On a functional equation related to roots of translations of positive integers

Bojan Bašić

Department of Mathematics and Informatics, University of Novi Sad,

Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia

bojan.basic@dmi.uns.ac.rs

Abstract

We consider the functional equation $f^q(n) = f(n+1) + k$, where $q \geqslant 2$ and $k \in \mathbb{Z}$ are given, and $f : \mathbb{N} \to \mathbb{N}$. This functional equation is related to roots of translations of positive integers, and another motivation for studying this functional equation is the fact that it can be thought of as the "prototypical case" of a more general functional equation of a very broad scope. Our main result is that the considered functional equation has a solution if and only if either $k = 0$ or $k \geqslant -1$ and $q - 1 \mid k + 1$. We further find all solutions for the case $q = 3$ and $k = 1$, which is an example that illustrates that the considered functional equation can have a very unexpected set of solutions even with quite small parameters.

Mathematics Subject Classification (2010): 39B12

Keywords: functional equation, iterative root, translation

1 Introduction

Given a function f, by f^q we denote the composition $\underbrace{f \circ f \circ \cdots \circ f}_{q \text{ times}}$. For functions f and g such that $f^q = g$, we say that the function f is a *q-th iterative root* of the function g. When g is a translation, that is, $g(n) = n+k$, Sarkaria [5] showed that the functional equation $f^q(n) = n + k$ (where q and k are fixed) has a solution $f : \mathbb{N} \to \mathbb{N}$ if and only if $q \mid k$, and noted that the same conclusion holds for $f : \mathbb{Z} \to \mathbb{Z}$, while for $f : \mathbb{R} \to \mathbb{R}$ the solution always exists. A few years before that, a "multiplicative" variant of this equation, namely $f^q(n) = kn$ where $(f(n))_{n\geqslant 0}$ is an increasing sequence of nonnegative integers, was analyzed for $q = 2$ [1], and very recently the case for a general q has also been analyzed [3]. There are also some general results on the functional equation $f^q = g$: this equation was solved for $q = 2$ by Isaacs [2], and for a general q by Łojasiewicz [4] when g is a bijection and finally by Zimmermann [6] for any g.

We here consider the functional equation

$$f^q(n) = f(n+1) + k, \tag{1.1}$$

where $q \geqslant 2$ and $k \in \mathbb{Z}$ are given, and $f : \mathbb{N} \to \mathbb{N}$ (here and onward, $\mathbb{N}$ denotes the set of positive integers). The functional equation (1.1) is a modification of the already mentioned functional equation determining roots of translation, $f^q(n) = n + k$ (note, however, that the functional equation (1.1) is not of the form $f^q(n) = g(n)$ for some function g given in advance). The motivation for studying the functional equation (1.1) is not only its relation to roots of translations, but also the fact that the functional equation (1.1) can be thought of as the "prototypical case" of the more general functional equation $f^q(n) = f(n+l)+k$, or even $f^q(n) = f^r(n+l)+k$; these functional equations are of a very broad scope and yet quite natural looking, and thus gaining a better knowledge about the behavior of their solutions might be very interesting.

Our main result is the following theorem.

Theorem 1.1. *Given $q \geqslant 2$ and $k \in \mathbb{Z}$, the functional equation*

$$f^q(n) = f(n+1) + k \tag{1.1 revisited}$$

has a solution $f : \mathbb{N} \to \mathbb{N}$ if and only if either $k = 0$ or $k \geqslant -1$ and $q - 1 \mid k + 1$.

In Section 2 we show that there are no solutions for $k \leqslant -2$, while in Section 3 we show that there are no solutions when $k \geqslant -1$ and $q - 1 \nmid k + 1$, which, together with explicit examples of solutions that we shall provide in all the other cases, completes the proof. In Section 4 we then completely solve the considered functional equation in the case $q = 3$ and $k = 1$, which is, as we shall see, an example that suggests that the solutions can have a very unexpected behavior and that it might be pretty hard to describe all solutions in the general case. In what follows, we let f denote a function from $\mathbb{N}$ to itself that satisfies the functional equation (1.1), where $q \geqslant 2$ and $k \in \mathbb{Z}$ are fixed. To introduce one more convenient notation, for any $U \subseteq \mathbb{N}$ we let $f[U]$ denote the set $\{f(u) : u \in U\}$.

