MAXIMAL CHAINS OF ISOMORPHIC SUBGRAPHS OF COUNTABLE ULTRAHOMOGENEOUS GRAPHS

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Abstract

For a countable ultrahomogeneous graph $\mathbb{G} = \langle G, \rho \rangle$ let $\mathbb{P}(\mathbb{G})$ denote the collection of sets $A \subset G$ such that $\langle A, \rho \cap [A]^2 \rangle \cong \mathbb{G}$. The order types of maximal chains in the poset $\langle \mathbb{P}(\mathbb{G}) \cup \{\emptyset\}, \subset \rangle$ are characterized as:

(I) the order types of compact sets of reals having the minimum non-isolated, if $\mathbb{G}$ is the Rado graph or the Henson graph $\mathbb{H}_n$, for some $n \geq 3$;

(II) the order types of compact nowhere dense sets of reals having the minimum non-isolated, if $\mathbb{G}$ is the union of μ disjoint complete graphs of size ν, where $\mu\nu = \omega$.

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1 Introduction

If $\mathbb{X}$ is a relational structure, $\mathbb{P}(\mathbb{X})$ will denote the set of domains of substructures of $\mathbb{X}$ which are isomorphic to $\mathbb{X}$. $\mathbb{X}$ is called *ultrahomogeneous* iff each isomorphism between two finite substructures of $\mathbb{X}$ can be extended to an automorphism of $\mathbb{X}$.

A structure $\mathbb{G} = \langle G, \rho \rangle$ is a *graph* iff G is a set and ρ a symmetric irreflexive binary relation on G. We will also use the following equivalent definition: a pair $\mathbb{G} = \langle G, \rho \rangle$ is a graph iff G is a set and $\rho \subset [G]^2$. Then for $H \subset G$, $\langle H, \rho \cap [H]^2 \rangle$ (or $\langle H, \rho \cap (H \times H) \rangle$, in the relational version) is the corresponding *subgraph* of $\mathbb{G}$. For a cardinal ν, $\mathbb{K}_\nu$ will denote the *complete graph* of size ν. A graph is called $\mathbb{K}_n$-free iff it has no subgraphs isomorphic to $\mathbb{K}_n$. We will use the following well-known classification of countable ultrahomogeneous graphs [9]:

Theorem 1.1 (Lachlan and Woodrow) Each countable ultrahomogeneous graph is isomorphic to one of the following graphs

- $\mathbb{G}_{\mu\nu}$, the union of μ disjoint copies of $\mathbb{K}_\nu$, where $\mu\nu = \omega$,
- $\mathbb{G}_{\mathrm{Rado}}$, the unique countable homogeneous universal graph, the Rado graph,
- $\mathbb{H}_n$, the unique countable homogeneous universal $\mathbb{K}_n$-free graph, for $n \geq 3$,
- the complements of these graphs.

Properties of maximal chains in posets are widely studied order invariants (see [1], [3], [4], [10], [11]) and, as a part of investigation of the partial orders of the form

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$\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, where $\mathbb{X}$ is a relational structure, the class of order types of maximal chains in the poset $\langle \mathbb{P}(\mathbb{G}_{\mathrm{Rado}}), \subset \rangle$ was characterized in [7]. The aim of this paper is to complete the picture for all countable ultrahomogeneous graphs in this context and, thus, the following theorem is our main result.

Theorem 1.2 Let $\mathbb{G}$ be a countable ultrahomogeneous graph. Then
(I) If $\mathbb{G} = \mathbb{G}_{\mathrm{Rado}}$ or $\mathbb{G} = \mathbb{H}_n$, for some $n \geq 3$, then for each linear order L the following conditions are equivalent:
(a) L is isomorphic to a maximal chain in the poset $\langle \mathbb{P}(\mathbb{G}) \cup \{\emptyset\}, \subset \rangle$;
(b) L is an $\mathbb{R}$-embeddable complete linear order with 0_L non-isolated;
(c) L is isomorphic to a compact set $K \subset \mathbb{R}$ having the minimum non-isolated.
(II) If $\mathbb{G} = \mathbb{G}_{\mu\nu}$, where $\mu\nu = \omega$, then for each linear order L the following conditions are equivalent:
(a) L is isomorphic to a maximal chain in the poset $\langle \mathbb{P}(\mathbb{G}) \cup \{\emptyset\}, \subset \rangle$;
(b) L is an $\mathbb{R}$-embeddable Boolean linear order with 0_L non-isolated;
(c) L is isomorphic to a compact nowhere dense set $K \subset \mathbb{R}$ having the minimum non-isolated.

