On highly palindromic words: the ternary case

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Abstract

A word over an n-ary alphabet is called *minimal-palindromic* if it does not contain palindromic subwords whose length is greater than $\lceil \frac{|w|}{n} \rceil$ (note that each n-ary word must contain a palindromic subword of at least that length: for example, a subword consisting of a prevalent letter, which explains the term "minimal-palindromic"). The *MP-ratio* of a given word w is defined as the quotient $\frac{|rws|}{|w|}$, where r and s are (possibly empty) words such that the word rws is minimal-palindromic and that the length $|r| + |s|$ is minimal possible. We show that the MP-ratio is well-defined in the ternary case (that is, that such words r and s always exist), as well as that it is bounded from above by the constant 6 and that 6 is the best possible upper bound.

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1 Introduction

In recent times, various questions that deal with structural properties of finite and infinite words and that are based on the notion of palindromes are a very active research field. We mention a few examples. Frid, Puzynina and Zamboni [14] defined the notion of a *palindromic length* of a finite word as the least number of palindromes whose concatenation is the given word; see [21, 8, 13] for some results on this topic, and see also [2, 1, 4, 9], where some variants of this concept have been introduced and studied. Another research direction is based on the result of Droubay, Justin and Pirillo [12], who proved that a word of length n can have at most $n+1$ different palindromic factors. The difference between this upper bound and the number of different palindromic factors of a given word w is called the *palindromic defect* of w; this proved to be a very fruitful research topic [15, 19, 6, 17, 22, 20, 23], where special attention has been paid to words whose palindromic defect is zero, called *rich* (the intuition being that such words are "rich" in palindromes).

Holub and Saari [18] introduced yet another way to measure how "rich" in palindromes a given word is, the so-called *MP-ratio*. MP-ratio is a rational number greater than or equal to 1 (a precise definition will be given in our Section 3) such that, the greater MP-ratio is, the given word is "richer" in palindromes (the authors of [18] say that such words are "highly palindromic"); those words whose MP-ratio equals 1 are called *minimal-palindromic*. It turns out that some properties of MP-ratio are not so easy to grasp, since, as shown in [5], it can behave in a quite unpredictable way. The concept of MP-ratio is based on palindromic *subwords* (and not factors) of a given word, which have been noticeably less considered in the literature. They, however, have some interesting properties. As shown in [18], a binary word can be reconstructed, up to reversal, from the set of its palindromic subwords. Also in [18], a property of a word being *abelian bordered* is defined, and it is shown that each binary minimal-palindromic word is abelian unbordered (which is a strong form of unborderedness); abelian (un)borderedness of words has attracted a growing attention in recent times [11, 16, 10, 3, 7]. However, the main drawback of the notion of MP-ratio is the fact that it is defined only for binary alphabet. Though there is a natural analogous way to extend the definition of MP-ratio to a larger alphabet, it is not clear whether in that case the notion is well-defined at all. For that reason, the authors of [18] left the question of well-definedness of MP-ratio for larger alphabets as an open problem.

In this article we solve that question for ternary alphabet. We show that the MP-ratio is well-defined in the ternary case, that it is bounded from above by the constant 6, and that this bound is the best possible (in the binary case, the best possible upper bound is 4).

2 Preliminaries

In this section we recall basic definitions and properties that will be needed through the article.

A *word* is a finite sequence of symbols taken from a nonempty finite set Σ, which is called the *alphabet*, and its elements are called *letters*. In the case $|\Sigma| = 2$ we speak about *binary* words, in the case $|\Sigma| = 3$ we speak about *ternary* words and, generally, in the case $|\Sigma| = n$ we speak about n*-ary* words. If $w = a_1 a_2 ... a_n$ with $a_1, a_2, \ldots, a_n \in \Sigma$, we say that the *length* of w is n, and write $|w| = n$. The unique word of length 0, called the *empty word*, is denoted by ε. The *concatenation* (or *product*) of words u and v, $u = a_1 a_2 \ldots a_n$ and $v = b_1 b_2 \ldots b_m$, is the word $a_1 a_2 \ldots a_n b_1 b_2 \ldots b_m$, denoted by uv. For a word w and a positive integer k we write w^k for the word $\underbrace{ww \ldots w}_{k}$. If $A \subseteq \Sigma$, we write A^* for the set $\{a_1 a_2 \ldots a_k : k \geqslant 0 \text{ and } a_i \in A \text{ for each } i\}$, and we write $A^+ = A^* \backslash \{\varepsilon\}$. If the set A has only one element, say $A = \{a\}$, we write a^* and a^+ instead of $\{a\}^*$ and $\{a\}^+$. If A and B are two sets of words, we write $AB = \{uv : u \in A, v \in B\}$. Since concatenation of words is an associative operation, the product of more than two sets of words is also well-defined.

A word $u \in \Sigma^*$ is called a *factor* (respectively *prefix*, *suffix*) of a word $w \in \Sigma^*$ if there exist words $x, y \in \Sigma^*$ such that $w = xuy$ (respectively $w = uy$, $w = xu$). A word $u \in \Sigma^*$ is a *subword* of $w \in \Sigma^*$ if there exist words $x_1, x_2, \ldots, x_n, x_{n+1} \in \Sigma^*$ and $y_1, y_2, \ldots, y_n \in \Sigma^*$ such that $u = y_1 y_2 \ldots y_n$ and $w = x_1 y_1 x_2 y_2 \ldots x_n y_n x_{n+1}$ (or, equivalently, u is a subword of w if u is its subsequence). The set of all factors (respectively prefixes, suffixes, subwords) of a word w is denoted by $\mathrm{Fact}(w)$ (respectively $\mathrm{Pref}(w)$, $\mathrm{Suff}(w)$, $\mathrm{Subw}(w)$).

We write $w[i]$ for the i^{th} letter of the word w, and for any pair (i, j) of integers such that $1 \leqslant i \leqslant j \leqslant |w|$ we write $w[i, j]$ for the factor of w that begins at the i^{th} position in w and ends at the j^{th} position in w (obviously, $w[i, i] = w[i]$). In the case $i > j$, as well as $i > |w|$ or $j < 1$, we define $w[i, j] = \varepsilon$. By convention, this operation has precedence over concatenation; in other words, $uv[i]$ (and similarly $uv[i, j]$) will always denote $u(v[i])$, not $(uv)[i]$.

If i and j are positive integers and $i \leqslant j$, $[i, j]_{\mathbb{N}}$ denotes the set $\{i, i + 1, i + 2, \ldots, j\}$.

For words u and v, we write $|u|_v$ for the number of distinct occurrences of v in u, that is:

$$|u|_v = |\{i : 1 \leqslant i \leqslant |u| - |v| + 1, u[i, i + |v| - 1] = v\}|.$$

We say that a letter c is *prevalent* in a word w if $|w|_c = \max\{|w|_a : a \in \Sigma\}$. (Note that a prevalent letter is not necessarily unique.)

We define the map $\widetilde{\ }: \Sigma^* \to \Sigma^*$, called *reversal*, as follows: if $w = a_1a_2\ldots a_n$, where $a_1, a_2, \ldots, a_n \in \Sigma$, then $\widetilde{w} = a_na_{n-1}\ldots a_1$. A word w is a *palindrome* (or *palindromic*) if $w = \widetilde{w}$. (The empty word is also a palindrome.) A palindromic subword of a given word will be called a *subpalindrome*.

3 MP-ratio

Clearly, each binary word w contains a subpalindrome of length at least $\lceil\frac{|w|}{2}\rceil$ (e.g., a subpalindrome consisting only of a prevalent letter of w). We say that a binary word w is *minimal-palindromic* if it does not contain a subpalindrome longer than $\lceil\frac{|w|}{2}\rceil$. For $w \in \{0,1\}^*$, a pair (r,s), where $r,s \in \{0,1\}^*$, such that rws is minimal-palindromic, is called an *MP-extension* of w, and if the length $|r|+|s|$ is the least possible, then the pair (r,s) is called a *shortest MP-extension*, or *SMP-extension* of w. The rational number $\frac{|rws|}{|w|}$, where (r,s) is an SMP-extension of w, is called the *MP-ratio* of w. As shown in [18], each binary word possesses an MP-extension (and thus also an SMP-extension, that is, the MP-ratio is well-defined); further, the MP-ratio of any binary word is bounded from above by 4, and this is the best possible upper bound.

Consider now the n-ary alphabet $\Sigma = \{0,1,\ldots,n-1\}$. Clearly, each $w \in \Sigma^*$ contains a subpalindrome of length at least $\lceil\frac{|w|}{n}\rceil$. Therefore, it is natural to say that a word $w \in \Sigma^*$ is *minimal-palindromic* if it does not contain a subpalindrome longer than $\lceil\frac{|w|}{n}\rceil$. For a word $w \in \Sigma^*$, a pair (r,s), where $r,s \in \Sigma^*$, such that rws is minimal-palindromic, is called an *MP-extension* of w, and we define an SMP-extension and the MP-ratio in the same way as in the binary case. However, as mentioned in the Introduction, in case of an arity greater than 2, it is not clear whether an MP-extension always exists, and thus whether the MP-ratio is well-defined. In this article we prove that this is true for ternary alphabet.

We first show an easy proposition that will be useful later.

Proposition 3.1. *Let $w \in \{0,1,2\}^*$, and let (r,s) be an SMP-extension of w and $|rs| \geqslant 2$. Then $|rws| = 3k-2$ for some positive integer k, and the values $|w|_0, |w|_1, |w|_2$ are (in some permutation) either $k-1, k-1, k$ or $k-2, k, k$.*

Proof. Suppose the contrary: (r,s) is an SMP-extension of w, $|rs| \geqslant 2$ and $|rws| = 3k-1$ (respectively $|rws| = 3k$). Let $r's'$ denote the word obtained by erasing any letter (respectively any two letters) from rs (where r' is a subword of r and s' of s). Clearly, the length of a longest subpalindrome of $r'ws'$ is not greater than the length of a longest subpalindrome of rws, which is at most $\lceil \frac{|rws|}{3} \rceil$. Since $\lceil \frac{|r'ws'|}{3} \rceil = \lceil \frac{3k-2}{3} \rceil = k = \lceil \frac{|rws|}{3} \rceil$, we conclude that (r', s') is an MP-extension, and $|r'| + |s'| < |r| + |s|$, a contradiction.

Therefore, we now know that $|rws| = 3k-2$. Let us show the second part of the statement. Let c be a prevalent letter in rws. Since $\lceil \frac{|rws|}{3} \rceil = \lceil \frac{3k-2}{3} \rceil = k$ and rws is minimal-palindromic, we have $|rws|_c \leqslant k$. If $|rws|_c < k$, then $|rws| \leqslant 3(k-1) < 3k-2$ would follow, which is a contradiction. Therefore, the only possibility is $|rws|_c = k$. If a prevalent letter is unique, then we see that each of the other two letters has to occur exactly $k-1$ times, while if there are two prevalent letters (both occurring k times), then the third letter has to occur $k-2$ times. ■

The rest of the article is organized as follows. In Section 4 we show that there always exists an MP-extension (r, s) of any ternary word w; in fact, since for our construction holds $|rws| = 6|w|$, we get that the MP-ratio is bounded from above by 6. During the course of the proof, two technical results are needed, and they are given as appendices in Sections 6 and 7 (where Section 6 is self-contained, and Section 7 relies only on Section 6; thus we believe that this will not cause confusion to the reader); further, those two results are essentially results on *binary* words (and there might be a slim chance that they could be also useful somewhere else), which again makes it natural to give them separated from the proof from Section 4. In Section 5 we show that the MP-ratio can be arbitrarily close to the constant 6, which gives that 6 is the best possible upper bound on the MP-ratio in the ternary case.