2 The case $k \leqslant -2$

Let $k \leqslant -2$, and let $k' = -k$; therefore, $k' \geqslant 2$. We prove the following lemma.

Lemma 2.1. *There does not exist a function $f : \mathbb{N} \to \mathbb{N}$ such that the equality*

$$f^q(n) = f(n+1) - k' \tag{2.1}$$

holds for each $n \in \mathbb{N}$.

Proof. We first prove that such a function f, if it existed, would have to be strictly increasing. By induction on n_1, we prove that for each n_1 and all $n_2 > n_1$ the inequality $f(n_2) > f(n_1)$ holds.

Let first $n_1 = 1$. Suppose the opposite: there exists $n_2 > 1$ such that $f(n_2) \leqslant f(1)$. Choose such n_2 for which $f(n_2)$ is minimal. Since $n_2 > 1$, we have $n_2 - 1 \in \mathbb{N}$. By the equation (2.1) for $n = n_2 - 1$ we obtain

$$f(f^{q-1}(n_2 - 1)) = f^q(n_2 - 1) = f(n_2) - k' < f(n_2) \leqslant f(1). \tag{2.2}$$

It follows that $f^{q-1}(n_2 - 1) \neq 1$, and therefore $f^{q-1}(n_2 - 1) > 1$. However, since by (2.2) we have $f(f^{q-1}(n_2 - 1)) < f(n_2)$, this contradicts the fact that n_2 is chosen among all the numbers greater than 1 in such a way that $f(n_2)$ is minimal. Therefore, the base of induction, for $n_1 = 1$, is proved.

Let now n_1 be given, and assume that for each $n_1' < n_1$ and all $n_2 > n_1'$ the inequality $f(n_2) > f(n_1')$ holds. We need to prove that for all $n_2 > n_1$ the inequality $f(n_2) > f(n_1)$ holds. Suppose the opposite: there exists $n_2 > n_1$ such that $f(n_2) \leqslant f(n_1)$. Choose such n_2 for which $f(n_2)$ is minimal. By the equation (2.1) for $n = n_2 - 1$ we obtain

$$f(f^{q-1}(n_2 - 1)) = f^q(n_2 - 1) = f(n_2) - k' < f(n_2) \leqslant f(n_1). \tag{2.3}$$

Note that for any $v \geqslant n_1$ the induction hypothesis gives $f(v) > f(n_1 - 1) > f(n_1 - 2) > \cdots > f(1) \geqslant 1$, from which we get $f(v) \geqslant n_1$. Since $n_2 > n_1$, it follows that $n_2 - 1 \geqslant n_1$, and thus $f(n_2 - 1) \geqslant n_1$. Repeating the same argument gives $f(f(n_2-1)) \geqslant n_1$, and after $q-1$ repetitions we get $f^{q-1}(n_2-1) \geqslant n_1$. Further, by (2.3) we see that $f^{q-1}(n_2 - 1) \neq n_1$, and therefore $f^{q-1}(n_2-1) > n_1$. However, since by (2.3) we have $f(f^{q-1}(n_2-1)) < f(n_2)$, this contradicts the fact that n_2 is chosen among all the numbers greater than n_1 in such a way that $f(n_2)$ is minimal. This completes the inductive step, and thus we get that f is strictly increasing.