It is easy to check that for a relational structure $\langle X, \rho \rangle$ we have $\mathbb{P}(\langle X, \rho \rangle) = \mathbb{P}(\langle X, \rho^c \rangle)$ and, hence, regarding Theorem 1.1, Theorem 1.2 in fact covers all countable ultrahomogeneous graphs. The statement (I) for the Rado graph is proved in [7] and in this paper we consider the Henson graphs $\mathbb{H}_n$ and the graphs $\mathbb{G}_{\mu\nu}$.

The outline of the paper is as follows. Section 2 contains necessary definitions and facts. The main result of Section 3, Theorem 3.2, is general and gives a condition (for a countable relational structure $\mathbb{X}$) providing that for each compact set $K \subset \mathbb{R}$ satisfying $\min K \in K'$ (e.g. for $[0,1]$ or the Cantor set) the poset of copies of $\mathbb{X}$ contains a maximal chain isomorphic to $K \setminus \{\min K\}$. In Section 4 we show that the graphs $\mathbb{H}_n$ satisfy this condition and, using Theorem 3.2, prove the corresponding part of Theorem 1.2 for the most complex class of countable ultrahomogeneous graphs - the class of Henson graphs. Essentially we construct a relational structure $\langle \mathbb{Q}, <, \rho \rangle$ satisfying conditions of Theorem 3.2 and such that $\langle \mathbb{Q}, \rho \rangle \cong \mathbb{H}_n$. Our construction is generic - we use the partial order of finite approximations of the structure $\langle \mathbb{Q}, \rho \rangle$ and, applying the Rasiowa-Sikorski theorem to a countable family of suitably chosen dense subsets of this partial order, we obtain a filter intersecting all of them and coding the relation ρ. In Section 5 we prove Theorem 1.2 for disjoint unions of complete graphs, which completes its proof.

2 Preliminaries

In this section we recall basic definitions and facts which will be used in the paper.

If $\langle P, \leq \rangle$ is a partial order, then the *smallest* and the *largest element* of P are denoted by 0_P and 1_P; the *intervals* $(x,y)_P$, $[x,y]_P$, $(-\infty, x)_P$ etc. are defined in the usual way. A set $D \subset P$ is *dense* iff for each $p \in P$ there is $q \in D$ such that

$q \leq p$. A set $G \subset P$ is a *filter* iff (F1) for each $p, q \in G$ there is $r \in G$ such that $r \leq p, q$ and (F2) $G \ni p \leq q$ implies $q \in G$.

Fact 2.1 (Rasiowa-Sikorski) If D_n, $n \in \omega$ are dense sets in a partial order $\langle P, \leq \rangle$, then there is a filter G in P intersecting all of them.

Proof. Let $p_0 \in D_0$ and, for $n \in \omega$, let us pick $p_{n+1} \in D_{n+1}$ such that $p_{n+1} \leq p_n$. Then $G = \{p \in P : \exists n \in \omega \; p_n \leq p\}$ is a filter intersecting all D_n's. □

A pair $\langle \mathcal{A}, \mathcal{B} \rangle$ is a *cut* in a linear order $\langle L, < \rangle$ iff $L = \mathcal{A} \dot{\cup} \mathcal{B}$, $\mathcal{A}, \mathcal{B} \neq \emptyset$ and $a < b$, for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$. A cut $\langle \mathcal{A}, \mathcal{B} \rangle$ is a *gap* iff neither $\max \mathcal{A}$ nor $\min \mathcal{B}$ exist. $\langle L, < \rangle$ is called: *complete* iff it has 0 and 1 and has no gaps; *dense* iff $(x, y)_L \neq \emptyset$, for each $x, y \in L$ satisfying $x < y$; $\mathbb{R}$-*embeddable* iff it is isomorphic to a subset of $\mathbb{R}$; *Boolean* iff it is complete and *has dense jumps*, which means that for each $x, y \in L$ satisfying $x < y$ there are $a, b \in L$ such that $x \leq a < b \leq y$ and $(a, b)_L = \emptyset$. A set $D \subset L$ is called *dense in* L iff for each $x, y \in L$ satisfying $x < y$ there is $z \in D$ such that $x < z < y$. If $\langle I, <_I \rangle$ and $\langle L_i, <_i \rangle$, $i \in I$, are linear orders and $L_i \cap L_j = \emptyset$, whenever $i \neq j$, then the corresponding *lexicographic sum* $\sum_{i \in I} L_i$ is the linear ordering $\langle \bigcup_{i \in I} L_i, < \rangle$ where the relation $<$ is defined by: $x < y \Leftrightarrow \exists i \in I \; (x, y \in L_i \wedge x <_i y) \vee \exists i, j \in I \; (i <_I j \wedge x \in L_i \wedge y \in L_j)$.

Fact 2.2 If $\langle L, < \rangle$ is an at most countable complete linear order, it is Boolean.

