A note on the paper "On Brlek-Reutenauer conjecture"

Bojan Bašić

Department of Mathematics and Informatics, University of Novi Sad,

Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia

bojan.basic@dmi.uns.ac.rs

Abstract

In this short note we point to an error in the proof of a theorem stated in [L. Balková & E. Pelantová & Š. Starosta, On Brlek-Reutenauer conjecture, *Theoret. Comput. Sci.* **412** (2011), 5649–5655]. By constructing a counterexample, we show that the assertion of the theorem is actually incorrect. Although this theorem is of a technical character, it was used in an argument leading to a corollary of a general interest to the Brlek-Reutenauer conjecture, and thus as a consequence of this note we have that the proof of the mentioned corollary is also flawed.

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1 Introduction

For any infinite word $\mathbf{u}$ having the language closed under reversal and containing infinitely many palindromes, it was claimed in [1, Theorem 5.7] that the following statements are equivalent:

(a) the defect of $\mathbf{u}$ is finite;

(b) there exists an integer H such that the longest palindromic suffix of any factor w of $\mathbf{u}$, of length $|w| \geqslant H$, occurs in w exactly once.

(For all the necessary notions, we refer the reader to [1].) The claimed proof briefly states that the equivalence follows by the definition of defect. In fact, by the definition of defect and [1, Corollary 2.3], it follows that the statements (a) and

(b_0) there exists an integer H such that the longest palindromic suffix of any *prefix* w of $\mathbf{u}$, of length $|w| \geqslant H$, occurs in w exactly once

are equivalent: the direction ($\Leftarrow$) is clear, while the direction ($\Rightarrow$) follows from the observation that, if v is a prefix of $\mathbf{u}$ such that $D(v) = D(\mathbf{u})$, then each prefix w of $\mathbf{u}$ longer than v contains v as a prefix, and thus the longest palindromic suffix of w must occur in w exactly once (since otherwise it would follow $D(w) \geqslant D(v) + 1 = D(\mathbf{u}) + 1$, a contradiction). Unfortunately, the same reasoning cannot be applied with factors in place of prefixes, and therefore the mentioned proof is erroneous (only the direction (b) $\Rightarrow$ (a) can be seen to hold, since we have (b) $\Rightarrow$ (b_0) $\Rightarrow$ (a)).

As we shall see, the assertion of the theorem is in fact incorrect. Since this theorem was used in a proof of [1, Corollary 5.10] (which is an important step towards a proof of the Brlek-Reutenauer conjecture), this proof is also flawed, and thus the mentioned corollary, cited below, is still open.

Still open problem. *Let* $\mathbf{u}$ *be an infinite word with the language closed under reversal. Then we have*

$$D(\mathbf{u}) < \infty \Rightarrow \sum_{n=0}^{\infty} T_{\mathbf{u}}(n) < \infty. \tag{1}$$

2 Construction of a counterexample

We shall now construct an infinite word $\mathbf{u}$ for which (a) holds but (b) does not. Let the morphism φ be defined by $\varphi(1) = 1213$, $\varphi(2) = \varepsilon$, $\varphi(3) = 23$, and let $\mathbf{u} = \varphi^{\infty}(1)$.

Claim 1. *For each* $i \geqslant 1$ *we have*

$$\varphi^{i+1}(1) = \varphi^{i}(1)\,\varphi^{i}(1)\,23.$$

Proof. Since $\varphi(1) = 1213$ and $\varphi^2(1) = \varphi(1)\varphi(2)\varphi(1)\varphi(3) = 1213\,1213\,23$, the assertion holds for $i = 1$. By induction, we have

$$\begin{aligned}\varphi^{i+1}(1) &= \varphi(\varphi^{i}(1)) = \varphi(\varphi^{i-1}(1)\,\varphi^{i-1}(1)\,23)\\ &= \varphi^{i}(1)\,\varphi^{i}(1)\,\varphi(2)\,\varphi(3) = \varphi^{i}(1)\,\varphi^{i}(1)\,23,\end{aligned}$$

which was to be proved. ■

Claim 2. *For each $i \geqslant 1$ we have*

$$\varphi^i(1) = p_i 3(23)^{i-1},$$

where each p_i is a palindrome that begins with 12 *(and thus ends with* 21*). Further, for $i \geqslant 2$, the largest power of* 23 *that is a factor of p_i is $(23)^{i-2}$, and for $i \geqslant 3$ this factor is unioccurrent in p_i.*