Let $(a_i)_{i=1}^{\infty}$ be the auxiliary sequence defined recursively by $a_1 = 1$ and $a_{i+1} = (q-1)a_i + 2$. By induction on i, we are going to prove that for each i and all $n \geqslant 3$ we have the inequality

$$f(n) \geqslant n + a_i k' - a_i. \tag{2.4}$$

Putting $n = 1$ into (2.1) gives $f^q(1) = f(2) - k'$, from which we get $f(2) \geqslant k' + 1 = 2 + k' - 1$. Since f is strictly increasing, we further get

$$f(n) \geqslant n + k' - 1 \text{ for all } n \geqslant 2. \tag{2.5}$$

This proves (2.4) for $i = 1$ (and actually for $n \geqslant 2$, which is more than claimed). Assume now that (2.4) holds for a given i, and let us prove that it also holds for $i+1$. Let $n \geqslant 3$ be given. Keeping in mind that f is strictly increasing, we get

$$\begin{aligned}
f(n) &\overset{(2.1)}{=} f^q(n-1) + k' = f^{q-1}(f(n-1)) + k' \overset{(2.5)}{\geqslant} f^{q-1}(n + k' - 2) + k' \\
&= f^{q-2}(f(n+k'-2)) + k' \overset{(2.4)}{\geqslant} f^{q-2}(n + k' - 2 + a_i k' - a_i) + k' \\
&= f^{q-2}(n + (a_i+1)k' - (a_i+2)) + k' \\
&\overset{(2.4)}{\geqslant} f^{q-3}(n + (2a_i+1)k' - (2a_i+2)) + k' \overset{(2.4)}{\geqslant} \cdots \\
&\overset{(2.4)}{\geqslant} f\big(n + ((q-2)a_i+1)k' - ((q-2)a_i+2)\big) + k' \\
&\overset{(2.4)}{\geqslant} n + ((q-1)a_i+1)k' - ((q-1)a_i+2) + k' \\
&= n + ((q-1)a_i+2)k' - ((q-1)a_i+2) = n + a_{i+1}k' - a_{i+1},
\end{aligned}$$

which was to be proved. (Here is a clarification of some steps in the preceding chain of inequalities. In the first line, where the inequality (2.5) is applied, the argument of f is $n-1$, which is indeed $\geqslant 2$, and thus the inequality (2.5) can be applied, but since this argument might not be $\geqslant 3$, we cannot apply the induction hypothesis (2.4) here. In the second line, where the inequality

(2.4) is applied, the argument of f is $n+k'-2$, and since $n \geqslant 3$ and $k' \geqslant 2$, this argument is indeed $\geqslant 3$ and thus the induction hypothesis (2.4) can be applied here; the same holds for further applications of (2.4).)

The fact that the inequality (2.4) holds for each i and all $n \geqslant 3$ is a clear contradiction: indeed, for any fixed n, the right-hand side of (2.4) is unbounded as $i \to \infty$, and thus it is impossible that the inequality (2.4) holds for each i. This contradiction completes the proof. ■

3 The case $k \geqslant -1$ and putting the pieces together

We first prove a lemma that gives a very useful conclusion when $q \geqslant 3$ and $k > 0$.

Lemma 3.1. *Let $q \geqslant 3$ and $k > 0$. Then any $f : \mathbb{N} \to \mathbb{N}$ that satisfies the functional equation (1.1) must be injective.*

Proof. Suppose the opposite: there exist $d, e \in \mathbb{N}$, $d \neq e$, such that $f(d) = f(e)$. Then $f^q(d) = f^q(e)$, and thus by (1.1) we get $f(d+1)+k = f(e+1)+k$, that is, $f(d+1) = f(e+1)$. Repeating the same argument leads to $f(d+j) = f(e+j)$ for each $j \in \mathbb{N}$, which implies that the function f takes only finitely many values.