Proof. Let $x, y \in L$ and $x < y$. Suppose that for each $a, b \in [x, y]_L$ satisfying $a < b$ we have $(a, b)_L \neq \emptyset$. Then $[x, y]_L$ would be a dense complete linear order, which is impossible because L is countable. Thus L has dense jumps. □

$\mathcal{P} \subset P(\omega)$ is called a *positive family* iff (P1) $\emptyset \notin \mathcal{P}$; (P2) $\mathcal{P} \ni A \subset B \subset \omega \Rightarrow B \in \mathcal{P}$; (P3) $A \in \mathcal{P} \wedge |F| < \omega \Rightarrow A \backslash F \in \mathcal{P}$; (P4) $\exists A \in \mathcal{P} \; |\omega \backslash A| = \omega$.

Fact 2.3 (See [6]) If $\mathcal{P} \subset P(\omega)$ is a positive family, then for each linear order L the following conditions are equivalent:

(a) L is isomorphic to a maximal chain in the poset $\langle \mathcal{P} \cup \{\emptyset\}, \subset \rangle$;

(b) L is an $\mathbb{R}$-embeddable Boolean linear order with 0_L non-isolated;

(c) L is isomorphic to a compact nowhere dense set $K \subset \mathbb{R}$ having the minimum non-isolated.

(d) L is isomorphic to a maximal chain $\mathcal{L}$ in the poset $\langle \mathcal{P} \cup \{\emptyset\}, \subset \rangle$ such that $\bigcap(\mathcal{L} \setminus \{\emptyset\}) = \emptyset$.

Fact 2.4 Let $A \subset B \subset \omega$ and let L be a complete linear ordering, such that $|B \setminus A| = |L| - 1$. Then there is a chain $\mathcal{L}$ in $[A, B]_{P(B)}$ satisfying $A, B \in \mathcal{L} \cong L$ and such that $\bigcup \mathcal{A}, \bigcap \mathcal{B} \in \mathcal{L}$ and $|\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1$, for each cut $\langle \mathcal{A}, \mathcal{B} \rangle$ in $\mathcal{L}$.

Proof. If $|B \setminus A|$ is a finite set, say $B = A \cup \{a_1, \ldots a_n\}$, then $|L| = n + 1$ and $\mathcal{L} = \{A, A \cup \{a_1\}, A \cup \{a_1, a_2\}, \ldots, B\}$ is a chain with the desired properties.

If $|B \backslash A| = \omega$, then L is a countable and, hence, $\mathbb{R}$-embeddable complete linear order. It is known that an infinite linear order is isomorphic to a maximal chain in

$P(\omega)$ iff it is $\mathbb{R}$-embeddable and Boolean (see, for example, [5]). By Fact 2.2 L is a Boolean order and, thus, there is a maximal chain $\mathcal{L}_1$ in $P(B \setminus A)$ isomorphic to L. Let $\mathcal{L} = \{A \cup C : C \in \mathcal{L}_1\}$. Since $\emptyset, B \setminus A \in \mathcal{L}_1$ we have $A, B \in \mathcal{L}$ and the function $f : \mathcal{L}_1 \to \mathcal{L}$, defined by $f(C) = A \cup C$, witnesses that $\langle \mathcal{L}_1, \subsetneq\rangle \cong \langle \mathcal{L}, \subsetneq\rangle$ so $\mathcal{L}$ is isomorphic to L. For each cut $\langle \mathcal{A}, \mathcal{B}\rangle$ in $\mathcal{L}_1$ we have $\bigcup \mathcal{A} \subset \bigcap \mathcal{B}$ and, by the maximality of $\mathcal{L}_1$, $\bigcup \mathcal{A}, \bigcap \mathcal{B} \in \mathcal{L}_1$ and $|\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1$. Clearly, the same is true for each cut in $\mathcal{L}$. $\Box$

3 General results

The following three general statements, concerning the class of the order types of maximal chains of copies of relational structures, will be used in the proof of Theorem 1.2. The first one gives a necessary condition for a type to be in the class corresponding to a countable ultrahomogeneous structure.

Theorem 3.1 ([8]) Let $\mathbb{X}$ be a countable ultrahomogeneous structure of an at most countable relational language and $\mathbb{P}(\mathbb{X}) \neq \{X\}$. Then for each linear order L we have (a) $\Rightarrow$ (b), where

(a) L is isomorphic to a maximal chain in the poset $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset\rangle$;

(b) L is an $\mathbb{R}$-embeddable complete linear order with 0_L non-isolated.

In particular, the union of a chain in $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ belongs to $\mathbb{P}(\mathbb{X})$.

The following statement describes a class of structures such that, regarding Theorem 3.1, the implication (b) $\Rightarrow$ (a) holds for each linear order L.

Theorem 3.2 Let $\mathbb{X}$ be a countable relational structure and $\mathbb{Q}$ the set of rationals.