4 An upper bound on the MP-ratio in the ternary case

Our aim is in this section to show that the MP-ratio of any ternary word w is at most 6. We fix the alphabet $\Sigma = \{0, 1, 2\}$.

The following functions will be needed. For $w \in \Sigma^*$ and $a, b \in \Sigma$, let

$$\gamma(w, a, b) = \min \left\{ 2\big|w[i, |w|]\big|_a - \big|w[i, |w|]\big|_b : i = 1, 2, \ldots, |w| + 1 \right\},$$

and let

$$g(w, a, b) = \max \left\{ 2\big|w[i, |w|]\big|_a - \big|w[i, |w|]\big|_b : i = 1, 2, \ldots, |w| + 1 \right\}.$$

Further, let $j(a,w)$ denote the position of the last occurrence of a in w (that is, $w[j(a,w)] = a$ and $w[k] \neq a$ for each k, $k > j(a,w)$), and $j(a,w) = 0$ if a does not occur in w. We define

$$g'(w,a,b) = \max\left(\left\{2\big|w[i,|w|]\big|_a - \big|w[i,|w|]\big|_b : i = 1,2,\dots,j(a,w)\right\} \cup \{0\}\right).$$

We first show two easy properties of these functions.

Lemma 4.1. *Let w be a finite word and let a and b be two distinct letters. Then:*

a) $g'(w,a,b) \leqslant g(w,a,b)$;

b) $\gamma(w,a,b) + g(\widetilde{w},a,b) = g(w,a,b) + \gamma(\widetilde{w},a,b) = 2|w|_a - |w|_b$.

Proof. a) Follows from the definitions of g and g'.

b) We first show that for each i, $1 \leqslant i \leqslant |w|+1$, we have the equality

$$\left(2\big|w[i,|w|]\big|_a - \big|w[i,|w|]\big|_b\right) + \left(2\big|\widetilde{w}[|w|-i+2,|w|]\big|_a - \big|\widetilde{w}[|w|-i+2,|w|]\big|_b\right) = 2|w|_a - |w|_b.$$

The equality follows by observing that each occurrence of the letter a is counted in exactly one of the parenthesis, and the same holds for each occurrence of the letter b. Note that, because of this equality, the first parenthesis reaches its minimum exactly when the second parenthesis reaches its maximum, and vice versa. When the first parenthesis reaches its minimum (and the second one its maximum), the expression on the left-hand side becomes $\gamma(w,a,b)+g(\widetilde{w},a,b)$ (by the definition of γ and g); when the first parenthesis reaches its maximum (and the second one its minimum), the expression on the left-hand side becomes $g(w,a,b)+\gamma(\widetilde{w},a,b)$. This proves the lemma. ∎

The following property of the function g is less obvious, but will also be very useful.

Lemma 4.2. *Let $w \in \Sigma^*$, let b be a prevalent letter in w, and let a be a letter distinct from b. We have:*

$$g(w,a,b) + g(\widetilde{w},a,b) \leqslant 3|w|_a.$$

Proof. First, we have the following sequence of equivalences (where Lemma 4.1b) is used in the first step):

$$\begin{aligned} g(w,a,b)+g(\widetilde{w},a,b) \leqslant 3|w|_a & \qquad \text{if and only if} \\ g(w,a,b)-\gamma(w,a,b)+2|w|_a-|w|_b \leqslant 3|w|_a & \qquad \text{if and only if} \\ g(w,a,b)-\gamma(w,a,b) \leqslant |w|_a+|w|_b. & \end{aligned}$$

Therefore, it is enough to show that $g(w,a,b)-\gamma(w,a,b) \leqslant |w|_a+|w|_b$.

Now, let K, respectively k, where $1 \leqslant K,k \leqslant |w|+1$, denote the value of i for which the expression

$$2\big|w[i,|w|]\big|_a - \big|w[i,|w|]\big|_b$$

reaches its maximal, respectively minimal, value. In other words,

$$g(w,a,b) = 2\big|w[K,|w|]\big|_a - \big|w[K,|w|]\big|_b$$

and

$$\gamma(w,a,b) = 2\big|w[k,|w|]\big|_a - \big|w[k,|w|]\big|_b.$$

We distinguish two cases depending on which one of k and K is greater, and show that in both cases the expected inequality holds.

Let first $K \leqslant k$. Now, let i transition gradually from K to k, and we monitor changes in the value $2\big|w[i,|w|]\big|_a - \big|w[i,|w|]\big|_b$. If $w[i]=a$, then the value of the expression $2\big|w[i,|w|]\big|_a - \big|w[i,|w|]\big|_b$ in the next step will decrease by 2 (in comparison to the current value); if $w[i]=b$ then the considered value will increase by 1; if $w[i] \notin \{a,b\}$, then the considered value will not change. Since $g(w,a,b) \geqslant \gamma(w,a,b)$, we conclude that the difference between them is at most twice the number of letters a in the factor $w[K,k-1]$ (that is, the maximum is reached when the considered value constantly decreases during the described process). Now we have:

$$g(w,a,b)-\gamma(w,a,b) \leqslant 2|w[K,k-1]|_a \leqslant 2|w|_a \leqslant |w|_a+|w|_b$$

(where the last inequality holds because of the assumption that b is a prevalent letter in w).

Let now $k \leqslant K$. In a similar manner as in the previous paragraph, we get that in this case the difference between $g(w,a,b)$ and $\gamma(w,a,b)$ is at most the number of letters b in the factor $w[k,K-1]$. Therefore, in this case we have:

$$g(w,a,b)-\gamma(w,a,b) \leqslant |w[k,K-1]|_b \leqslant |w|_b \leqslant |w|_a+|w|_b.$$

This completes the proof. ■

Now we are ready to construct an MP-extension of a given word w. For the rest of this section, without loss of generality, we assume $|w|_0 \leqslant |w|_1 \leqslant |w|_2$. We shall describe two extensions of the word w, denoted by $f(w)$ and $f'(w)$, and show that at least one of them is an MP-extension. Those two extensions are:

$$\begin{aligned} f(w) &= 0^{2|w|-|w|_0} 2^{2|w|-|w|_2-g'(w,0,2)} \;\; w \;\; 2^{g'(w,0,2)} 1^{2|w|-|w|_1}; \\ f'(w) &= 1^{2|w|-|w|_1} 2^{g'(\widetilde{w},0,2)} \;\; w \;\; 2^{2|w|-|w|_2-g'(\widetilde{w},0,2)} 0^{2|w|-|w|_0}. \end{aligned}$$

Note that $f'(w) = \widetilde{f(\widetilde{w})}$. By r and s, respectively r' and s', we shall refer to the prefix and the suffix attached to w in $f(w)$, respectively $f'(w)$.

In other words, the letters 1 and 0 are piled up at the ends, and the letter 2 is arranged around w in an asymmetric way. We shall later need a more precise "quantification" of this asymmetry, so let us show that

$$(2|w| - |w|_2 - g'(w,0,2)) - g'(w,0,2) \geqslant |w|_2 \tag{1}$$

(and the same holds with $\widetilde{w}$ in place of w), which reduces to

$$g'(w,0,2) + |w|_2 \leqslant |w|.$$

And indeed:

$$g'(w,0,2) + |w|_2 \leqslant 2|w|_0 + |w|_2 \leqslant |w|_0 + |w|_1 + |w|_2 = |w|,$$

which was to be proved.

Note. The presented construction is not the only one possible. Another possibility is to use the function g in place of g' (or any intermediate value), and the proof in that case is completely the same. We decided to present the version with g' because that version is exactly a "borderline" case in the sense that the letters 2 are arranged in the "mostly asymmetric" way possible; in other words, by transferring only one letter 2 from the "smaller pile" to the "larger pile" we would not have an MP-extension anymore.

As already announced, we claim that at least one of the pairs (r, s) and (r', s') represents an MP-extension of w; that is, at least one of the words $f(w)$ and $f'(w)$ does not have subpalindromes whose length exceeds $2|w|$ (having in mind that $|f(w)| = |f'(w)| = 6|w|$). The proof consists of a number of intermediate assertions.

Lemma 4.3. *The length of an arbitrary subpalindrome of the form* $0p0$ *in each of the words* $f(w)$ *and* $f'(w)$ *is less than or equal to* $2|w|$.

Proof. Without loss of generality, we prove the assertion for the word $f(w)$. (This indeed does not affect the generality: if we prove the claim for $f(w)$ for each w, then it also holds for each $f(\widetilde{w})$, and now we only need to recall the equality $f'(w) = \widetilde{f(\widetilde{w})}$ and the fact that the claimed property remains true for $\widetilde{f(\widetilde{w})}$ if it is true for $f(\widetilde{w})$.) Each subword of $f(w)$ of the form $0p0$ must be a subword of

$$rw = 0^{2|w|-|w|_0}2^{2|w|-|w|_2-g'(w,0,2)}w,$$

because s obviously does not contain the letter 0.

If at least $\frac{|0p0|}{2}$ letters from w participate in the palindrome $0p0$ (which means: $0p0$ is a subword of rw obtained by selecting at least $\frac{|0p0|}{2}$ letters from w, while the rest of the letters are selected from r), then, clearly, $|0p0| \leqslant 2|w|$, which was to be proved. Assume now that more than $\frac{|0p0|}{2}$ letters from r participate in the palindrome $0p0$ (it must be so if the assumption from the previous sentence is not true). Then, clearly, $0p0 \in 0^*2^*0^*$.

If $0p0 \in 0^*$, then we immediately have

$$|0p0| \leqslant |rw|_0 = (2|w| - |w|_0) + |w|_0 = 2|w|,$$

which was to be proved. Therefore, it remains to check the case $0p0 \in 0^*2^+0^*$. Note that then there exists a position i in the word w such that among the letters at the positions $1, 2, \ldots, i-1$, respectively $i, i+1, \ldots, |w|$, only the letters 2, respectively the letters 0, can participate in the palindrome $0p0$. Hence, there can be at most $\big|w[i, |w|]\big|_0$ zeros at the end of $0p0$, and therefore also at the beginning. Altogether, we conclude $|0p0| \leqslant |r|_2 + \big(|w|_2 - \big|w[i, |w|]\big|_2\big) + 2\big|w[i, |w|]\big|_0$. Since $0p0$ ends with 0, we have that i is at most the position of the rightmost letter 0 in w; this gives that the expression from the previous sentence is bounded from above by $|r|_2 + |w|_2 + g'(w, 0, 2)$ (by the definition of g'). In other words, we again have

$$\begin{aligned} |0p0| &\leqslant |r|_2 + |w|_2 + g'(w,0,2) \\ &= (2|w| - |w|_2 - g'(w,0,2)) + |w|_2 + g'(w,0,2) = 2|w|, \end{aligned}$$

which completes the proof. ■

Lemma 4.4. *The length of an arbitrary subpalindrome of the form* $1p1$ *in each of the words* $f(w)$ *and* $f'(w)$ *is less than or equal to* $2|w|$.