Let $f(z)$ be the greatest value of the function f. If it were $z > 1$, putting $n = z-1$ in (1.1) gives $f^q(z-1) = f(z)+k > f(z)$, a contradiction. Therefore, $f(1)$ is the unique greatest value of the function f. Let $f(z_2)$ be the second largest value of the function f (we do not assume that it is unique). Since

$$f^q(z_2-1) = f(z_2)+k, \tag{3.1}$$

that is, $f(z_2)+k$ belongs to the set of values of f, and since $f(z_2)$ is the second largest value, we conclude that $f(z_2)+k$ must be the greatest value of f, that is, $f(z_2)+k = f(1)$. In a similar manner, if $f(z_3)$ is the third largest value of f, we have $f^q(z_3-1) = f(z_3)+k$, from which we conclude $f(z_3)+k = f(z_2)$. By continuing this reasoning, we conclude that the set of values of the function f forms an arithmetic progression with step k. Finally, by (3.1) we get $f^q(z_2-1) = f(1)$; since we earlier obtained that there does not exist $z \neq 1$ such that $f(z) = f(1)$, it follows that

$$f^{q-1}(z_2-1) = 1, \tag{3.2}$$

that is, the number 1 belongs to the set of values of f. Altogether, we conclude that the set of values of the function f is

$$\{1 + ik : 0 \leqslant i \leqslant m\} \tag{3.3}$$

for some $m \in \mathbb{N}$.

Since $f(1) \neq 1$, by (3.2) we get $f^{q-2}(z_2 - 1) > 1$. Since $q \geqslant 3$, it follows that $f^{q-2}(z_2 - 1)$ belongs to the set of values of f. By the fact that this set is of the form (3.3), we get that there exists $v \in \mathbb{N}$ such that $f(v) = f^{q-2}(z_2 - 1) - k$. Since $f(1)$ is the greatest value of f, we conclude that $v \neq 1$, and thus we may substitute $n = v - 1$ in (1.1), which gives $f^q(v - 1) = f(v) + k = f^{q-2}(z_2 - 1)$. By (3.2), this implies

$$f^{q+1}(v - 1) = f^{q-1}(z_2 - 1) = 1.$$

On the other hand, (1.1) gives

$$f^{q+1}(v - 1) = f^q(f(v - 1)) = f(f(v - 1) + 1) + k > 1.$$

The contradiction from the last two conclusions proves that f is an injection. ■

The following lemma establishes a restriction on the relationship between q and k.

Lemma 3.2. *If there exists a solution of the functional equation (1.1), where $k \geqslant -1$ and $k \neq 0$, then $q - 1 \mid k + 1$.*

Proof. Let f be a solution of the functional equation (1.1). If $q = 2$ or $k = -1$, then the conclusion holds. Therefore, we may assume $q \geqslant 3$ and $k > 0$. By Lemma 3.1, f is injective.

Let

$$A_1 = \mathbb{N} \setminus f[\mathbb{N}] \text{ and } A_i = f[A_{i-1}] \text{ for } 2 \leqslant i \leqslant q.$$

Since f is injective, we have

$$|A_1| = |A_2| = \cdots = |A_q|. \tag{3.4}$$

Also by the injectivity of f we get

$$A_i = f^{i-1}[\mathbb{N}] \setminus f^i[\mathbb{N}],$$

which implies that the sets $A_1, A_2, \ldots, A_q$ are pairwise disjoint. Since $f[\mathbb{N}] \supseteq f^2[\mathbb{N}] \supseteq \cdots \supseteq f^q[\mathbb{N}]$, we easily get

$$A_2 \cup A_3 \cup \cdots \cup A_q = f[\mathbb{N}] \setminus f^q[\mathbb{N}]. \tag{3.5}$$

The following observation, which immediately follows from (1.1), will be useful:

$$s \in f^q[\mathbb{N}] \text{ if and only if } s - k \in f[\mathbb{N}] \setminus \{f(1)\}. \tag{3.6}$$

For a given i, $0 \leqslant i \leqslant k-1$, let us define

$$s_i = \min\{f(n) : f(n) \equiv i \pmod{k}\}.$$

We also define

$$s_k = \min\{f(n) : f(n) > f(1) \text{ and } f(n) \equiv f(1) \pmod{k}\}.$$

In case that any of these values is undefined (because the set at the right-hand side is empty), any further reference to such s_i should be simply ignored. We claim that

$$A_2 \cup A_3 \cup \cdots \cup A_q = \{s_i : 0 \leqslant i \leqslant k\}. \tag{3.7}$$

