works on Heesch number utilize the approach by "bumps" and "nicks" (that is, marking the edges of some "nice" polygons by matching bumps and nicks, thus forbidding some possible configurations of tiles; see again Figure 2 for an example). Although this approach proved to be very fruitful, it seems that its potential has come close to its limits and that some modifications are necessary in order to obtain some new results on this topic. By marking edges of a certain polygon by bumps (and matching nicks) of different shapes (where each bump can be matched only with a nick of the corresponding shape), Tarasov [13] has recently managed to elegantly prove that, in the hyperbolic plane, the Heesch number for protosets consisting of one prototile is unbounded.

Here we make use of prototiles that are merely rectangles with some bumps and nicks of *asymmetric* shape on their sides. This enables us to disprove the conjecture of Grünbaum and Shephard in a surprisingly simple way.

Note. Following the definition from the Introduction, it is possible that a given protoset does not admit any tiling, although some of its subsets does. In other words, in order for a protoset $\mathscr{P}$ to admit some tiling, there has to exist a tiling $\mathscr{T}$ such that not only each tile from $\mathscr{T}$ is congruent to a prototile from $\mathscr{P}$, but also for each prototile from $\mathscr{P}$ a congruent copy of it appears at least once among the tiles from $\mathscr{T}$. This definition follows [5] and in fact perfectly agrees with the conjecture of Grünbaum and Shephard, since their idea is that, given a protoset, we can get "close" to a tiling if *each* prototile can be surrounded many times (and hence, if a protoset $\mathscr{P}$ contains one "bad" prototile B, which cannot be surrounded, then because solely of the prototile B we shall think of $\mathscr{P}$ as being far from admit a tiling, even if, say, $\mathscr{P} \setminus \{B\}$ admits a tiling). Some may argue that it would be more natural to define that a protoset $\mathscr{P}$ admits some tiling if there exists a tiling in which each tile is congruent to a prototile from $\mathscr{P}$, without further ado. The analogue of the conjecture of Grünbaum and Shephard that would agree with this definition would then be based around the maximal integer n such that *at least one* prototile from $\mathscr{P}$ can be surrounded n times but not $n+1$ times, but this is completely another question (which, in fact, has also been asked in the literature [4, Question 4.1(a)]—where the Heesch number of a protoset is defined differently, more appropriately for that purpose), and it seems that not much from the present paper can be applied in that case.

3 Proof of the unboundedness

We first treat the case $k = 3$.

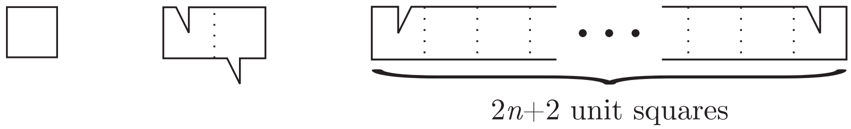


Figure 3: The prototiles S, D and L.

Theorem 3.1. *For any integer $n \geqslant 1$, there exists a protoset consisting of 3 prototiles whose Heesch number is n.*

Proof. Let $n \geqslant 1$ be given. Let $\mathscr{P} = \{S, D, L\}$, where S, D and L are the prototiles shown in Figure 3, from left to right (S, D and L stand for "square", "domino" and "long", respectively). Let us describe these prototiles. Prototile S is a unit square. Prototile D is comprised of two unit squares, with asymmetric, matching bump and nick added as in Figure 3. Prototile L is a $(2n+2) \times 1$ rectangle with two nicks, which are mirror images of each other, added at the first and the last square as in Figure 3 (there are no bumps on this prototile). Of course, the nicks on prototile L match the bump on prototile D.

We claim that the Heesch number of $\mathscr{P}$ equals n. Let us first notice that prototiles S and D tile the plane easily, and thus they can be surrounded infinitely many times. Further, Figure 4 shows how prototile L can be surrounded n times (Figure 4 shows the case $n = 3$, but the generalization for any given n is obvious).

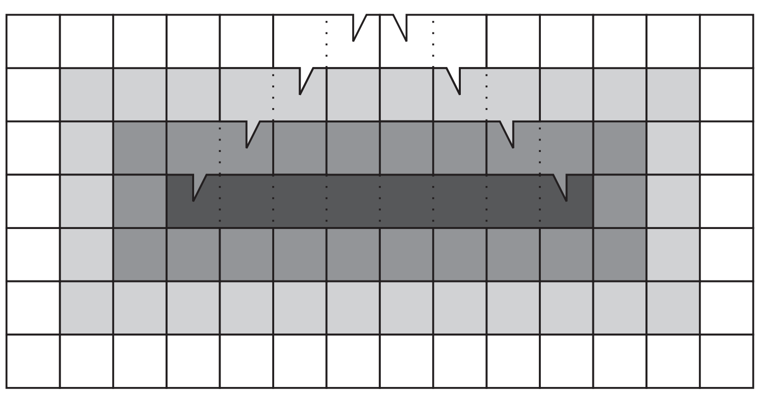


Figure 4: The prototile L surrounded n times.

Therefore, we are left to prove that at least one of the prototiles S, D or L cannot be surrounded more than n times. We shall show this for prototile

L (of course, since the other two prototiles can be surrounded infinitely many times, as we have already noted).