Proof. We again prove the assertion only for the word $f(w)$. We may assume $1p1 \in 1^*2^+1^*$ (everything else can be dealt with in a completely analogous way like in Lemma 4.3). Then we can write the palindrome $1p1$ in the form $1p_wp_2p_11$, where $1p_w \in \mathrm{Subw}(w)$, $p_2 \in 2^*$ and $p_11 \in 1^*$. Since there are at most $|w|_1$ letters 1 to the left of p_2, we conclude $|p_11| \leqslant |w|_1$. Now we have

$$\begin{aligned}|1p1| = |1p_wp_2p_11| &= |1p_w| + |p_2| + |p_11| \leqslant |w| + g'(w,0,2) + |w|_1 \\ &\leqslant |w| + 2|w|_0 + |w|_1 \leqslant |w| + |w|_0 + |w|_2 + |w|_1 \\ &= 2|w|,\end{aligned}$$

which completes the proof. ■

Lemma 4.5. *Let p and q be two nonempty subpalindromes of w. Let w_p, v, w_q and t be such that $w = w_pv = tw_q$, p is a subword of w_p, and q is a subword of w_q. Then*

$$|p| + 2|v|_2 + |q| + 2|t|_2 \leqslant 4|w|_2 + |w|_1 + |w|_0. \tag{2}$$

Proof. Define the word w', $|w'| = |w|$, in the following way:

$$w'[i] = \begin{cases} 1, & \text{if } w[i] = 0 \text{ or } w[i] = 1; \\ 2, & \text{if } w[i] = 2. \end{cases}$$

We obviously have $|w'|_2 = |w|_2$ and $|w'|_1 = |w|_1 + |w|_0$. By the assumption $|w|_2 \geqslant |w|_1 \geqslant |w|_0$ we get $2|w'|_2 \geqslant |w'|_1$. Similarly, let p', v', q', t', be the words obtained from p, v, q, t, respectively, by replacing all 0s by 1s. Then p' and q' are subpalindromes of the word w', and by applying Theorem 7.1 (formulated and proved later in Section 7) we get

$$|p'| + 2|v'|_2 + |q'| + 2|t'|_2 \leqslant 4|w'|_2 + |w'|_1.$$

Note that the left-hand side is the left-hand side of (2), and the right-hand side is the right-hand side of (2), which proves the lemma. ■

Lemma 4.6. *At least one among the words $f(w)$ and $f'(w)$ does not contain a subpalindrome of the form $2p2$ longer than $2|w|$.*

Proof. Suppose the contrary: in both the words $f(w)$ and $f'(w)$ the length of a longest subpalindrome of the form $2p2$ is greater than $2|w|$. Consider the word $f(w)$. Since $|f(w)|_2 = 2|w|$, such a longest subpalindrome of

$f(w)$ contains a letter different from 2, and thus, by (1), we can write it as $2^{l+|s|_2}p_w2^{l+|s|_2}$ where $p_w = \widetilde{p_w}$ and $p_w2^l \in \mathrm{Subw}(w)$. This palindrome has length $|p_w| + 2l + 2|s|_2$, which equals $|p_w| + 2l + 2g'(w, 0, 2)$. Therefore, the assumption from the beginning reduces to:

$$|p_w| + 2l + 2g'(w, 0, 2) > 2|w|.$$

In a similar manner, considering $f'(w)$, we get:

$$|q_w| + 2l' + 2g'(\widetilde{w}, 0, 2) > 2|w|$$

(where q_w and l' are defined analogously).

Summing the last two inequalities yields:

$$|p_w| + 2l + |q_w| + 2l' + 2g'(w, 0, 2) + 2g'(\widetilde{w}, 0, 2) > 4|w|,$$

which is equivalent to:

$$|p_w| + 2l + |q_w| + 2l' > 4|w| - 2g'(w, 0, 2) - 2g'(\widetilde{w}, 0, 2). \tag{3}$$

Note that the left-hand side of (3) equals the left-hand side of (2), which is, by Lemma 4.5, less than or equal to $4|w|_2+|w|_1+|w|_0$. On the other hand, for the right-hand side of (3) we have:

$$\begin{aligned}
&4|w| - 2g'(w, 0, 2) - 2g'(\widetilde{w}, 0, 2)\\
&\qquad \geqslant 4|w| - 2g(w, 0, 2) - 2g(\widetilde{w}, 0, 2) \geqslant 4|w| - 6|w|_0\\
&\qquad = 4|w|_2 + 4|w|_1 - 2|w|_0 \geqslant 4|w|_2 + |w|_1 + 3|w|_0 - 2|w|_0\\
&\qquad = 4|w|_2 + |w|_1 + |w|_0,
\end{aligned}$$

where we used Lemma 4.1a) and Lemma 4.2. This gives a contradiction, and the lemma is thus proved. ∎

We are now ready for the main theorem of this section.

Theorem 4.7. *The MP-ratio of any ternary word is at most* 6*.*

Proof. The assertion follows directly from Lemmas 4.3, 4.4 and 4.6. ∎

Note. We make no claim that the considered extension is an SMP-extension. In fact, having in mind Proposition 3.1, we see that this is certainly *not* the case; by erasing any two letters from r and s, we would get a shorter MP-extension, which at the same time shows that the MP-ratio of any ternary word is *strictly* less than 6. However, because of the following section, this does not make any crucial difference. We chose to write the proof in the presented way since we felt that it was a little bit easier (from a technical point of view) if each letter in rws had the same number of occurrences. In any case, an MP-extension obtained by erasing two letters from our extension still does not have to be an SMP-extension. The question of constructing an SMP-extension of a given word is much harder, and seems to be far out of reach even in the binary case [5].

5 Optimality of the upper bound

We shall now show that the constant 6 from the previous section is optimal.

The authors in [18] introduced the properties of a binary word being economic and k-economic. We slightly modify their definition to make an appropriate adaptation for the ternary case. We say that a word $w \in \{0,1,2\}^*$ is *k-economic* (with respect to the letter 1) if w is a palindrome and the word $w1^k$ contains a subpalindrome of length at least $|w|_1 + k + 3$. Each such subpalindrome can be written in the form $1^m q 1^m$ where $0 \leqslant m \leqslant k$ and $1^m q \in \mathrm{Subw}(w)$; the pair (q, m) is then called a *k-witness* of w.

We say that w is *economic* if it is k-economic for every k, $k = 0, 1, \ldots, |w|_1$.

The following three lemmas are (more or less) direct adaptations of Lemma 6, Lemma 7 and Lemma 8 from [18].

Lemma 5.1. *Let $w \in \{0,1,2\}^*$, and let (r,s) be an MP-extension of w. If w is economic, then $|rs|_1 > |w|_1$.*

Proof. Suppose the contrary: $|rs|_1 \leqslant |w|_1$. Let $|r|_1 = i$ and $|s|_1 = j$, and assume, without loss of generality, $i \leqslant j$. Since w is economic and $j - i \leqslant |s|_1 \leqslant |rs|_1 \leqslant |w|_1$, it follows that w is $(j-i)$-economic. Therefore, $w1^{j-i}$ contains a subpalindrome of length at least $|w|_1 + j - i + 3$, and that subpalindrome can be written in the form $1^m q 1^m$ for $m \leqslant j - i$ and $1^m q \in \mathrm{Subw}(w)$. But we now have that $1^{m+i} q 1^{m+i}$ is a subpalindrome of rws, and we calculate:

$$\begin{aligned} |1^{m+i} q 1^{m+i}| &= 2i + |1^m q 1^m| \geqslant 2i + |w|_1 + j - i + 3 = |w|_1 + i + j + 3 \\ &= |rws|_1 + 3 > |rws|_1 + 2 \geqslant \left\lceil \frac{|rws|}{3} \right\rceil \end{aligned}$$

(the last inequality follows from Proposition 3.1). Contradiction, since the word rws is minimal-palindromic. This proves the lemma. ■

Lemma 5.2. *Let $w \in \{0,1,2\}^*$, and let (r,s) be an MP-extension of w. If w is economic, then $|rws| > 6|w|_1$.*

Proof. The proof is a straightforward computation that relies on Proposition 3.1 and the previous lemma:

$$\begin{aligned}|rws| &= |rws|_0 + |rws|_1 + |rws|_2 \geqslant 3|rws|_1 - 2 = 3|w|_1 + 3|rs|_1 - 2 \\ &\geqslant 3|w|_1 + 3(|w|_1 + 1) - 2 > 6|w|_1.\end{aligned}$$

■

Lemma 5.3. *Let w_0 be an economic word and let the sequence $(w_i)_{i\geqslant 0}$ be defined recursively by $w_{i+1} = w_i 1^{t_i} w_i$, where $(t_i)_{i\geqslant 0}$ is a given sequence of positive integers. If for each nonnegative integer i we have $t_i < |w_i|_0$, then all the words w_i are economic.*

Proof. We proceed by induction on i. The base is clear (there is nothing to prove for $i = 0$). We now assume that w_i is economic and prove that then w_{i+1} is also economic. We should prove that w_{i+1} is k-economic for each k, $k = 0, 1, \ldots, |w_{i+1}|_1$.

Assume first $0 \leqslant k \leqslant |w_i|_1$. By the inductive assumption, w_i is k-economic. Let (q,m) be a k-witness of w_i. Recall that $m \leqslant k$ and $2m + |q| \geqslant |w_i|_1 + k + 3$. Let

$$p = 1^m q 1^{t_i + m} q 1^m.$$

Since $1^m q \in \mathrm{Subw}(w_i)$ and $1^m \in \mathrm{Subw}(1^k)$, we have $p \in \mathrm{Subw}(w_i 1^{t_i} w_i 1^k) = \mathrm{Subw}(w_{i+1} 1^k)$. Furthermore,

$$\begin{aligned}|p| &= 3m + 2|q| + t_i = 2(2m + |q|) - m + t_i \geqslant 2(|w_i|_1 + k + 3) - k + t_i \\ &= |w_{i+1}|_1 + k + 6 > |w_{i+1}|_1 + k + 3.\end{aligned}$$

This gives that w_{i+1} is k-economic.

Assume now $k = |w_i|_1 + 1$. Then the word w_i is $(k-1)$-economic. Let (q,m) be a $(k-1)$-witness of w_i (now $m \leqslant k-1$). Let (again) $p = 1^m q 1^{t_i+m} q 1^m$. Then $p \in \mathrm{Subw}(w_{i+1} 1^k)$ and

$$\begin{aligned}|p| &= 3m + 2|q| + t_i = 2(2m + |q|) - m + t_i \\ &\geqslant 2(|w_i|_1 + k - 1 + 3) - (k-1) + t_i = |w_{i+1}|_1 + k + 5 > |w_{i+1}|_1 + k + 3;\end{aligned}$$

therefore, w_{i+1} is k-economic.

Let now $|w_i|_1 + 1 < k \leqslant |w_i|_1 + t_i$. Then we write

$$p = 1^k w_i 1^k.$$

Clearly, $p \in \mathrm{Subw}(w_{i+1}1^k)$, and since $t_i + 1 \leqslant |w_i|_0$ and $|w_i|_1 + 2 \leqslant k$, we have

$$|p| = 2k + |w_i|_1 + |w_i|_0 \geqslant k + 2|w_i|_1 + 2 + t_i + 1 = |w_{i+1}|_1 + k + 3,$$

which means that w_{i+1} is k-economic.