To show the direction ($\supseteq$), by (3.5) we see that it is enough to prove that for each i, $0 \leqslant i \leqslant k$, we have $s_i \in f[\mathbb{N}]$ and $s_i \notin f^q[\mathbb{N}]$. It is clear that $s_i \in f[\mathbb{N}]$. Suppose now that for some i we have $s_i \in f^q[\mathbb{N}]$. By (3.6) we then get $s_i - k \in f[\mathbb{N}] \setminus \{f(1)\}$, but this contradicts the minimality of s_i. This proves the direction ($\supseteq$) in (3.7).

Before we show the direction ($\subseteq$), we need a few preparatory observations. Note that from (3.6) we get that if $s \in f[\mathbb{N}]$ and $s \neq f(1)$, then $s+k \in f[\mathbb{N}]$. Repeating this argument leads to $s+jk \in f[\mathbb{N}]$ for each nonnegative j, unless $f(1)$ is among these values. In particular, since $s_i \in f[\mathbb{N}]$, we conclude that $s_i+jk \in f[\mathbb{N}]$ for each nonnegative j and each i, $0 \leqslant i \leqslant k$, with at most one possible exception for i: namely, $i = i_0$ such that $0 \leqslant i_0 \leqslant k-1$ and $f(1) \equiv i_0 \pmod{k}$. Furthermore, we get that $s_{i_0} + jk \in f[\mathbb{N}]$ for each nonnegative j such that $s_{i_0} + jk \leqslant f(1)$.

Let us now return to the direction ($\subseteq$). Suppose the opposite, that an element x different from all s_i belongs to the left-hand side. If $x > f(1)$ and $x \equiv f(1) \pmod{k}$, choose $j \in \mathbb{N}$ such that $x = s_k + jk$; otherwise, we may choose a suitable i, $0 \leqslant i \leqslant k-1$, and $j \in \mathbb{N}$ such that $x = s_i+jk$. (Of course, j is nonzero since we supposed that x is none of the s_i's.) In the first case, since $x-k = s_k+(j-1)k$, by the previous paragraph we get $x-k \in f[\mathbb{N}]$, and in fact $x-k \in f[\mathbb{N}] \setminus \{f(1)\}$ since $x-k = s_k+(j-1)k \geqslant s_k > f(1)$. In the other cases, if $i \neq i_0$ then $x-k = s_i+(j-1)k$ and clearly $x-k \in f[\mathbb{N}] \setminus \{f(1)\}$ (again by the previous paragraph), while if $x = s_{i_0} + jk$, then we have $x \leqslant f(1)$ (since otherwise, as $x \equiv s_{i_0} \equiv i_0 \equiv f(1) \pmod k$, x would fall under the first case), and thus $x-k = s_{i_0}+(j-1)k < f(1)$, from which we again conclude $x-k \in f[\mathbb{N}] \setminus \{f(1)\}$. Therefore, the conclusion is always $x-k \in f[\mathbb{N}] \setminus \{f(1)\}$, and now (3.6) gives $x \in f^q[\mathbb{N}]$. However, by (3.5) then x cannot belong to the left hand side of (3.7), a contradiction. This proves the equality (3.7).

The equality (3.7) implies that $A_2 \cup A_3 \cup \cdots \cup A_q$ has only finitely many elements. By (3.4), the set A_1, that is, $\mathbb{N} \setminus f[\mathbb{N}]$ is also finite. We thus get that each s_i, $0 \leqslant i \leqslant k$, is indeed well defined (since if any of the sets used in the definition of some s_i were empty, it would follow that $\mathbb{N} \setminus f[\mathbb{N}]$ is infinite), and now by (3.7) and the observation that all s_i's are clearly pairwise different we get

$$|A_2 \cup A_3 \cup \cdots \cup A_q| = k+1. \tag{3.8}$$

Together with (3.4) and the fact that the sets A_1, A_2, …, A_q are pairwise disjoint, this implies $q-1 \mid k+1$. The proof is completed. ∎

We are now ready for the proof of the main result.