Proof. Since $\varphi(1) = 1213$, the assertion holds for $i = 1$ (with $p_1 = 121$). Further, since $\varphi^2(1) = 1213121323$, the second part of the assertion holds for $i = 2$ (with $p_2 = 1213121$). By induction, using Claim 1, we have

$$\varphi^{i+1}(1) = p_i 3(23)^{i-1} p_i 3(23)^{i-1} 23 = p_i 3(23)^{i-1} p_i 3(23)^i, \tag{2}$$

and since

$$p_{i+1} = p_i 3(23)^{i-1} p_i \tag{3}$$

is a palindrome, the first part of the claim is proved. Further, since p_i ends with 1 and begins with 1, the largest power of 23 that is a factor of p_{i+1} is $(23)^{i-1}$, which is unioccurrent in p_{i+1} for $i + 1 \geqslant 3$, and thus the proof is finished. ■

Claim 3. *The language of* **u** *is closed under reversal, and* **u** *contains infinitely many palindromes.*

Proof. Each factor w of **u** is a factor of $\varphi^i(1)$ for i large enough. Since $\varphi^i(1)$ is a factor of p_{i+1} (see Claim 2), it follows that w is a factor of p_{i+1}, and thus its reversal is also a factor of p_{i+1} and in turn a factor of **u**.

The second part is clear by Claim 2. ■

Claim 4. *The word* **u** *does not satisfy the statement (b).*

Proof. By (3), for each $i \geqslant 1$ we have that $(23)^i 12$ is a factor of p_{i+2} and in turn a factor of **u**. The longest palindromic suffix of this word is clearly only the letter 2, having $i + 1$ occurrences in $(23)^i 12$. Thus, there are arbitrarily large factors w of **u** such that the longest palindromic suffix of w occurs in w more than once. Therefore, (b) fails. ■

Claim 5. *The defect of* **u** *is finite.*

Proof. We shall prove that the longest palindromic suffix of any prefix w of $\mathbf{u}$, of length $|w| \geqslant 10$, is unioccurrent in w. Therefore, $\mathbf{u}$ satisfies the statement (b_0), which is equivalent to (a).

Let w be a prefix of $\mathbf{u}$, $|w| \geqslant 10$. Choose i such that w is not a prefix of $\varphi^i(1)$ (also not equal to it), but is a prefix of $\varphi^{i+1}(1)$. If $|w| = 10$, then $w = 1213121323 = \varphi^2(1)$, and the longest palindromic suffix of w is 323, which is indeed unioccurrent in w. Thus, assume that $|w| \geqslant 11$. It now follows that $i \geqslant 2$.

By (2), w is a prefix of $p_i3(23)^{i-1}p_i3(23)^i$ longer than $p_i3(23)^{i-1}$. Let us first consider the case when w is a prefix of $p_i3(23)^{i-1}p_i$. In this case, it holds that $w = p_i3(23)^{i-1}v$, where v is a prefix of p_i. Therefore, $\overline{v}$ is a suffix of p_i, and thus $\overline{v}3(23)^{i-1}v$ is a palindromic suffix of w. This suffix is also the longest palindromic suffix of w, since if there were a longer one, there would be at least two occurrences of $3(23)^{i-1}$ in it and thus also in $p_{i+1} = p_i3(23)^{i-1}p_i$, contradicting Claim 2. For the same reason, the suffix $\overline{v}3(23)^{i-1}v$ is unioccurrent in w, which was to be proved.

Assume now that w is longer than $p_i3(23)^{i-1}p_i$. Therefore, it holds that either $w = p_i3(23)^{i-1}p_i3(23)^j$ for $0 \leqslant j \leqslant i$, or $w = p_i3(23)^{i-1}p_i(32)^j$ for $1 \leqslant j \leqslant i$.