Finally, assume $|w_i|_1 + t_i < k \leqslant |w_{i+1}|_1$. Let $j = |w_i|_1 + t_i$ and $l = k - j$. Since $k - j \leqslant |w_i|_1$, we conclude that w_i is l-economic. Let (q, m) be an l-witness of w_i. Write

$$p = 1^{j+m} q 1^{j+m}.$$

Since $1^j \in \mathrm{Subw}(w_i 1^{t_i})$, $1^m q \in \mathrm{Subw}(w_i)$ and $j + m \leqslant k$, we have $p \in \mathrm{Subw}(w_{i+1}1^k)$. Furthermore,

$$|p| = 2j + |1^m q 1^m| \geqslant 2j + |w_i|_1 + l + 3 = |w_{i+1}|_1 + k + 3.$$

Therefore, w_{i+1} is economic also in this case, which completes the proof. ■

We now recall a construction by Holub and Saari that will be useful in our case, too. For a sequence $(t_i)_{i\geqslant 0}$, let $w(t_0, t_1, \ldots, t_{j-1})$ denote the word w_j from the statement of Lemma 5.3, with the initial term $w_0 = 0000$ (we observe that w_0 is economic as a ternary word; indeed, since $|w_0|_1 = 0$, we only have to check whether w_0 is 0-economic, and it clearly is since w_0 itself is a palindrome of length 4). Note that, if the sequence $(t_i)_{i\geqslant 0}$ satisfies $2^i \leqslant t_i < 2^{i+2}$ for each i, then we easily see $t_j < 2^{j+2} = |w(t_0, t_1, \ldots, t_{j-1})|_0$, and thus, by Lemma 5.3, the word $w(t_0, t_1, \ldots, t_{j-1})$ is economic (for each j and for each sequence $(t_i)_{i\geqslant 0}$ satisfying the required property). Holub and Saari proved [18, Lemma 9] that for every large enough integer k (in particular, they proved for $k \geqslant 448$, but the exact bound is of no relevance) there exists a word, say v_k, that can be obtained by the described construction, such that $|v_k| = k$; further, we have

$$\lim_{k\to\infty} \frac{|v_k|_1}{|v_k|} = 1. \tag{4}$$

We now have enough prerequisites to prove the main theorem of this section.

Theorem 5.4. *Let $R(n)$ denote the maximal MP-ratio over all the words $w \in \{0,1,2\}^*$, $|w| = n$. We have*

$$\lim_{n\to\infty} R(n) = 6.$$

Proof. Given a positive real number η, choose an integer k_0 such that, for each $k \geqslant k_0$, we have

$$\frac{|v_k|_1}{|v_k|} > 1 - \frac{\eta}{6}$$

(such k_0 exists because of (4)). Let a pair (r, s) be an MP-extension of v_k, $k \geqslant k_0$. By Lemma 5.2, due to the fact that the word v_k is economic, we have

$$\frac{|rv_ks|}{|v_k|} > \frac{6|v_k|_1}{|v_k|} > 6 - \eta;$$

therefore, the MP-ratio of v_k is greater than $6 - \eta$. This completes the proof. ∎

6 A postponed technical theorem

Theorem 6.1. *Let $u \in \{1,2\}^*$, let $t, v \in 2^*$, and let p and q be subpalindromes of tu and uv, respectively. If*

$$|p| + |q| > 2|u|,$$

then

$$|u|_1 \leqslant \frac{|tv| - 1}{|tv|}|tuv|_2. \tag{5}$$

Before we begin the proof, we shall show that it is enough to prove the theorem in the special case when the subpalindrome p, respectively q, starts (and ends) with t, respectively v. Assume that this case of the theorem is proved. Let now t, u, v, p and q be as in the statement of the theorem, but not satisfying the conditions of the described special case. Let t_0, respectively v_0, be the longest prefix (and suffix) of p, respectively q, that is a subword of t, respectively v; note that $|t_0v_0| < |tv|$. Then p and q are subpalindromes of t_0u and uv_0, respectively, $|p| + |q| > 2|u|$, and t_0, u, v_0, p and q satisfy the condition of the described special case. Since the theorem is assumed to hold in this case, we have

$$|u|_1 \leqslant \frac{|t_0v_0|-1}{|t_0v_0|}|t_0uv_0|_2 < \frac{|tv|-1}{|tv|}|tuv|_2 \tag{6}$$

(where the second inequality follows from $\frac{|t_0v_0|-1}{|t_0v_0|} = 1-\frac{1}{|t_0v_0|} < 1-\frac{1}{|tv|} = \frac{|tv|-1}{|tv|}$ and $|t_0uv_0|_2 < |tuv|_2$); therefore, the theorem holds for t, u, v, p and q.

From now onward we assume that p, respectively q, contains all the letters from t, respectively v.

In the following two subsections we shall give two (very) different proofs of Theorem 6.1. The second proof is (much) shorter than the first one, and many would probably agree that it is also more elegant. However, we feel that the second proof is a neat little "trick" that works almost by a coincidence, while the first proof presents a deep structural analysis and gives some insight into *why* the theorem is true (we actually feel that the first proof is more intuitive than the second one, despite some quite heavy expressions at some places). In case that a result similar to Theorem 6.1 turns out to be needed to deal with (for example) the MP-ratio for alphabets of larger arities, we think that it would not be surprising if the (suitably modified) first proof would then still work, but the second one would not. Therefore, it is our belief that, despite the evident disparity in their lengths, both proofs have their own merits, and thus we decided to present them both.

6.1 First proof

We first define sequences $P_1, P_2, \ldots, P_{|p|}$ and $Q_1, Q_2, \ldots, Q_{|q|}$ such that $1 \leqslant P_1 < P_2 < \cdots < P_{|p|} \leqslant |tu|$ and $|t|+1 \leqslant Q_1 < Q_2 < \cdots < Q_{|q|} \leqslant |tuv|$,

$$p = (tuv)[P_1](tuv)[P_2]\ldots(tuv)[P_{|p|}]$$

and

$$q = (tuv)[Q_1](tuv)[Q_2]\ldots(tuv)[Q_{|q|}].$$

We write $P = \{P_1, P_2, \ldots, P_{|p|}\}$ and $Q = \{Q_1, Q_2, \ldots, Q_{|q|}\}$.

We define $\sigma_P : P \to P$ by $\sigma_P : P_s \mapsto P_{|P|-s+1}$ and $\sigma_Q : Q \to Q$ by $\sigma_Q : Q_s \mapsto Q_{|Q|-s+1}$. Note that σ_P and σ_Q are bijections, their squares are identical mappings.

For $1 \leqslant n \leqslant |t|$, let $\sigma_0(n) = n$ and

$$\sigma_{i+1}(n) = \begin{cases} \sigma_P(\sigma_i(n)), & \text{for } 2 \mid i \text{ and } \sigma_i(n) \in P; \\ \sigma_Q(\sigma_i(n)), & \text{for } 2 \nmid i \text{ and } \sigma_i(n) \in Q; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

In a similar manner, for $|tu| + 1 \leqslant n \leqslant |tuv|$, let $\sigma_0(n) = n$ and

$$\sigma_{i+1}(n) = \begin{cases} \sigma_Q(\sigma_i(n)), & \text{for } 2 \mid i \text{ and } \sigma_i(n) \in Q; \\ \sigma_P(\sigma_i(n)), & \text{for } 2 \nmid i \text{ and } \sigma_i(n) \in P; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

We now show a few properties of the defined notions.

Proposition 6.2. *a) For every $n, m \in Q$ (respectively, $n, m \in P$), if $n < m$, then $\sigma_Q(n) > \sigma_Q(m)$ (respectively, $\sigma_P(n) > \sigma_P(m)$).*

b) We have $\sigma_0(n) > \sigma_2(n) > \sigma_4(n) > \cdots$ and $\sigma_1(n) < \sigma_3(n) < \sigma_5(n) < \cdots$ for $n \geqslant |tu| + 1$, and $\sigma_0(n) < \sigma_2(n) < \sigma_4(n) < \cdots$ and $\sigma_1(n) > \sigma_3(n) > \sigma_5(n) > \cdots$ for $n \leqslant |t|$. (The inequalities are extended as long as the terms are defined.)

c) If one of the following holds:

1) n and m simultaneously belong to the interval $[1, |t|]_{\mathbb{N}}$ or the interval $[|tu| + 1, |tuv|]_{\mathbb{N}}$, and i and j are of the same parity; or

2) n and m are in different intervals and i and j are of opposite parities,

then $\sigma_i(n) = \sigma_j(m)$ implies $n = m$ and $i = j$. (In particular, $\sigma_i(n) = \sigma_j(m)$ is impossible in the case 2).)

d) For each n such that $n \leqslant |t|$ or $n \geqslant |tu|+1$, there exists $z \in \mathbb{N}$ such that $\sigma_z(n)$ is the last defined term in the sequence $\sigma_0(n), \sigma_1(n), \sigma_2(n) \ldots$

Proof. a) Let $n, m \in Q$ and $n < m$. Write $n = Q_s$ and $m = Q_r$. Since $s < r$, we have $|Q| - s + 1 > |Q| - r + 1$, and thus

$$\sigma_Q(n) = \sigma_Q(Q_s) = Q_{|Q|-s+1} > Q_{|Q|-r+1} = \sigma_Q(Q_r) = \sigma_Q(m),$$

which was to be proved. The proof of the claim for $n, m \in P$ and σ_P is analogous.

b) We consider only the case $n \geqslant |tu| + 1$ (the other one is analogous). Since $\sigma_2(n) = \sigma_P(\sigma_Q(n)) \in P$ (assuming, of course, that $\sigma_2(n)$ is defined), we have

$$\sigma_2(n) \leqslant |tu| < |tu| + 1 \leqslant n = \sigma_0(n).$$

Iteratively applying a), we get

$$\sigma_3(n) = \sigma_Q(\sigma_2(n)) > \sigma_Q(\sigma_0(n)) = \sigma_1(n),$$

then

$$\sigma_4(n) = \sigma_P(\sigma_3(n)) < \sigma_P(\sigma_1(n)) = \sigma_2(n)$$

etc., which was to be proved.

c) Let $\sigma_i(n) = \sigma_j(m)$. Assume first that 1) holds. Without loss of generality, let $m, n \geqslant |tu| + 1$ (the case $m, n \leqslant |t|$ is analogous), and let $i \geqslant j$. If $i > j = 0$, then $2 \mid i$, and we have $\sigma_P(\sigma_{i-1}(n)) = \sigma_i(n) = \sigma_j(m) = m$, which is impossible since the left-hand side is in P, and thus no greater than $|tu|$, while $m \geqslant |tu| + 1$. Therefore, if $j = 0$, then $i = 0$, and we then immediately have $n = m$, which was to be proved. Assume now $i \geqslant j > 0$. If i and j are even, then $\sigma_P(\sigma_{i-1}(n)) = \sigma_i(n) = \sigma_j(m) = \sigma_P(\sigma_{j-1}(m))$, which implies $\sigma_{i-1}(n) = \sigma_{j-1}(m)$; if i and j are odd, we get the same conclusion in a similar manner. Iterating this, we obtain $\sigma_{i-j}(n) = \sigma_0(m)$. By the previous case, we get $n = m$ and $i - j = 0$, that is, $i = j$, which was to be proved.

Assume now that 2) holds and let $i > j$. In the same way as in the previous paragraph we conclude that $\sigma_i(n) = \sigma_j(m)$ implies $\sigma_{i-j}(n) = \sigma_0(m) = m$. However, if $|n| \geqslant |tu| + 1$ (and then $m \leqslant |t|$), then, since $2 \nmid i - j$, we have $\sigma_{i-j}(n) = \sigma_Q(\sigma_{i-j-1}(n)) \geqslant |t| + 1$, a contradiction; if $n \leqslant |t|$, we again get a contradiction in a similar manner.

Therefore, we have proved that, under any of the assumptions 1) or 2), $\sigma_i(n) = \sigma_j(m)$ implies $n = m$ and $i = j$.

d) This is a direct consequence of b). ■

The following lemma will be useful.