Proof of Theorem 1.1. If $k=0$, the functional equation (1.1) has a solution $f(n) \equiv c$, where c is an arbitrary constant. If $k \geqslant -1$ and $q-1 \mid k+1$, the functional equation (1.1) has a solution $f(n) = n + \frac{k+1}{q-1}$ (indeed, both sides evaluate to $n + \frac{q(k+1)}{q-1}$). If $k \leqslant -2$ then there is no solution by Lemma 2.1, while if $k \geqslant -1$ but $q-1 \nmid k+1$, then there is no solution by Lemma 3.2. ∎

4 Obtaining all solutions for the case $q = 3$ and $k = 1$

In this section we completely solve the example $q = 3$, $k = 1$. These values are chosen because they neatly illustrate that the considered functional equation can have a very unexpected set of solutions even with quite small parameters.

Example 4.1. *There exist exactly two functions that satisfy the functional equation*

$$f(f(f(n))) = f(n+1) + 1, \tag{4.1}$$

namely

$$f(n) = n+1 \qquad \text{and} \qquad f(n) = \begin{cases} n+1, & \text{for } n \equiv 1,3 \pmod 4; \\ n+5, & \text{for } n \equiv 2 \pmod 4; \\ n-3, & \text{for } n \equiv 0 \pmod 4. \end{cases} \tag{4.2}$$

Proof. Let f be a function that satisfies the functional equation (4.1). By Lemma 3.1, f is an injection. (Note: in this special case the injectivity of f can also be established in a somewhat simpler manner than in Lemma 3.1. Namely, as in Lemma 3.1 we get that f takes only finitely many values. On the other hand, by the equality (4.1) applied twice, we get

$$f(f(f(f(n+1)))) = f(f(n+1)+1) + 1 = f(f(f(f(n)))) + 1,$$

from which it follows that the set of values of the function f^4, and thus also of the function f, is infinite. This contradiction proves that f is injective.)

As in the proof of Lemma 3.2, the observation

$$s \in f[f[f[\mathbb{N}]]] \text{ if and only if } s-1 \in f[\mathbb{N}] \setminus \{f(1)\} \tag{4.3}$$

will be very useful. We shall employ some more ideas from the proof of Lemma 3.2. We let

$$A_1 = \mathbb{N} \setminus f[\mathbb{N}],\ A_2 = f[\mathbb{N}] \setminus f^2[\mathbb{N}] \text{ and } A_3 = f^2[\mathbb{N}] \setminus f^3[\mathbb{N}].$$

As in the proof of Lemma 3.2 we get $|A_2 \cup A_3| = 2$ (namely, this is the equation (3.8)), that is, $|A_2| = |A_3| = 1$, and now also $|A_1| = 1$, that is,

$$\mathbb{N} \setminus f[\mathbb{N}] = \{a\}$$

for some $a \in \mathbb{N}$. In the rest of the proof we distinguish two cases: $a = 1$ and $a > 1$.

The case $a = 1$**.** Choose $b \in \mathbb{N}$ such that $f(b) = 2$. If $b \neq 1$, then there exists $b' \in \mathbb{N}$ such that $f(b') = b$, that is, $f(f(b')) = 2$. Since, by (4.3), $2 \notin f[f[f[\mathbb{N}]]]$, the only possibility is $b' = 1$. Then $f(f(f(1))) =$