(A) If there exist a partition $\{J_n : n \in \omega\}$ of $\mathbb{Q}$ and a structure with the domain $\mathbb{Q}$ of the same signature as $\mathbb{X}$ such that

(i) J_0 is a dense subset of $\mathbb{Q}$,

(ii) J_n, $n \in \mathbb{N}$, are coinitial subsets of $\mathbb{Q}$,

(iii) $J_0 \cap (-\infty, x) \subset A \subset \mathbb{Q} \cap (-\infty, x)$ implies $A \cong \mathbb{X}$, for all $x \in \mathbb{R} \cup \{\infty\}$,

(iv) $J_0 \cap (-\infty, q] \subset C \subset \mathbb{Q} \cap (-\infty, q]$ implies $C \not\cong \mathbb{X}$, for each $q \in J_0$,

then for each uncountable $\mathbb{R}$-embeddable complete linear order L with 0_L non-isolated and such that all initial segments of $L \setminus \{0_L\}$ are uncountable there is a maximal chain in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset\rangle$ isomorphic to L.

(B) If, in addition,

(v) for each countable complete linear order L with 0_L non-isolated there is a maximal chain in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset\rangle$ isomorphic to L, then for each $\mathbb{R}$-embeddable complete linear order L with 0_L non-isolated there is a maximal chain in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$ isomorphic to L.

Proof. Let L be an uncountable $\mathbb{R}$-embeddable complete linear order with 0_L non-isolated.

Claim 3.3 $L \cong \sum_{x\in[-\infty,\infty]} L_x$, where
(L1) L_x, $x \in [-\infty, \infty]$, are at most countable complete linear orders,
(L2) The set $M = \{x \in [-\infty, \infty] : |L_x| > 1\}$ is at most countable,
(L3) $|L_{-\infty}| = 1$ or $0_{L_{-\infty}}$ is non-isolated.

Proof. $L = \sum_{i\in I} L_i$, where L_i are the equivalence classes corresponding to the condensation relation $\sim$ on L given by: $x \sim y \Leftrightarrow |[\min\{x, y\}, \max\{x, y\}]| \leq \omega$ (see [12]). Since L is complete and $\mathbb{R}$-embeddable I is too and, since the cofinalities and coinitialities of L_i's are countable, I is a dense linear order; so $I \cong [0, 1] \cong [-\infty, \infty]$. Hence L_i's are complete and, since $\min L_i \sim \max L_i$, countable. If $|L_i| > 1$, L_i has a jump (Fact 2.2) so, $L \hookrightarrow \mathbb{R}$ gives $|M| \leq \omega$. □

(A) Let all initial segments of $L \setminus \{0_L\}$ be uncountable. Then, by Claim 3.3, $|L_{-\infty}| = 1$, that is $-\infty \notin M$, and we have two cases.
Case I: $\infty \in M$. By (L2) there is an injection $\varphi : M \to \mathbb{N}$. By (L1), for $y \in M$ we have $|L_y| \leq \omega$ and by (ii) $|J_{\varphi(y)} \cap (-\infty, y)| = \omega$ so we take $I_y \in [J_{\varphi(y)} \cap (-\infty, y)]^{|L_y|-1}$. Let us define the sets A_x, $x \in [-\infty, \infty]$ and A^+_x, $x \in M$, by

$$A_x = \begin{cases} \emptyset, & \text{for } x = -\infty, \\ (J_0 \cap (-\infty, x)) \cup \bigcup_{y\in M\cap(-\infty,x)} I_y, & \text{for } x \in (-\infty, \infty]; \end{cases}$$

$$A^+_x = A_x \cup I_x, \quad \text{for } x \in M.$$

Since $J_0 \subset A^+_\infty = J_0 \cup \bigcup_{y\in M} I_y \subset \mathbb{Q}$, by (iii) we have $A^+_\infty \cong \mathbb{X}$ and we construct a maximal chain $\mathcal{L}$ in $\langle \mathbb{P}(A^+_\infty) \cup \{\emptyset\}, \subset \rangle$, such that $\mathcal{L} \cong L$.

Claim 3.4 The sets A_x, $x \in [-\infty, \infty]$ and A^+_x, $x \in M$ are subsets of the set A^+_∞. In addition, for each $x, x_1, x_2 \in [-\infty, \infty]$ we have
(a) $A_x \subset (-\infty, x)$;
(b) $A^+_x \subset (-\infty, x)$, if $x \in M$;
(c) $x_1 < x_2 \Rightarrow A_{x_1} \subsetneq A_{x_2}$;
(d) $M \ni x_1 < x_2 \Rightarrow A^+_{x_1} \subsetneq A_{x_2}$;
(e) $|A^+_x \setminus A_x| = |L_x| - 1$, if $x \in M$;
(f) $A_x \in \mathbb{P}(A^+_\infty)$, for each $x \in (-\infty, \infty]$.
(g) $A^+_x \in \mathbb{P}(A^+_\infty)$ and $[A_x, A^+_x]_{\mathbb{P}(A^+_\infty)} = [A_x, A^+_x]_{P(A^+_x)}$, for each $x \in M$.