Consider the two nicks on prototile L. Note that there is a total of $2n$ unit squares between those two unit squares that have nicks on them (this will be needed later). Since the only prototile from $\mathscr{P}$ that has a matching bump is prototile D, the two nicks on L have to be "filled" by two bumps from two copies of D. Because of the asymmetry of nicks, the position of these two copies of D is uniquely determined. Also note that, since these two copies of D have common boundary points with the "zeroth corona" (the original tile L), they must belong to the first corona. Finally, note that these two copies of D now contribute a "new" nick each, and that between the unit squares that have these two nicks on them there is a distance of $2n-2$ unit squares (that is, 2 unit squares less than it was at the original tile L). Everything being said here is shown in Figure 5.

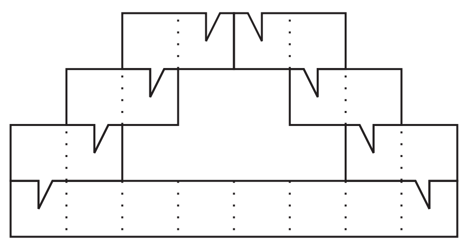


Figure 5: Placements of all these tiles are uniquely determined.

In the same way, we conclude that the two "new" nicks force the unique placement of another two copies of D. Since they have common boundary points with the first corona, they must belong to the second corona at most. Also note that they contribute another two nicks to the configuration obtained so far, and the distance between the unit squares that have these nicks on them is $2n-4$ unit squares. Repeating the process, we get that at the i^{th} step we add two copies of D that belong to the i^{th} corona at most, and the distance between their nicked squares equals $2n-2i$ unit squares. This implies that, after n steps, after adding two copies of D belonging to the n^{th} corona at most, we have two neighboring unit squares with nicks (see Figure 5 again). However, there is no way to "fill" these nicks simultaneously, which implies that no more than n coronas can be formed. This completes the proof. ∎

It is now easy to modify the above proof for any $k \geqslant 3$.

Corollary 3.2. *For any integers $k \geqslant 3$ and $n \geqslant 1$, there exists a protoset consisting of k prototiles whose Heesch number is n.*

Proof. Let $k \geqslant 3$ and $n \geqslant 1$ be given. Let $\mathscr{P}_k = \{S, D, L_1, L_2, \dots, L_{k-2}\}$, where S and D are prototiles from the previous proof, and $L_1, L_2, \dots, L_{k-2}$ are $k-2$ prototiles obtained by dividing the prototile L from the previous proof (corresponding to the given n) into $k-2$ parts in an "irregular enough" way (by "irregular enough" we mean that the placement of any prototile among $L_1, L_2, \dots, L_{k-2}$ in the plane uniquely forces placements of the other $k-3$ prototiles in such a way that they altogether form the prototile L; see Figure 6). In the same way as in the previous proof, we show that the Heesch number of $\mathscr{P}_k$ equals n. ■

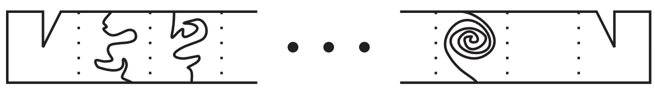


Figure 6: Dividing the prototile L in an "irregular enough" way.

Acknowledgments

The research was supported by the Ministry of Science and Technological Development of Serbia (project 174006) and by the Provincial Secretariat for Science and Technological Development, Autonomous Province of Vojvodina (project "Ordered structures and applications").

References

[1] L. Adleman & L. Kari & J. Kari & D. Reishus, On the decidability of self-assembly of infinite ribbons, in: *Proceedings of the 43rd Symposium on Foundations of Computer Science (FOCS'02)*, 2002, pp. 530–537.

[2] F. Ardila & R. P. Stanley, Tilings, *Math. Intelligencer* **32** (2010), 32–43.

[3] A. Fontaine, An infinite number of plane figures with Heesch number two, *J. Combin. Theory Ser. A* **57** (1991), 151–156.

[4] C. Goodman-Strauss, *Open Questions in Tiling*, unpublished.

[5] B. Grünbaum & G. C. Shephard, *Tilings and patterns*, W. H. Freeman and Company, New York, 1987.

[6] H. Heesch, *Reguläres Parkettierungsproblem*, Westdeutscher Verlag, Cologne, 1968.

[7] C. S. Kaplan, *Introductory Tiling Theory for Computer Graphics*, Morgan & Claypool Publishers, 2009.

[8] W. Lietzmann, *Lustiges und Merkwürdiges von Zahlen und Formen*, Hirt, Breslau, 1928.

[9] C. Mann, *Heesch's Problem and Other Tiling Problems*, PhD thesis, University of Arkansas, 2001.

[10] C. Mann, Heesch's Tiling Problem, *Amer. Math. Monthly* **111** (2004), 509–517.

[11] M. Senechal, *Quasicrystals and geometry*, Cambridge University Press, Cambridge, 1995.

[12] S. K. Stein & S. Szabó, *Algebra and Tiling: Homomorphisms in the Service of Geometry*, Mathematical Association of America, Washington, DC, 1994.

[13] A. S. Tarasov, On the Heesch number for the Lobachevskiĭ plane, *Mat. Zametki* **88** (2010), 97–104 (Russian); English translation in *Math. Notes* **88** (2010), 97–102.

[14] W. P. Thurston, *Groups, tilings and finite state automata*, Summer 1989 AMS Colloquium Lectures, Res. Rep., Geometry Computing Group, Minneapolis, MN, 1989.