Lemma 6.3. *a) For each $n \in Q$ such that $\sigma_P(\sigma_Q(n))$ is defined (that is, $\sigma_Q(n) \in P$), we have*

$$\begin{aligned} n - \sigma_P(\sigma_Q(n)) \leqslant 2(|t| + |v|) - 1 \\ &- |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q| \\ &- |[1, \sigma_Q(n)]_{\mathbb{N}} \setminus P| - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P|. \end{aligned}$$

Also, for each $n \in P$ such that $\sigma_Q(\sigma_P(n))$ is defined (that is, $\sigma_P(n) \in Q$), we have

$$\begin{aligned}\sigma_Q(\sigma_P(n)) - n \leqslant 2(|t| + |v|) - 1 \\ &- |[1, n]_{\mathbb{N}} \setminus P| - |[1, \sigma_P(n)]_{\mathbb{N}} \setminus P| \\ &- |[\sigma_P(n), |tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(\sigma_P(n)), |tuv|]_{\mathbb{N}} \setminus Q|.\end{aligned}$$

b) For each $n \in Q$ such that $\sigma_P(\sigma_Q(n))$ is undefined (that is, $\sigma_Q(n) \notin P$), we have

$$n \leqslant 2(|t| + |v|) + |P| - \sigma_Q(n) - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q|.$$

Also, for each $n \in P$ such that $\sigma_Q(\sigma_P(n))$ is undefined (that is, $\sigma_P(n) \notin Q$), we have

$$\begin{aligned}|tuv| + 1 - n \leqslant 2(|t| + |v|) + |Q| - (|tuv| + 1 - \sigma_P(n)) \\ &- |[1, n]_{\mathbb{N}} \setminus P| - |[1, \sigma_P(n)]_{\mathbb{N}} \setminus P|.\end{aligned}$$

Proof. a) We shall prove only the first statement (the second one is analogous).

The following equalities will be used repeatedly: for any $x \in [1, |P|]_{\mathbb{N}}$ we have

$$P_x = x + |[1, P_x]_{\mathbb{N}} \setminus P|,$$

and for any $x \in [1, |Q|]_{\mathbb{N}}$ we have

$$\begin{aligned}Q_x &= |t| + x + |[|t| + 1, Q_x]_{\mathbb{N}} \setminus Q| \\ &= |t| + x + (|[|t| + 1, |tuv|]_{\mathbb{N}} \setminus Q| - |[Q_x, |tuv|]_{\mathbb{N}} \setminus Q|) \\ &= |t| + x + ((|uv| - |Q|) - |[Q_x, |tuv|]_{\mathbb{N}} \setminus Q|) \\ &= |tuv| + x - |Q| - |[Q_x, |tuv|]_{\mathbb{N}} \setminus Q|.\end{aligned}$$

Let us now proceed to the proof. Since $n \in Q$ and $\sigma_Q(n) \in P$, we may write $n = Q_s$ and $\sigma_Q(n) = Q_{|Q|-s+1} = P_r$ (and also $\sigma_P(\sigma_Q(n)) = P_{|P|-r+1}$). We then have:

$$\begin{aligned}n - \sigma_P(\sigma_Q(n)) &= Q_s - P_{|P|-r+1} \\ &= |tuv| + s - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| \\ &\quad - (|P| - r + 1 + |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P|)\end{aligned}$$

$$
\begin{aligned}
&= |tuv| + s - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |P| - 1 \\
&\quad - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P| + (P_r - |[1, P_r]_{\mathbb{N}} \setminus P|) \\
&= |tuv| + s - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |P| - 1 \\
&\quad - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P| + Q_{|Q|-s+1} - |[1, \sigma_Q(n)]_{\mathbb{N}} \setminus P| \\
&= |tuv| + s - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |P| - 1 \\
&\quad - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P| - |[1, \sigma_Q(n)]_{\mathbb{N}} \setminus P| \\
&\quad + (|tuv| + |Q| - s + 1 - |Q| - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q|) \\
&= 2(|t| + |v|) + 2|u| - |P| - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| \\
&\quad - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P| - |[1, \sigma_Q(n)]_{\mathbb{N}} \setminus P| \\
&\quad - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q| \\
&\leqslant 2(|t| + |v|) - 1 - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P| \\
&\quad - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q| - |[1, \sigma_Q(n)]_{\mathbb{N}} \setminus P|,
\end{aligned} \tag{7}
$$

which was to be proved.

b) We shall prove only the first statement (the second one is analogous). In addition to the two equalities from the part a), we shall also use the following one: for any $x \in [1, |Q|]_{\mathbb{N}}$ we have

$$
\begin{aligned}
\sigma_Q(Q_x) = Q_{|Q|-x+1} &= |tuv| + (|Q| - x + 1) - |Q| - |[\sigma_Q(Q_x), |tuv|]_{\mathbb{N}} \setminus Q| \\
&= |tuv| - x + 1 - |[\sigma_Q(Q_x), |tuv|]_{\mathbb{N}} \setminus Q|.
\end{aligned}
$$

Let us now proceed to the proof. Since $n \in Q$, we may write $n = Q_s$. We then have:

$$
\begin{aligned}
n = Q_s &= |tuv| + s - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| \\
&= |tuv| + (|tuv| - \sigma_Q(n) + 1 - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q|) \\
&\quad - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| \\
&= 2(|t| + |v|) + 2|u| + 1 \\
&\quad - |Q| - \sigma_Q(n) - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q| \\
&\leqslant 2(|t| + |v|) + |P| - \sigma_Q(n) - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q|,
\end{aligned}
$$

which was to be proved. ■

Given a number n, $n \leqslant |t|$ or $n \geqslant |tu| + 1$, let $\operatorname{end}(n)$ denote the number z whose existence was shown in Proposition 6.2d). We say that n *dies* if

$|t|+1 \leqslant \sigma_{\mathrm{end}(n)}(n) \leqslant |tu|$. Let n be such that n dies and that $\sigma_{\mathrm{end}(n)}(n) \notin P$. Note that it is impossible for any m, $m \neq n$, to have $\sigma_{\mathrm{end}(m)}(m) = \sigma_{\mathrm{end}(n)}(n)$ (and thus $\sigma_{\mathrm{end}(m)}(m) \notin P$). Indeed, in that case n and m would satisfy one of the conditions 1) or 2) from Proposition 6.2c), which would imply $m = n$, a contradiction. Therefore,

$$\begin{aligned}|\{n : n \text{ dies and } \sigma_{\mathrm{end}(n)}(n) \notin P\}| &\leqslant |[|t|+1, |tu|]_{\mathbb{N}} \setminus P| \\ &= |[1, |tu|]_{\mathbb{N}} \setminus P| = |tu| - |p|.\end{aligned}$$

Analogously, we prove

$$\begin{aligned}|\{n : n \text{ dies and } \sigma_{\mathrm{end}(n)}(n) \notin Q\}| &\leqslant |[|t|+1, |tu|]_{\mathbb{N}} \setminus Q| \\ &= |[|t|+1, |tuv|]_{\mathbb{N}} \setminus Q| = |uv| - |q|.\end{aligned}$$

Therefore, there are at most $|tu| - |p| + |uv| - |q| < |t| + |v|$ numbers n that die. This implies that there exists n, $n \leqslant |t|$ or $n \geqslant |tu|+1$, that does not die. Let n_0 be any such number. Without loss of generality, we may assume $n_0 \geqslant |tu|+1$. By the choice of n_0, we have either $\sigma_{\mathrm{end}(n_0)}(n_0) \leqslant |t|$ or $\sigma_{\mathrm{end}(n_0)}(n_0) \geqslant |tu|+1$.

Lemma 6.4. *Let i, $i \geqslant 0$, be such that $\sigma_{2i+2}(n_0)$ is defined.*

a) For each m, $m \geqslant |tu|+1$, we have one of the following:

- *there exists j such that $\sigma_{2i+2}(n_0) < \sigma_j(m) \leqslant \sigma_{2i}(n_0)$;*
- *$2 \mid \mathrm{end}(m)$ and $\sigma_{\mathrm{end}(m)}(m) > \max\{\sigma_{2i}(n_0), \sigma_{2i+1}(n_0)\}$;*
- *$2 \nmid \mathrm{end}(m)$ and $\sigma_{\mathrm{end}(m)}(m) < \min\{\sigma_{2i+1}(n_0), \sigma_{2i+2}(n_0)\}$.*

Also, for each m, $m \leqslant |t|$, we have one of the following:

- *there exists j such that $\sigma_{2i+2}(n_0) < \sigma_j(m) \leqslant \sigma_{2i}(n_0)$;*
- *$2 \nmid \mathrm{end}(m)$ and $\sigma_{\mathrm{end}(m)}(m) > \max\{\sigma_{2i}(n_0), \sigma_{2i+1}(n_0)\}$;*
- *$2 \mid \mathrm{end}(m)$ and $\sigma_{\mathrm{end}(m)}(m) < \min\{\sigma_{2i+1}(n_0), \sigma_{2i+2}(n_0)\}$.*

b) To each m, $m \leqslant |t|$ or $m \geqslant |tu|+1$, for which there exists j described in a) we can assign one such j in such a way that all the corresponding values $\sigma_j(m)$ are different.

Proof. a) We shall prove only the first assertion (the second one is analogous). Choose the least even j such that $\sigma_j(m) \leqslant \sigma_{2i}(n_0)$, or the least odd j such that $\sigma_j(m) > \sigma_{2i+2}(n_0)$, assuming that there exists j that satisfies either of these two conditions. We claim that in that case we have

$$\sigma_{2i+2}(n_0) < \sigma_j(m) \leqslant \sigma_{2i}(n_0).$$

Assume first that j is even. If $j = 0$, then

$$\sigma_{2i+2}(n_0) = \sigma_P(\sigma_{2i+1}(n_0)) \leqslant |tu| < m = \sigma_j(m),$$

which was to be proved. If $j > 0$, then $\sigma_{2i}(n_0) < \sigma_{j-2}(m)$ (by the minimality of j), and Proposition 6.2a) (applied twice) now gives

$$\sigma_{2i+2}(n_0) < \sigma_j(m),$$

which was to be proved. Assume now that j is odd. If $j = 1$, then, having in mind that $\sigma_{2i+2}(n_0)$ is defined, that is, $\sigma_{2i+1}(n_0) \in P$, we obtain $m > |tu| \geqslant \sigma_{2i+1}(n_0)$, and now Proposition 6.2a) gives

$$\sigma_1(m) = \sigma_Q(m) < \sigma_Q(\sigma_{2i+1}(n_0)) = \sigma_Q(\sigma_Q(\sigma_{2i}(n_0))) = \sigma_{2i}(n_0),$$

which was to be proved. If $j > 1$, then $\sigma_{j-2}(m) \leqslant \sigma_{2i+2}(n_0)$ (by the minimality of j), and Proposition 6.2a) (applied twice) now gives

$$\begin{aligned}\sigma_j(m) &= \sigma_Q(\sigma_P(\sigma_{j-2}(m))) \leqslant \sigma_Q(\sigma_P(\sigma_{2i+2}(n_0))) \\ &= \sigma_Q(\sigma_P(\sigma_P(\sigma_Q(\sigma_{2i}(n_0))))) = \sigma_{2i}(n_0),\end{aligned}$$

which was to be proved.