Proof. Statements (c) and (d) are true since J_0 is a dense subset of $\mathbb{Q}$; (a), (b) and (e) follow from the definitions of A_x and A^+_x and the choice of the sets I_y. For $x \in (-\infty, \infty]$ we have $J_0 \cap (-\infty, x) \subset A_x \subset \mathbb{Q} \cap (-\infty, x)$ so, by (iii), $A_x \cong \mathbb{X} \cong A^+_\infty$

and (f) is true. If $x \in M$, then $J_0 \cap (-\infty, x) \subset A_x \subset A_x^+ \subset \mathbb{Q} \cap (-\infty, x)$ so, by (iii), $A_x \subset A \subset A_x^+$ implies $A \cong \mathbb{X} \cong A_\infty^+$ and (g) is true as well. $\Box$

Now, for $x \in [-\infty, \infty]$ we define chains $\mathcal{L}_x$ in $\langle \mathbb{P}(A_\infty^+) \cup \{\emptyset\}, \subset \rangle$ as follows.

For $x \notin M$ we define $\mathcal{L}_x = \{A_x\}$. In particular, $\mathcal{L}_{-\infty} = \{\emptyset\}$.

For $x \in M$, using Claim 3.4(g) and Fact 2.4 we obtain $\mathcal{L}_x \subset [A_x, A_x^+]_{P(A_x^+)}$ such that $\langle \mathcal{L}_x, \varsubsetneq \rangle \cong \langle L_x, <_x \rangle$ and

$$A_x, A_x^+ \in \mathcal{L}_x \subset [A_x, A_x^+]_{\mathbb{P}(A_\infty^+)}, \tag{1}$$

$$\bigcup \mathcal{A}, \bigcap \mathcal{B} \in \mathcal{L}_x \quad \text{and} \quad |\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1, \text{ for each cut } \langle \mathcal{A}, \mathcal{B} \rangle \text{ in } \mathcal{L}_x. \tag{2}$$

For $\mathcal{A}, \mathcal{B} \subset \mathbb{P}(A_\infty^+)$ we will write $\mathcal{A} \prec \mathcal{B}$ iff $A \varsubsetneq B$, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Claim 3.5 Let $\mathcal{L} = \bigcup_{x \in [-\infty, \infty]} \mathcal{L}_x$. Then

(a) If $-\infty \leq x_1 < x_2 \leq \infty$, then $\mathcal{L}_{x_1} \prec \mathcal{L}_{x_2}$ and $\bigcup \mathcal{L}_{x_1} \subset A_{x_2} \subset \bigcup \mathcal{L}_{x_2}$.

(b) $\mathcal{L}$ is a chain in $\langle \mathbb{P}(A_\infty^+) \cup \{\emptyset\}, \subset \rangle$ isomorphic to $L = \sum_{x \in [-\infty, \infty]} L_x$.

(c) $\mathcal{L}$ is a maximal chain in $\langle \mathbb{P}(A_\infty^+) \cup \{\emptyset\}, \subset \rangle$.

Proof. (a) Let $A \in \mathcal{L}_{x_1}$ and $B \in \mathcal{L}_{x_2}$. If $x_1 \in (-\infty, \infty] \setminus M$, then, by (1) and Claim 3.4(c) we have $A = A_{x_1} \varsubsetneq A_{x_2} \subset B$. If $x_1 \in M$, then, by (1) and Claim 3.4(d), $A \subset A_{x_1}^+ \varsubsetneq A_{x_2} \subset B$. The second statement follows from $A_{x_2} \in \mathcal{L}_{x_2}$.

(b) By (a), $\langle [-\infty, \infty], < \rangle \cong \langle \{\mathcal{L}_x : x \in [-\infty, \infty]\}, \prec \rangle$. Since $\mathcal{L}_x \cong L_x$, for $x \in [-\infty, \infty]$, we have $\langle \mathcal{L}, \varsubsetneq \rangle \cong \sum_{x \in [-\infty, \infty]} \langle \mathcal{L}_x, \varsubsetneq \rangle \cong \sum_{x \in [-\infty, \infty]} L_x = L$.