$f(f(b)) = f(2)$, which contradicts (4.1). Hence, the assumption $b \neq 1$ leads to a contradiction, and we thus conclude $b = 1$, that is,

$$f(1) = 2. \tag{4.4}$$

Choose $c \in \mathbb{N}$ such that $f(c) = 3$. Clearly, $c \neq 1$, and therefore there exists $c' \in \mathbb{N}$ such that $f(c') = c$, that is, $f(f(c')) = 3$. If $c' \neq 1$, then there exists $c'' \in \mathbb{N}$ such that $f(c'') = c'$, that is, $f(f(f(c''))) = 3$. The equality (4.1) now gives $f(f(f(c''))) = f(c'' + 1) + 1$, from which it follows that $f(c'' + 1) = 2$, but this contradicts (4.4) and the injectivity of f. We thus conclude $c' = 1$, $f(c') = f(1) = 2$ and

$$f(2) = f(f(c')) = 3.$$

Let us now show by induction that $f(n) = n + 1$ for each $n \in \mathbb{N}$. We already have this for $n = 1$ and $n = 2$. Assume now that the assertion holds for all the numbers less than a given $n \geqslant 3$, and let us prove the assertion for n. From the equality (4.1) we obtain

$$f(f(f(n-2))) = f(n-1) + 1,$$

and by the induction hypothesis we get $f(f(f(n-2))) = f(f(n-1)) = f(n)$ and $f(n-1) + 1 = n + 1$. To summarize, if $a = 1$, then

$$f(n) = n + 1.$$

The case $a > 1$. Note that $f(1) \neq 1$, since otherwise it would follow that $f(f(f(1))) = 1$, which contradicts (4.1). Therefore, $1 \in f[\mathbb{N}] \setminus \{f(1)\}$, and now by (4.3) we obtain $2 \in f[f[f[\mathbb{N}]]]$, that is, $a \geqslant 3$. Since $a - 1 \in f[\mathbb{N}]$ but $a \notin f[f[f[\mathbb{N}]]]$, by (4.3) we conclude

$$a - 1 = f(1). \tag{4.5}$$

Further, we have $a - 2 \in f[\mathbb{N}] \setminus \{f(1)\}$, and again by (4.3) we conclude $a - 1 \in f[f[f[\mathbb{N}]]]$. The last two conclusion, together with the injectivity of f, imply that there exists $b \in \mathbb{N}$ such that $f(f(b)) = 1$. If $b \neq a$, then $b \in f[\mathbb{N}]$ and thus $1 \in f[f[f[\mathbb{N}]]]$, which we have already seen to be impossible. Therefore, $b = a$, that is,

$$f(f(a)) = 1. \tag{4.6}$$

Choose $c \in \mathbb{N}$ such that $f(c) = a + 1$. If $c \neq a$, then there exists $c' \in \mathbb{N}$ such that $f(c') = c$, that is, $f(f(c')) = a + 1$. By (4.6) it follows that $c' \neq a$, from which we get $c' \in f[\mathbb{N}]$ and then $a + 1 \in f[f[f[\mathbb{N}]]]$, which by (4.3) implies $a \in f[\mathbb{N}]$, a contradiction. Hence, the assumption $c \neq a$ leads to a contradiction, and we thus conclude $c = a$, that is,

$$f(a) = a + 1. \tag{4.7}$$

Now from the equality $f(f(f(a))) = f(a+1)+1$, together with $f(f(f(a))) = f(1)$ (by (4.6)) and $f(a + 1) = f(f(a)) = 1$ (by (4.7) and (4.6)), we get $f(1) = 2$. Together with (4.5) this implies

$$a = 3.$$

Therefore, so far we have $f(1) = 2$, $f(3) = 4$ and $f(4) = 1$. We further obtain