Assume now that j from the previous paragraph does not exist. Let $2 \mid \mathrm{end}(m)$. Then clearly

$$\sigma_{\mathrm{end}(m)}(m) > \sigma_{2i}(n_0),$$

since otherwise there would exist the even j from the previous paragraph. We now prove

$$\sigma_{\mathrm{end}(m)}(m) > \sigma_{2i+1}(n_0).$$

Suppose $\sigma_{2i+1}(n_0) \geqslant \sigma_{\mathrm{end}(m)}(m)$. In fact, the inequality must be strict, since the right-hand side does not belong to Q (by the definition of $\mathrm{end}(m)$), while the left-hand side does. Then, by Proposition 6.2a), we have

$$\begin{aligned}\sigma_{2i+2}(n_0) &= \sigma_P(\sigma_{2i+1}(n_0)) < \sigma_P(\sigma_{\mathrm{end}(m)}(m)) \\ &= \sigma_P(\sigma_P(\sigma_{\mathrm{end}(m)-1}(m))) = \sigma_{\mathrm{end}(m)-1}(m),\end{aligned}$$

and thus there would exist the odd j from the previous paragraph, a contradiction. The case $2 \nmid \mathrm{end}(m)$ is similar. Indeed, the inequality

$$\sigma_{\mathrm{end}(m)}(m) \geqslant \sigma_{2i+2}(n_0)$$

is impossible, since it would have to be strict (the right-hand side is in P, the left-hand side is not), and thus the odd j from the previous paragraph would exist; the inequality

$$\sigma_{\mathrm{end}(m)}(m) \geqslant \sigma_{2i+1}(n_0)$$

is also impossible, since applying σ_Q to the both sides gives, by Proposition 6.2a), $\sigma_{\mathrm{end}(m)-1}(m) \leqslant \sigma_{2i}(n_0)$, and thus the even j from the previous paragraph would exist. This completes the proof.

b) Define the following relation on the set $[1,|t|]_{\mathbb{N}} \cup [|tu|+1,|tuv|]_{\mathbb{N}}$:

$$m \sim m' \text{ if and only if there exists } l \text{ such that } m' = \sigma_l(m).$$

Let us show that "$\sim$" is an equivalence relation. Indeed, it is clearly reflexive and symmetric (if $m' = \sigma_l(m)$, then it is easily checked that $m = \sigma_l(m')$), and if $m' = \sigma_l(m)$ and $m'' = \sigma_{l'}(m')$, then $m'' = \sigma_{l'}(\sigma_l(m))$, while it is not hard to see that either $\sigma_{l'}(\sigma_l(m)) = \sigma_{l'+l}(m)$ or $\sigma_{l'}(\sigma_l(m)) = \sigma_{|l'-l|}(m)$; this proves the assertion.

We claim that each equivalence class is of size either 1 or 2. This is implied by the following observation: if $m' \neq m$, $m' = \sigma_l(m)$ and m' and m simultaneously belong to the interval $[1,|t|]_{\mathbb{N}}$ or the interval $[|tu|+1,|tuv|]_{\mathbb{N}}$, then l is odd, while if m' and m are in different intervals, then l is even.

We are now ready to prove b). In the rest of the proof, m (and also m') will always denote a value from $[1,|t|]_{\mathbb{N}} \cup [|tu|+1,|tuv|]_{\mathbb{N}}$ such that there exists j for which $\sigma_{2i+2}(n_0) < \sigma_j(m) \leqslant \sigma_{2i}(n_0)$.

Note that, if $\sigma_j(m) = \sigma_{j'}(m')$, then $m \sim m'$ (since either $m' = \sigma_{j+j'}(m)$ or $m' = \sigma_{|j-j'|}(m)$). Therefore, if m is alone in its class, then its assigned j (whichever we choose if there is a choice) will not collide with the other assignments. Let now $m \sim m'$, $m \neq m'$. We prove that there exist j and j' such that

$$\sigma_{2i+2}(n_0) < \sigma_j(m), \sigma_{j'}(m') \leqslant \sigma_{2i}(n_0)$$

and

$$\sigma_j(m) \neq \sigma_{j'}(m').$$

Note that from $m \sim m'$ we get that neither m nor m' dies. It is enough to prove that for any m that does not die there exist both an even j and an odd j that satisfy the requirement. Let us first show why this is enough. Since neither m nor m' dies, if, say, both $m, m' \geqslant |tu|+1$ (the other cases are similar), then we can choose j and j' to be of the same parity, and Proposition 6.2c) implies $\sigma_j(m) \neq \sigma_{j'}(m')$, which was to be proved.

Therefore, assume that m does not die, and let, without loss of generality, $m \geqslant |tu| + 1$. If $2 \mid \mathrm{end}(m)$, then, because of $\sigma_{\mathrm{end}(m)}(m) \in P$, we have $\sigma_{\mathrm{end}(m)}(m) \leqslant |t|$. Since $\sigma_{2i}(n_0) \in Q$ (because $\sigma_{2i+1}(n_0)$ is defined) and $\sigma_{2i+1}(n_0) \in Q$ (because it equals $\sigma_Q(\sigma_{2i}(n_0))$), we have that $\sigma_{\mathrm{end}(m)}(m)$ is less than both of these values. But this implies, as seen during the proof of the part a), that there exist both an even j and an odd j that satisfy the requirement, which was to be proved. The case $2 \nmid \mathrm{end}(m)$ is analogous. This completes the proof. ∎

Finally, we shall need the following lemma.

Lemma 6.5. *Let n be such that $n \geqslant |tu|+1$ and $\sigma_{\mathrm{end}(n)}(n) \geqslant |tu|+1$. Then:*

$$\begin{aligned} &2|tuv| + 1 - n - \sigma_{\mathrm{end}(n)}(n) \\ &\qquad \geqslant |\{m : m \geqslant |tu| + 1,\ \mathrm{end}(m) \geqslant \mathrm{end}(n) \textit{ and } \sigma_{\mathrm{end}(m)}(m) \geqslant |tu| + 1\}| \\ &\qquad\quad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n) \textit{ and } m \textit{ dies}\}| \end{aligned}$$

Proof. Let us first prove the following: if $|tu| + 1 \leqslant m < m'$ and both $\sigma_{\mathrm{end}(m)}(m), \sigma_{\mathrm{end}(m')}(m') \geqslant |tu| + 1$, then $\mathrm{end}(m) \leqslant \mathrm{end}(m')$. Suppose the contrary: $\mathrm{end}(m) > \mathrm{end}(m')$. We get $2 \nmid \mathrm{end}(m')$ (because of $\sigma_{\mathrm{end}(m')}(m') \notin P$); therefore, by Proposition 6.2a), from $m < m'$ we obtain $\sigma_{\mathrm{end}(m')}(m) > \sigma_{\mathrm{end}(m')}(m')$ (the left-hand side is defined since $\mathrm{end}(m) > \mathrm{end}(m')$). However, this implies $\sigma_{\mathrm{end}(m')}(m) > |tu|+1$ and thus $\sigma_{\mathrm{end}(m')}(m) \notin P$, which contradicts $\mathrm{end}(m) > \mathrm{end}(m')$. This proves the assertion.

Now, let n be as in the lemma's statement. In the calculations below we shall need only one more observation: the function $m \mapsto \sigma_{\mathrm{end}(m)}(m)$ bijectively maps the set

$$\{m : m > n,\ \mathrm{end}(m) = \mathrm{end}(n) \text{ and } \sigma_{\mathrm{end}(m)}(m) \geqslant |tu| + 1\}$$

to the set

$$\{m : m < \sigma_{\mathrm{end}(n)}(n),\ \mathrm{end}(m) = \mathrm{end}(n) \text{ and } \sigma_{\mathrm{end}(m)}(m) \geqslant |tu| + 1\}$$

(indeed, this follows by Proposition 6.2a), having in mind that $2 \nmid \mathrm{end}(n)$, and we see that the considered function is its own inverse). For the sake of brevity, we say that m, $m \geqslant |tu|+1$, is *pleasant* if $\mathrm{end}(m) > \mathrm{end}(n)$ and $\sigma_{\mathrm{end}(m)}(m) \geqslant |tu|+1$, and is *delightful* if $\mathrm{end}(m) = \mathrm{end}(n)$ and $\sigma_{\mathrm{end}(m)}(m) \geqslant |tu|+1$. Note that, by the assertion from the first paragraph, there are no pleasant numbers less than n, nor less than $\sigma_{\mathrm{end}(n)}(n)$ (since $\mathrm{end}(\sigma_{\mathrm{end}(n)}(n)) = \mathrm{end}(n)$). We finally have:

$$\begin{aligned}
&2|tuv| + 1 - n - \sigma_{\mathrm{end}(n)}(n) \\
&\quad = |[n+1, |tuv|]_{\mathbb{N}}| + |[\sigma_{\mathrm{end}(n)}(n), |tuv|]_{\mathbb{N}}| \\
&\quad \geqslant |\{m : m > n,\ m \text{ is pleasant or delightful}\}| \\
&\qquad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n),\ m \text{ is pleasant or delightful, or } m \text{ dies}\}| \\
&\quad = |\{m : m > n,\ m \text{ is delightful}\}| + 2|\{m : m \text{ is pleasant}\}| \\
&\qquad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n),\ m \text{ is delightful or } m \text{ dies}\}| \\
&\quad = |\{m : m < \sigma_{\mathrm{end}(n)}(n),\ m \text{ is delightful}\}| + 2|\{m : m \text{ is pleasant}\}| \\
&\qquad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n),\ m \text{ is delightful or } m \text{ dies}\}| \\
&\quad = |\{m : m \text{ is delightful}\}| + 2|\{m : m \text{ is pleasant}\}| \\
&\qquad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n) \text{ and } m \text{ dies}\}| \\
&\quad \geqslant |\{m : m \text{ is pleasant or delightful}\}| \\
&\qquad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n) \text{ and } m \text{ dies}\}|,
\end{aligned} \tag{8}$$

which was to be proved. ■

Finally, we are ready to prove Theorem 6.1.

First proof of Theorem 6.1. First of all, we make a (trivial) observation that, whenever $m \in [1, |t|]_{\mathbb{N}} \cup [|tu|+1, |tuv|]_{\mathbb{N}}$ and $\sigma_i(m)$ is defined, then holds $(tuv)[\sigma_i(m)] = 2$.

Assume first $2 \mid \mathrm{end}(n_0)$ (where n_0 is chosen as described earlier). Then $\sigma_{\mathrm{end}(n_0)}(n_0) \in P$, and therefore $\sigma_{\mathrm{end}(n_0)}(n_0) \leqslant |t|$. By Proposition 6.2b), we may write

$$
\begin{aligned}
|tuv|_2 &= \big|(tuv)[n_0+1,|tuv|]\big|_2 \\
&\quad + \sum_{i=0}^{\frac{\mathrm{end}(n_0)}{2}-1} \big|(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]\big|_2 + \big|(tuv)[1,\sigma_{\mathrm{end}(n_0)}(n_0)]\big|_2 \\
&= (|tuv|-n_0) + \sum_{i=0}^{\frac{\mathrm{end}(n_0)}{2}-1} \big|(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]\big|_2 + \sigma_{\mathrm{end}(n_0)}(n_0).
\end{aligned} \tag{9}
$$

Write $k=|tv|$. Let us first prove that, for each i such that $\sigma_{2i+2}(n_0)$ is defined, we have

$$
|(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]|_2 \geqslant \frac{k}{k-1}|(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]|_1. \tag{10}
$$

It is enough to prove

$$
|(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]|_2 \geqslant \frac{\sigma_{2i}(n_0)-\sigma_{2i+2}(n_0)+1}{2};
$$

indeed, we note that then we would have $|(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]|_1 \leqslant \frac{\sigma_{2i}(n_0)-\sigma_{2i+2}(n_0)-1}{2}$, and thus

$$
\begin{aligned}
\frac{|(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]|_2}{|(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]|_1} &\geqslant \frac{\sigma_{2i}(n_0)-\sigma_{2i+2}(n_0)+1}{\sigma_{2i}(n_0)-\sigma_{2i+2}(n_0)-1} \\
&= 1+\frac{2}{\sigma_{2i}(n_0)-\sigma_{2i+2}(n_0)-1} \\
&\geqslant 1+\frac{2}{(2k-1)-1} = 1+\frac{1}{k-1} = \frac{k}{k-1}
\end{aligned} \tag{11}
$$

(the last inequality follows by Lemma 6.3a) for $n=\sigma_{2i}(n_0)$), which is what we want to prove. Therefore, let us prove the claimed inequality.