(c) Suppose that $C \in \mathbb{P}(A_\infty^+) \cup \{\emptyset\}$ witnesses that $\mathcal{L}$ is not maximal. Clearly $\mathcal{L} = \mathcal{A} \dot{\cup} \mathcal{B}$ and $\mathcal{A} \prec \mathcal{B}$, where $\mathcal{A} = \{A \in \mathcal{L} : A \varsubsetneq C\}$ and $\mathcal{B} = \{B \in \mathcal{L} : C \varsubsetneq B\}$. Now $\emptyset \in \mathcal{L}_{-\infty}$ and, since $\infty \in M$, by (1) we have $A_\infty^+ \in \mathcal{L}_\infty$. Thus $\emptyset, A_\infty^+ \in \mathcal{L}$, which implies $\mathcal{A}, \mathcal{B} \neq \emptyset$ and, hence, $\langle \mathcal{A}, \mathcal{B} \rangle$ is a cut in $\langle \mathcal{L}, \varsubsetneq \rangle$. By (1) we have $\{A_x : x \in (-\infty, \infty]\} \subset \mathcal{L} \setminus \{\emptyset\}$ and, by Claim 3.4(a), $\bigcap (\mathcal{L} \setminus \{\emptyset\}) \subset \bigcap_{x \in (-\infty, \infty]} A_x \subset \bigcap_{x \in (-\infty, \infty]} (-\infty, x) = \emptyset$, which implies $\mathcal{A} \neq \{\emptyset\}$. Clearly,

$$\bigcup \mathcal{A} \subset C \subset \bigcap \mathcal{B}. \tag{3}$$

Case 1: $\mathcal{A} \cap \mathcal{L}_{x_0} \neq \emptyset$ and $\mathcal{B} \cap \mathcal{L}_{x_0} \neq \emptyset$, for some $x_0 \in (-\infty, \infty]$. Then $|\mathcal{L}_{x_0}| > 1$, $x_0 \in M$ and $\langle \mathcal{A} \cap \mathcal{L}_{x_0}, \mathcal{B} \cap \mathcal{L}_{x_0} \rangle$ is a cut in $\mathcal{L}_{x_0}$ satisfying (2). By (a), $\mathcal{A} = \bigcup_{x < x_0} \mathcal{L}_x \cup (\mathcal{A} \cap \mathcal{L}_{x_0})$ and, consequently, $\bigcup \mathcal{A} = \bigcup (\mathcal{A} \cap \mathcal{L}_{x_0}) \in \mathcal{L}$. Similarly, $\bigcap \mathcal{B} = \bigcap (\mathcal{B} \cap \mathcal{L}_{x_0}) \in \mathcal{L}$ and, since $|\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1$, by (3) we have $C \in \mathcal{L}$. A contradiction.

Case 2: $\neg$ Case 1. Then for each $x \in (-\infty, \infty]$ we have $\mathcal{L}_x \subset \mathcal{A}$ or $\mathcal{L}_x \subset \mathcal{B}$. Since $\mathcal{L} = \mathcal{A} \dot{\cup} \mathcal{B}$, $\mathcal{A} \neq \{\emptyset\}$ and $\mathcal{A}, \mathcal{B} \neq \emptyset$, the sets $\mathcal{A}' = \{x \in (-\infty, \infty] : \mathcal{L}_x \subset \mathcal{A}\}$ and $\mathcal{B}' = \{x \in (-\infty, \infty] : \mathcal{L}_x \subset \mathcal{B}\}$ are non-empty and $(-\infty, \infty] = \mathcal{A}' \dot{\cup} \mathcal{B}'$. Since $\mathcal{A} \prec \mathcal{B}$, for $x_1 \in \mathcal{A}'$ and $x_2 \in \mathcal{B}'$ we have $\mathcal{L}_{x_1} \prec \mathcal{L}_{x_2}$ so, by (a), $x_1 < x_2$. Thus $\langle \mathcal{A}', \mathcal{B}' \rangle$ is a cut in $(-\infty, \infty]$ and, consequently, there is $x_0 \in (-\infty, \infty]$ such that $x_0 = \max \mathcal{A}'$ or $x_0 = \min \mathcal{B}'$.

Subcase 2.1: $x_0 = \max \mathcal{A}'$. Then $x_0 < \infty$ because $\mathcal{B} \neq \emptyset$ and $\mathcal{A} = \bigcup_{x \leq x_0} \mathcal{L}_x$ so, by (a), $\bigcup \mathcal{A} = \bigcup_{x \leq x_0} \bigcup \mathcal{L}_x = \bigcup_{x < x_0} \bigcup \mathcal{L}_x \cup \bigcup \mathcal{L}_{x_0} = \bigcup \mathcal{L}_{x_0}$ which, together with (1) implies

$$\bigcup \mathcal{A} = \begin{cases} A_{x_0} & \text{if } x_0 \notin M, \\ A^+_{x_0} & \text{if } x_0 \in M. \end{cases} \tag{4}$$