$$f(f(2)) = f(f(f(1))) = f(2) + 1$$

(we have used (4.1) for $n = 1$), then

$$f(f(2) + 1) = f(f(f(2))) = f(3) + 1 = 5,$$

and now

$$f(5) = f(f(f(f(2)))) = f(f(2) + 1) + 1 = 5 + 1 = 6$$

(the second equality follows from (4.1) for $n = f(2)$). Finally, we get

$$f(2) = f(f(1)) = f(f(f(4))) = f(5) + 1 = 7.$$

Our aim here is to prove that in the current case (that is, $a > 1$) the only solution is the second function from (4.2). Since the values of f obtained so far support this conclusion for $n \leqslant 5$, in order to finish the proof it is enough to show the following inductive step: if $f(4t - 2)$, $f(4t - 1)$, $f(4t)$, $f(4t + 1)$ equal $4t + 3$, $4t$, $4t - 3$, $4t + 2$ for all $t \leqslant t_0$, respectively, then $f(4t_0 + 2)$, $f(4t_0 + 3)$, $f(4t_0 + 4)$, $f(4t_0 + 5)$ equal $4t_0 + 7$, $4t_0 + 4$, $4t_0 + 1$, $4t_0 + 6$, respectively. We note

$$f(4t_0+3) = f(f(4t_0-2)) = f(f(f(4t_0-3))) = f(4t_0-2)+1 = 4t_0+4, \tag{4.8}$$

from which it follows that

$$f(4t_0+4) = f(f(4t_0+3)) = f(f(f(4t_0-2))) = f(4t_0-1)+1 = 4t_0+1. \tag{4.9}$$

We further note

$$f(f(4t_0+2)) = f(f(f(4t_0+1))) = f(4t_0+2)+1,$$

then

$$f(f(4t_0+2)+1) = f(f(f(4t_0+2))) = f(4t_0+3)+1 = 4t_0+5,$$

and now

$$f(4t_0+5) = f(f(f(f(4t_0+2)))) = f(f(4t_0+2)+1)+1 = 4t_0+5+1 = 4t_0+6. \tag{4.10}$$

Finally, we get

$$f(4t_0+2) = f(f(4t_0+1)) = f(f(f(4t_0+4))) = f(4t_0+5)+1 = 4t_0+7. \tag{4.11}$$

The equalities (4.8), (4.9), (4.10) and (4.11) are what was needed to complete the proof. ■

5 Future directions

As mentioned in the Introduction, establishing a necessary and sufficient condition for the existence of solution of the more general functional equation

$$f^q(n) = f(n+l)+k$$

or even

$$f^q(n) = f^r(n+l)+k$$

is a natural next step. The present author believes that the condition $q-r \mid k+l$ has some role in that necessary and sufficient condition, but it seems that there is a lot of work still to be done here.

The question after that is whether the set of all solutions of these functional equations, at least in some special case, admits some nice description. Judging by Example 4.1, this might be harder than it seems.

Another interesting line of research is to analyze these functional equations when $f : \mathbb{Z} \to \mathbb{Z}$. It seems that, unfortunately, not many ideas from this work are applicable in that case.

Acknowledgments

The author would like to thank the anonymous referee for valuable comments.

The research was supported by the Ministry of Science and Technological Development of Serbia (project 174006) and by the Provincial Secretariat for Science and Technological Development, Autonomous Province of Vojvodina (project "Ordered structures and applications").

References

[1] J.-P. Allouche & N. Rampersad & J. Shallit, On integer sequences whose first iterates are linear, *Aequationes Math.* **69** (2005), 114–127.

[2] R. Isaacs, Iterates of fractional order, *Canadian J. Math.* **2** (1950), 409–416.

[3] V. Laohakosol & B. Yuttanan, Iterates of increasing sequences of positive integers, *Aequationes Math.* **87** (2014), 89–103.

[4] S. Łojasiewicz, Solution générale de l'équation fonctionelle $f(f(\cdots f(x)\cdots)) = g(x)$, *Ann. Soc. Polon. Math.* **24** (1951) 88–91.

[5] K. S. Sarkaria, Roots of translations, *Aequationes Math.* **75** (2008), 304–307.

[6] G. Zimmermann, *Über die Existenz iterativer Wurzeln von Abbildungen*, Inaugural-Dissertation, Philipps-Universität Marburg/Lahn, 1978.