We shall use Lemma 6.4 here. If $\sigma_{\mathrm{end}(m)}(m) > \max\{\sigma_{2i}(n_0),\sigma_{2i+1}(n_0)\}$ for either $m \geqslant |tu|+1$ and $2 \mid \mathrm{end}(m)$, or $m \leqslant |t|$ and $2 \nmid \mathrm{end}(m)$, then we have $\sigma_{\mathrm{end}(m)}(m) \notin Q$; therefore, there are at most

$$
\min\{|[\sigma_{2i}(n_0),|tuv|]_{\mathbb{N}} \setminus Q|, |[\sigma_{2i+1}(n_0),|tuv|]_{\mathbb{N}} \setminus Q|\}
$$

such values m (we recall that, for any two such different values m and m', we have $\sigma_{\mathrm{end}(m)}(m) \neq \sigma_{\mathrm{end}(m')}(m')$, which was necessary for the last conclusion). In a similar manner, we see that there are at most

$$\min\{|[1, \sigma_{2i+1}(n_0)]_{\mathbb{N}} \setminus P|, |[1, \sigma_{2i+2}(n_0)]_{\mathbb{N}} \setminus P|\}$$

values m such that $\sigma_{\mathrm{end}(m)}(m) < \min\{\sigma_{2i+1}(n_0), \sigma_{2i+2}(n_0)\}$ and either $m \geqslant |tu| + 1$ and $2 \nmid \mathrm{end}(m)$, or $m \leqslant |t|$ and $2 \mid \mathrm{end}(m)$. Altogether, in the set $[1, |t|]_{\mathbb{N}} \cup [|tu| + 1, |tuv|]_{\mathbb{N}}$ there are at least

$$\begin{aligned} k &- \min\{|[\sigma_{2i}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|, |[\sigma_{2i+1}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|\} \\ &- \min\{|[1, \sigma_{2i+1}(n_0)]_{\mathbb{N}} \setminus P|, |[1, \sigma_{2i+2}(n_0)]_{\mathbb{N}} \setminus P|\} \end{aligned}$$

values m for which there exists a corresponding j from Lemma 6.4. But then Lemma 6.4b) immediately implies that this bound is also a lower bound for $|(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]|_2$. And now, by Lemma 6.3a) for $n = \sigma_{2i}(n_0)$, we obtain

$$\begin{aligned} &\frac{\sigma_{2i}(n_0) - \sigma_{2i+2}(n_0) + 1}{2} \\ &\qquad \leqslant k - \frac{|[\sigma_{2i}(n_0), |tuv|]_{\mathbb{N}} \setminus Q| + |[\sigma_{2i+1}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|}{2} \\ &\qquad\quad - \frac{|[1, \sigma_{2i+1}(n_0)]_{\mathbb{N}} \setminus P| + |[1, \sigma_{2i+2}(n_0)]_{\mathbb{N}} \setminus P|}{2} \\ &\qquad \leqslant k - \min\{|[\sigma_{2i}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|, |[\sigma_{2i+1}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|\} \\ &\qquad\quad - \min\{|[1, \sigma_{2i+1}(n_0)]_{\mathbb{N}} \setminus P|, |[1, \sigma_{2i+2}(n_0)]_{\mathbb{N}} \setminus P|\} \\ &\qquad \leqslant |(tuv)[\sigma_{2i+2}(n_0) + 1, \sigma_{2i}(n_0)]|_2, \end{aligned}$$

which proves the claim.

Using (10), from (9) we get

$$\begin{aligned} |tuv|_2 &> \frac{k}{k-1} \sum_{i=0}^{\frac{\mathrm{end}(n_0)}{2}-1} |(tuv)[\sigma_{2i+2}(n_0) + 1, \sigma_{2i}(n_0)]|_1 \\ &= \frac{k}{k-1} |(tuv)[\sigma_{\mathrm{end}(n_0)}(n_0), n_0]|_1 = \frac{k}{k-1} |u|_1 \end{aligned} \tag{12}$$

(the inequality is strict since the rightmost summand at (9) is nonzero; the last equality follows from $\sigma_{\mathrm{end}(n_0)}(n_0) \leqslant |t|$ and $n_0 \geqslant |tu|+1$), which completes the case $2 \mid \mathrm{end}(n_0)$.

We can now assume that not only $2 \nmid \mathrm{end}(n_0)$, but also $2 \nmid \mathrm{end}(n)$ for any n that does not die (otherwise, if there were such n, we could take it for n_0, and the above proof would work). We still assume $n_0 \geqslant |tu| + 1$, without loss of generality. Finally, a further assumption we can make is that $\mathrm{end}(n_0)$ is no greater than $\mathrm{end}(n)$ for any n that does not die and that $n \geqslant |tu| + 1$ (since otherwise we could again rechoose n_0). With all these assumptions, we proceed to the rest of the proof.

Using (10), we get

$$\begin{aligned}
|tuv|_2 &= \big|(tuv)[n_0+1, |tuv|]\big|_2 + \sum_{i=0}^{\frac{\mathrm{end}(n_0)-3}{2}} \big|(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]\big|_2 \\
&\qquad + \big|(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]\big|_2 \\
&\geqslant (|tuv| - n_0) + \frac{k}{k-1} \sum_{i=0}^{\frac{\mathrm{end}(n_0)-3}{2}} |(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]|_1 \\
&\qquad + |(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \\
&= |tuv| - n_0 + |(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \\
&\qquad + \frac{k}{k-1} |(tuv)[\sigma_{\mathrm{end}(n_0)-1}(n_0)+1, n_0]|_1.
\end{aligned}$$

Therefore, it is enough to prove

$$|tuv| - n_0 + |(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \geqslant \frac{k}{k-1} |(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_1$$

(in that case the calculations above would give $|tuv|_2 \geqslant \frac{k}{k-1}|u|_1$, which is what we need).

The following sets will be needed through the proof, and for the sake of brevity, we name them as follows:

$$A = \{m : |tu|+1 \leqslant m < \sigma_{\mathrm{end}(n_0)}(n_0) \text{ and } |t|+1 \leqslant \sigma_{\mathrm{end}(m)}(m) \leqslant \sigma_{\mathrm{end}(n_0)-1}(n_0)\};$$

$$B = \{m : |tu|+1 \leqslant m < \sigma_{\mathrm{end}(n_0)}(n_0) \text{ and } \sigma_{\mathrm{end}(n_0)-1}(n_0) < \sigma_{\mathrm{end}(m)}(m) \leqslant |tu|\};$$

$$C = \{m : m \geqslant \sigma_{\mathrm{end}(n_0)}(n_0) \text{ and } m \text{ dies}\};$$

$$\begin{aligned} D &= \{m : m \geqslant |tu|+1, \mathrm{end}(m) \geqslant \mathrm{end}(n_0) \text{ and } \sigma_{\mathrm{end}(m)}(m) \geqslant |tu|+1\} \\ &= \{m : m \geqslant |tu|+1 \text{ and } m \text{ does not die}\}; \end{aligned}$$

$$E = [\sigma_{\mathrm{end}(n_0)-1}(n_0)+1, |tu|]_{\mathbb{N}} \setminus P;$$

$$F = [\sigma_{\mathrm{end}(n_0)-1}(n_0)+1, |tu|]_{\mathbb{N}} \setminus Q = [\sigma_{\mathrm{end}(n_0)-1}(n_0), |tuv|]_{\mathbb{N}} \setminus Q.$$

The equality between the two forms of D follows by the assumption introduced above, and the one between the two forms of F is clear. We shall also use the equality

$$|A| + |B| + |C| + |D| = |v|$$

(which is easily seen), as well as the inequality

$$|E| + |F| \geqslant |B|$$

(which follows by the observation that the function $m \mapsto \sigma_{\mathrm{end}(m)}(m)$ injectively maps the set B to the set $E \cup F$).

Note that, for each m where $\sigma_{\mathrm{end}(n_0)}(n_0) \leqslant m \leqslant |tuv|$, we have $|t| + 1 \leqslant \sigma_Q(m) \leqslant \sigma_{\mathrm{end}(n_0)-1}(n_0)$; that makes for $|tuv| - \sigma_{\mathrm{end}(n_0)}(n_0) + 1$ letters 2 in the word $(tuv)[|t| + 1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]$. Further, for each $m \in A$, the value $\sigma_{\mathrm{end}(m)}(m)$ marks the position of another letter 2 in the word $(tuv)[|t| + 1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]$ (and all these positions are pairwise different, and also different from the positions from the previous sentence, since we recall that all the positions "generated" by a number that dies are unique). Therefore:

$$|(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \geqslant |t| + (|tuv| - \sigma_{\mathrm{end}(n_0)}(n_0) + 1) + |A|.$$

From this inequality we get

$$\begin{aligned} |tuv| - n_0 + &|(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \\ &\geqslant |tuv| - n_0 + |t| + (|tuv| - \sigma_{\mathrm{end}(n_0)}(n_0) + 1) + |A| \\ &\geqslant |D| + |C| + |t| + |A| = |t| + |v| - |B| = k - |B| \end{aligned} \tag{13}$$

(the second inequality is due to Lemma 6.5), and

$$\begin{aligned}
&|(tuv)[1,\sigma_{\mathrm{end}(n_0)-1}(n_0)]|_1\\
&\qquad = \sigma_{\mathrm{end}(n_0)-1}(n_0) - |(tuv)[1,\sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2\\
&\qquad \leqslant 2(|t|+|v|) + |P| - \sigma_{\mathrm{end}(n_0)}(n_0) - |[\sigma_{\mathrm{end}(n_0)-1}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|\\
&\qquad\quad - (|t| + (|tuv| - \sigma_{\mathrm{end}(n_0)}(n_0) + 1) + |A|)\\
&\qquad = |t| + |v| + (|P| + |v| - |tuv|) - |F| - 1 - |A|\\
&\qquad = k - |[1,|tu|]_{\mathbb{N}} \setminus P| - |F| - 1 - |A|\\
&\qquad \leqslant k - |E| - |F| - 1 - |A| \leqslant k - 1 - |A| - |B|
\end{aligned} \tag{14}$$

(the first inequality has been obtained with the help of Lemma 6.3b) for $n = \sigma_{\mathrm{end}(n_0)-1}(n_0)$; note that then $|[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q| = |[\sigma_{\mathrm{end}(n_0)}(n_0), |tuv|]_{\mathbb{N}} \setminus Q| = 0$, since $\sigma_{\mathrm{end}(n_0)}(n_0) \geqslant |tu| + 1$). Finally,

$$\begin{aligned}
\frac{|tuv| - n_0 + |(tuv)[1,\sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2}{|(tuv)[1,\sigma_{\mathrm{end}(n_0)-1}(n_0)]|_1} &\geqslant \frac{k-|B|}{k-1-|A|-|B|} \geqslant \frac{k-|B|}{k-1-|B|}\\
&= 1 + \frac{1}{k-1-|B|}\\
&\geqslant 1 + \frac{1}{k-1} = \frac{k}{k-1},
\end{aligned} \tag{15}$$

which completes the proof. ■

It turns out that the case when the equality in (5) is reached has an interesting characterization, which can be obtained from the above proof in the (more or less) straightforward manner.

Proposition 6.6. *Under the conditions of Theorem 6.1, the equality in (5) is reached if and only if, for a positive integer k and a nonnegative integer l, we have $u = (1^{k-1}2^k)^l 1^{k-1}$, $t = \varepsilon$ and $v = 2^k$ (or vice versa).*

Proof. Assume that u, t, v, p and q are such that the equality is reached. We may also assume that p and q are *longest* subpalindromes of tu and uv, respectively.