Since $\mathcal{B} = \bigcup_{x \in (x_0, \infty]} \mathcal{L}_x$, we have $\bigcap \mathcal{B} = \bigcap_{x \in (x_0, \infty]} \bigcap \mathcal{L}_x$. By (1) $\bigcap \mathcal{L}_x = A_x$, so we have $\bigcap \mathcal{B} = (\bigcap_{x \in (x_0, \infty]} (-\infty, x) \cap J_0) \cup (\bigcap_{x \in (x_0, \infty]} \bigcup_{y \in M \cap (-\infty, x)} I_y) = ((-\infty, x_0] \cap J_0) \cup \bigcup_{y \in M \cap (-\infty, x_0]} I_y = A_{x_0} \cup (\{x_0\} \cap J_0) \cup \bigcup_{y \in M \cap \{x_0\}} I_y$, so

$$\bigcap \mathcal{B} = \begin{cases} A_{x_0} & \text{if } \; x_0 \notin J_0 \;\; \wedge \;\; x_0 \notin M, \\ A_{x_0} \cup \{x_0\} & \text{if } \; x_0 \in J_0 \;\; \wedge \;\; x_0 \notin M, \\ A^+_{x_0} & \text{if } \; x_0 \notin J_0 \;\; \wedge \;\; x_0 \in M, \\ A^+_{x_0} \cup \{x_0\} & \text{if } \; x_0 \in J_0 \;\; \wedge \;\; x_0 \in M. \end{cases} \tag{5}$$

If $x_0 \notin J_0$, then, by (3), (4) and (5), we have $\bigcup \mathcal{A} = \bigcap \mathcal{B} = C \in \mathcal{L}$. A contradiction.

If $x_0 \in J_0$ and $x_0 \notin M$, then $\bigcup \mathcal{A} = A_{x_0}$ and $\bigcap \mathcal{B} = A_{x_0} \cup \{x_0\}$. So, by (3) and since $C \notin \mathcal{L}$ we have $C = \bigcap \mathcal{B} = A_{x_0} \cup \{x_0\}$. Thus $J_0 \cap (-\infty, x_0] \subset C$ and, by Claim 3.4(a), $C \subset (-\infty, x_0]$. But by (iv) we have $C \ncong \mathbb{X} (\cong A^+_\infty)$. A contradiction.

If $x_0 \in J$ and $x_0 \in M$, then $\bigcup \mathcal{A} = A^+_{x_0}$ and $\bigcap \mathcal{B} = A^+_{x_0} \cup \{x_0\}$. Again, by (3) and since $C \notin \mathcal{L}$ we have $C = \bigcap \mathcal{B} = A^+_{x_0} \cup \{x_0\}$. Thus $J_0 \cap (-\infty, x_0] \subset C$ and, by Claim 3.4(b), $C \subset (-\infty, x_0]$. Again, by (iv) we have $C \ncong \mathbb{X} (\cong A^+_\infty)$, a contradiction.

Subcase 2.2: $x_0 = \min \mathcal{B}'$. Then, by (1), $A_{x_0} \in \mathcal{L}_{x_0} \subset \mathcal{B}$ which, by (a), implies $\bigcap \mathcal{B} = A_{x_0}$. Since $A_x \in \mathcal{L}_x$, for $x \in (-\infty, \infty]$ and $\mathcal{A} = \bigcup_{x < x_0} \mathcal{L}_x$ we have $\bigcup \mathcal{A} = \bigcup_{x < x_0} \bigcup \mathcal{L}_x \supset \bigcup_{x < x_0} A_x = \bigcup_{x < x_0} ((-\infty, x) \cap J_0) \cup \bigcup_{x < x_0} \bigcup_{y \in M \cap (-\infty, x)} I_y = ((-\infty, x_0) \cap J_0) \cup \bigcup_{y \in M \cap (-\infty, x_0)} I_y = A_{x_0}$ so $A_{x_0} \subset \bigcup \mathcal{A} \subset \bigcap \mathcal{B} = A_{x_0}$, which implies $C = A_{x_0} \in \mathcal{L}$. A contradiction. □

Case II: $\infty \notin M$. Then $L_\infty = \{\max L\}$ and the sum $L + 1$ belongs to Case I. So, there exists a maximal chain $\mathcal{L}$ in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$ and an isomorphism $f : \langle L + 1, < \rangle \to \langle \mathcal{L}, \subset \rangle$. Then $A = f(\max L) \in \mathbb{P}(\mathbb{X})$ and $\mathcal{L}' = f[L] \cong L$. By the maximality of $\mathcal{L}$, $\mathcal{L}'$ is a maximal chain in $\langle \mathbb{P}(A) \cup \{\emptyset\}, \subset \rangle \cong \langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$.