First, let

$$w = p_i3(23)^{i-1}p_i3(23)^j \tag{4}$$

for $0 \leqslant j \leqslant i$. If $j = i$, we claim that the longest palindromic suffix of w is $3(23)^i$. Since this suffix is indeed palindromic, it is enough to show that there does not exist a longer one. Suppose that v is a longer palindromic suffix. Since, by Claim 2, p_i ends with 1, we see that $v = \ldots 13(23)^i$, and by the fact that v is palindromic we now get $v = 3(23)^i \ldots 13(23)^i$. It follows that $3(23)^i$ is a factor of $p_i3(23)^{i-1}p_i = p_{i+1}$, while by Claim 2 we have that the largest power of 23 that is a factor of p_{i+1} is $(23)^{i-1}$, a contradiction. Therefore, $3(23)^i$ is indeed the longest palindromic suffix of w, and it has to be unioccurrent in w since otherwise it would again follow that $3(23)^i$ is a factor of $p_i3(23)^{i-1}p_i$, an already seen contradiction. We shall now treat the case $0 \leqslant j \leqslant i-1$. In this case, the suffix $3(23)^jp_i3(23)^j$ of w is clearly palindromic, and we show that there does not exist a longer one. Suppose that v is a longer palindromic suffix. We see that, in the word v, the letter at the position $2j+2$ from the right is 1 (because p_i ends with 1), and thus, by the fact that v is palindromic, the letter at the position $2j+2$ from the left also has to be 1. Since v ends with $3(23)^jp_i3(23)^j$ and is longer than it, it follows that there has to be the letter 1 in v before $3(23)^j p_i 3(23)^j$. Recalling that w is of the form (4), we conclude that v encompasses the whole factor $3(23)^{i-1}$, that is, $v = \dots 13(23)^{i-1} p_i 3(23)^j$. However, in the word v, there are at most $|p_i|$ letters before $3(23)^{i-1}$ (since there are no more letters in w), and there are $|p_i| + 2j + 1 > |p_i|$ letters after it. By this and the fact that v is a palindrome, it follows that $\overline{13(23)^{i-1}} = 3(23)^{i-1}1$ must be a factor of $(23)^{i-1} p_i 3(23)^j$, and therefore a factor of $p_i 3(23)^j$. This is a contradiction (by Claim 2, the largest power of 23 that is a factor of p_i is $(23)^{i-2}$). Therefore, $3(23)^j p_i 3(23)^j$ is indeed the longest palindromic suffix of w, and it has to be unioccurrent in w since there are only two occurrences of p_i in w and the first one has no letters preceding it.

We now check the case

$$w = p_i 3(23)^{i-1} p_i (32)^j$$

for $1 \leqslant j \leqslant i$. If $j = i$, we claim that the longest palindromic suffix of w is $2(32)^{i-1}$. And indeed, this suffix is indeed palindromic, and in a similar manner as in the previous paragraph we see that there does not exist a longer one (since it would have to be of the form $(23)^i \dots 1(32)^i$, and a contradiction would be reached). Further, it has to be unioccurrent in w, since otherwise it would follow that $2(32)^{i-1}$ is a factor of either p_i or $3(23)^{i-1}$, a contradiction (the first possibility cannot hold because of Claim 2 and $i \geqslant 2$, while the second one clearly is not true). We shall now treat the case $1 \leqslant j \leqslant i - 1$. In this case, the suffix $(23)^j p_i (32)^j$ of w is clearly palindromic, and we show that there does not exist a longer one. Suppose that v is a longer palindromic suffix. In a similar manner as in the previous paragraph, noting that, in the word v, the letter at the position $2j + 1$ from the right is 1, we conclude that v encompasses the whole factor $3(23)^{i-1}$, and get a contradiction as before. Therefore, $(23)^j p_i (32)^j$ is indeed the longest palindromic suffix of w, and it has to be unioccurrent in w since, again, there are only two occurrences of p_i in w and the first one has no letters preceding it. ■

In conclusion: by Claims 3, 5 and 4, $\mathbf{u}$ is a counterexample to the assertion of the considered theorem.

3 Further comments

We may note that the defect of $\mathbf{u}$ equals 1. Indeed, by the proof of Claim 5, it is seen that $D(\mathbf{u}) = D(121312132)$. Since the word 121312132 is of length 9 and has 9 palindromic factors: ε, 1, 2, 3, 121, 131, 21312, 31213, 1213121, the assertion follows (by definition, $D(w)$ equals the difference between $|w| + 1$ and the number of palindromic factors of w).

It may be asked whether the word $\mathbf{u}$ perhaps disproves even the implication (1). That said, nothing in this paper suggests so. And actually, the present author has managed to prove that the constructed word $\mathbf{u}$ indeed satisfies $\sum_{n=0}^{\infty} T_{\mathbf{u}}(n) < \infty$. However, the proof is quite long and tedious, while the result does not seem to be of a significant importance (that is: the conjecture survives, and the word $\mathbf{u}$ turns out to be just one more word obeying it). That is why this question has not been dealt with here.

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References

[1] L. Balková & E. Pelantová & Š. Starosta, On Brlek-Reutenauer conjecture, *Theoret. Comput. Sci.* **412** (2011), 5649–5655.