We first note that p, respectively q, contains all the letters from t, respectively v (since otherwise the equality cannot be reached because of the strict inequality in (6)). Let n_0 be as in the proof. Since in the case $2 \mid \mathrm{end}(n_0)$ we have a strict inequality in (12), we conclude $2 \nmid \mathrm{end}(n_0)$, and we then also recall all the assumptions from the paragraph following (12). Now, in order for the second and the third inequalities in (15) to be equalities, we conclude $A = \varnothing$ and $B = \varnothing$, and then, in order for the last inequality in (14) to be equality, we conclude $|E| + |F| = |B|$, that is, $E = F = \varnothing$. The penultimate inequality in (14) (when converted to equality) now gives $|[1, |tu|]_{\mathbb{N}} \setminus P| = |E| = 0$, that is,

$$p = tu. \tag{16}$$

Since Lemma 6.5 for n_0 was used in (13), we need to have equality in that lemma. Looking at (8), we see that there must not be any pleasant number (in order to reach the equality at the end), as well as that there must not be any number greater than n_0 that dies (in order to reach the equality at the beginning). In particular, $|tuv|$ does not die (and this implies $2 \nmid \mathrm{end}(|tuv|)$, that is, $\sigma_{\mathrm{end}(|tuv|)}(|tuv|) \geqslant |tu| + 1$, because of the recalled assumptions about n_0), $\mathrm{end}(|tuv|) \leqslant \mathrm{end}(n_0)$ (because there are no pleasant numbers), and from this we conclude that the only possibility is $\mathrm{end}(|tuv|) = \mathrm{end}(n_0)$ (again because of the assumptions about n_0). Therefore, $|tuv|$ satisfies all the same assumptions as n_0 does, and thus for the rest of the proof we may assume $n_0 = |tuv|$ (we rechoose n_0 if necessary).

Assume first $\mathrm{end}(|tuv|) = 1$, that is, $\sigma_Q(|tuv|) \geqslant |tu| + 1$. Then it is easy to see that $q = v$, and furthermore, u contains only 1s (since otherwise $2^{|uv|_2}$ would be a subpalindrome of uv longer than q). Now from (16) we get $t = \varepsilon$, and thus the equality in (5) reduces to $|u| = \frac{|v|-1}{|v|}|uv|_2 = \frac{|v|-1}{|v|}|v| = |v| - 1$; in other words, $v = 2^k$ and $u = 1^{k-1}$, which was to be proved.

Assume now $\mathrm{end}(|tuv|) > 1$. Let $k = |tv|$. Then the second inequality in (11) for $i = 0$ (when converted to equality) gives $|tuv| - \sigma_2(|tuv|) = 2k - 1$, and the first inequality gives

$$\big|(tuv)[\sigma_2(|tuv|) + 1, |tuv|]\big|_2 = \frac{|tuv| - \sigma_2(|tuv|) + 1}{2} = k \tag{17}$$

and

$$\big|(tuv)[\sigma_2(|tuv|) + 1, |tuv|]\big|_1 = \frac{|tuv| - \sigma_2(|tuv|) - 1}{2} = k - 1. \tag{18}$$

Also note that the second inequality in (11) is an application of Lemma 6.3a), and in order for the equality to hold in that lemma, from the last lines of (7) we see that $|p|+|q| = 2|u|+1$ must hold; (16) reduces this to $|t|+|q| = |u|+1$, which implies

$$|uv| - |q| = |t| + |v| - 1 = k - 1. \tag{19}$$

Further, since (16) implies that tu ends with $|t|$ letters 2, that is, tuv ends with k letters 2, by (17) and (18) we conclude that there is an array of $k-1$ letters 1 immediately preceding those 2s. In other words, $1^{k-1}2^{|t|} \in \mathrm{Suff}(tu)$, and now because of (16) we have $2^{|t|}1^{k-1} \in \mathrm{Pref}(tu)$, that is, $1^{k-1} \in \mathrm{Pref}(u)$. Those 1s clearly do not participate in the palindrome q, and now (19) implies that everything else has to, that is,

$$q = (uv)[k, |uv|]. \tag{20}$$

Starting from $tuv = 2^{|t|}1^{k-1} \ldots 1^{k-1}2^k$, by (20) we get

$$tuv = 2^{|t|}1^{k-1}2^k1^{k-1} \ldots 1^{k-1}2^k,$$

then by (16)

$$tuv = 2^{|t|}1^{k-1}2^k1^{k-1} \ldots 1^{k-1}2^k1^{k-1}2^k,$$

then we again use (20) etc. To conclude,

$$tuv = 2^{|t|}(1^{k-1}2^k)^l1^{k-1}2^k \quad \text{for a nonnegative integer } l.$$

Now we evaluate $|u|_1 = (l+1)(k-1)$ and $|tuv|_2 = |t|+(l+1)k$. The equality case in (5) is now reduced to $(l+1)(k-1) = \frac{k-1}{k}(|t| + (l+1)k)$, that is, $k(l+1) = |t| + (l+1)k$, which gives $|t| = 0$, that is, $t = \varepsilon$. We then have $v = 2^k$ and $u = (1^{k-1}2^k)^l1^{k-1}$, and it is straightforward to check that these words indeed satisfy the required equality. ■

6.2 Second proof

Second proof of Theorem 6.1. We prove the theorem by induction on $|u|$. If $|u| = 0$, then (5) trivially holds, since the left-hand side is 0 while the right-hand side is always nonnegative. Now we assume that the assertion holds for each word shorter than u, and prove that it holds for u. Let v' denote the shortest prefix of uv such that $|v'|_2 = |v|$, and let t' denote the shortest suffix of tu such that $|t'|_2 = |t|$.

Assume first that $|v'| + |t'| < |u|$. Let $u = v'u't'$, and let p' and q' be longest subpalindromes of $u't$ and vu', respectively (note that we now put

t to the right and v to the left of u', not vice versa, as it was before!). We claim that

$$|q'| \geqslant |p| - 2|t| - 2(|v'| - |v|).$$

We can write $p = 2^{|t|} p_1 p_2 2^{|t|}$, where $p_1 \in \mathrm{Subw}(v')$ and $p_2 \in \mathrm{Subw}(u')$. We have that $p_1 p_2$ is a palindrome of length $|p| - 2|t|$; erasing all the letters 1 from p_1 (and there are at most $|v'|_1$, which is $|v'| - |v|$, of them), and additionally erasing (if necessary) all the "mirror images" (with respect to the midpoint of p) of all these 1s, we obtain a subpalindrome of vu' of length at least $|p| - 2|t| - 2(|v'| - |v|)$, which proves the claim. Analogously, we also obtain

$$|p'| \geqslant |q| - 2|v| - 2(|t'| - |t|).$$

We aim to use the inductive assumption on the words v, u' and t. Clearly, $|u'| = |u| - |v'| - |t'| \leqslant |u| - |v| - |t| < |u|$. Let us now show that the condition of the theorem is satisfied:

$$\begin{aligned} |q'| + |p'| &\geqslant |p| - 2|t| - 2(|v'| - |v|) + |q| - 2|v| - 2(|t'| - |t|) \\ &= |p| + |q| - 2|v'| - 2|t'| > 2|u| - 2|v'| - 2|t'| = 2|u'|. \end{aligned}$$

Therefore, by the inductive assumption, we get

$$|u'|_1 \leqslant \frac{|vt| - 1}{|vt|} |vu't|_2. \tag{21}$$

Note that, since 1s inside the word v' do not participate in the palindrome q (since the first $|v|$ letters of q are 2), we have $|v'|_1 \leqslant |uv| - |q|$. Analogously, $|t'|_1 \leqslant |tu| - |p|$. Therefore,

$$|v'|_1 + |t'|_1 \leqslant |uv| - |q| + |tu| - |p| < |t| + |v|. \tag{22}$$

Now, (21) and (22) yield:

$$\begin{aligned} |u|_1 = |v'|_1 + |u'|_1 + |t'|_1 \leqslant |tv| - 1 + |u|_1 \leqslant \frac{|tv| - 1}{|tv|}(|tv| + |vu't|_2) \\ = \frac{|tv| - 1}{|tv|}(|t'|_2 + |v'|_2 + |vu't|_2) = \frac{|tv| - 1}{|tv|}|tuv|_2, \end{aligned}$$

which was to be proved.

Finally, we need to take care of the case $|v'| + |t'| \geqslant |u|$. Then we have:

$$|u|_1 \leqslant |v'|_1 + |t'|_1 \leqslant |t| + |v| - 1 = \frac{|tv| - 1}{|tv|}|tv| \leqslant \frac{|tv| - 1}{|tv|}|tuv|_2,$$

(where the second inequality was already seen at (22)), which was to be proved. ∎

7 Another postponed technical theorem

Theorem 7.1. *Let $w \in \{1,2\}^*$ be such that $2|w|_2 \geqslant |w|_1$. Let p and q be two nonempty subpalindromes of w. Let w_p, v, w_q and t be such that $w = w_p v = t w_q$, p is a subword of w_p, and q is a subword of w_q. Then*

$$|p| + 2|v|_2 + |q| + 2|t|_2 \leqslant 4|w|_2 + |w|_1.$$

Proof. We distinguish two cases:

- Case 1°: $|w_p| \leqslant |t|$;
- Case 2°: $|t| < |w_p|$.

Case 1°. In this case we have $|p|_1 + |q|_1 \leqslant |w|_1$. Furthermore, we have:

$$|p|_2 + 2|v|_2 \leqslant |w|_2 + |v|_2 \leqslant 2|w|_2.$$

In an analogous way we obtain $|q|_2 + 2|t|_2 \leqslant 2|w|_2$. Now, we get the required inequality directly:

$$\begin{aligned} |p| + 2|v|_2 + |q| + 2|t|_2 &\leqslant |p|_1 + |q|_1 + |p|_2 + 2|v|_2 + |q|_2 + 2|t|_2 \\ &\leqslant |w|_1 + 4|w|_2. \end{aligned}$$

Case 2°. In this case we may write $w = tuv$, where u is a nonempty word. Now suppose that the required inequality does not hold, that is,

$$|p| + 2|v|_2 + |q| + 2|t|_2 > |w|_1 + 4|w|_2.$$

This reduces to

$$|p| + |q| > |w|_1 + 2|w|_2 + 2|u|_2 \geqslant 2|w|_1 + 2|u|_2. \tag{23}$$

Let $\widehat{t}$ and $\widehat{v}$ be the words obtained from the words t and v, respectively, by erasing all the letters 1 (or, equivalently, $\widehat{t} = 2^{|t|_2}$ and $\widehat{v} = 2^{|v|_2}$), and let $\widehat{p}$ and $\widehat{q}$ be the longest subpalindromes of $\widehat{t}u$ and $u\widehat{v}$, respectively. We then have:

$$\begin{aligned} |\widehat{p}| &\geqslant |p| - 2|t|_1; \\ |\widehat{q}| &\geqslant |q| - 2|v|_1. \end{aligned}$$

Therefore,

$$|\widehat{p}| + |\widehat{q}| \geqslant |p| - 2|t|_1 + |q| - 2|v|_1 > (2|w|_1 + 2|u|_2) - 2|t|_1 - 2|v|_1 = 2|u|,$$

which means that the conditions of Theorem 6.1 are satisfied (for u, $\widehat{t}$, $\widehat{v}$, $\widehat{p}$ and $\widehat{q}$); by that theorem we obtain

$$|u|_1 \leqslant \frac{|\widehat{t}\widehat{v}| - 1}{|\widehat{t}\widehat{v}|} |\widehat{t}u\widehat{v}|_2 < |\widehat{t}u\widehat{v}|_2 \leqslant |tuv|_2 = |w|_2.$$

On the other hand, since $|p| + |q| \leqslant |w| + |u|$, by the first inequality in (23) we have $|w| + |u| > |w|_1 + 2|w|_2 + 2|u|_2$, that is, $|u|_1 > |w|_2 + |u|_2 \geqslant |w|_2$, a contradiction. The proof is completed. ∎

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