(B) Since (v) holds we assume that L is uncountable. If all initial segments of L are uncountable, the statement is proved in (A). Otherwise, by Claim 3.3 we have $L = \sum_{x \in [-\infty, \infty]} L_x$, (L1) and (L2) hold and

(L3′) $L_{-\infty}$ is a countable complete linear order with $0_{L_{-\infty}}$ non-isolated.

Clearly $L = L_{-\infty} + L^+$, where $L^+ = \sum_{x \in (-\infty, \infty]} L_x = \sum_{y \in (0, \infty]} L_{\ln y}$ (here $\ln \infty = \infty$). Let L'_y, $y \in [-\infty, \infty]$, be disjoint linear orders such that $L'_y \cong 1$, for $y \in [-\infty, 0]$, and $L'_y \cong L_{\ln y}$, for $y \in (0, \infty]$. Now $\sum_{y \in [-\infty, \infty]} L'_y \cong [-\infty, 0] +$

L^+ and by (A) we obtain a maximal chain $\mathcal{L}$ in $\mathbb{P}(\mathbb{X}) \cup \{\emptyset\}$ and an isomorphism $f : \langle[-\infty, 0] + L^+, <\rangle \to \langle\mathcal{L}, \subset\rangle$. Clearly, for $A_0 = f(0)$ and $\mathcal{L}^+ = f[L^+]$ we have $A_0 \in \mathcal{L}$ and $\mathcal{L}^+ \cong L^+$.

By the assumption and (L3′), $\mathbb{P}(A_0) \cup \{\emptyset\}$ contains a maximal chain $\mathcal{L}_{-\infty} \cong L_{-\infty}$. Clearly $A_0 \in \mathcal{L}_{-\infty}$ and $\mathcal{L}_{-\infty} \cup \mathcal{L}^+ \cong L_{-\infty} + L^+ = L$. Suppose that B witnesses that $\mathcal{L}_{-\infty} \cup \mathcal{L}^+$ is not a maximal chain in $\mathbb{P}(\mathbb{X}) \cup \{\emptyset\}$. Then either $A_0 \subsetneq B$, which is impossible since $\mathcal{L}$ is maximal in $\mathbb{P}(\mathbb{X}) \cup \{\emptyset\}$, or $B \subsetneq A_0$, which is impossible since $\mathcal{L}_{-\infty}$ is maximal in $\mathbb{P}(A_0) \cup \{\emptyset\}$. □

The following theorem gives a sufficient condition for (v) of Theorem 3.2.

Theorem 3.6 Let $\mathbb{X} = \langle X, \langle\sigma_i : i \in I\rangle\rangle$ be a countable relational structure. If there is a positive family $\mathcal{P}$ in $P(X)$ such that $\mathcal{P} \subset \mathbb{P}(\mathbb{X})$ and $\bigcap \mathcal{P} = \emptyset$, then

(a) For each $\mathbb{R}$-embeddable Boolean linear order L with 0_L non-isolated there is a maximal chain in $\langle\mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset\rangle$ isomorphic to L;

(b) For each countable complete linear order L with 0_L non-isolated there is a maximal chain in $\langle\mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset\rangle$ isomorphic to L.

Proof. (a) By Fact 2.3 there is a maximal chain $\mathcal{L}$ in $\mathcal{P} \cup \{\emptyset\}$ isomorphic to L and satisfying $\bigcap(\mathcal{L} \setminus \{\emptyset\}) = \emptyset$. Suppose that $C \in \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}$ witnesses that $\mathcal{L}$ is not a maximal chain in $\langle\mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset\rangle$. Since $C \neq \emptyset$ there is $A \in \mathcal{L} \setminus \{\emptyset\}$ such that $A \subset C$ and, hence, $C \in \mathcal{P}$. Thus $\mathcal{L} \cup \{C\}$ is a chain in $\mathcal{P} \cup \{\emptyset\}$ bigger than $\mathcal{L}$. A contradiction.

(b) follows from (a) and Fact 2.2. □

4 Maximal chains of copies of $\mathbb{H}_n$

The graphs $\mathbb{H}_n$, $n \geq 3$, were constructed by Henson in [2]. By [2], $\mathbb{H}_n$ is the unique (up to isomorphism) countable ultrahomogeneous universal $\mathbb{K}_n$-free graph.

In order to cite a characterization of $\mathbb{H}_n$ which is more convenient for our construction, we introduce the following notation. If $\mathbb{G} = \langle G, \rho\rangle$ is a graph and $n \geq 3$ let $C_n(\mathbb{G})$ denote the set of all pairs $\langle H, K\rangle$ of finite subsets of G such that:

(C1) $K \subset H$ and

(C2) K does not contain a copy of $\mathbb{K}_{n-1}$.

For $\langle H, K\rangle \in C_n(\mathbb{G})$, let G^H_K denote the set of all $v \in G \setminus H$ satisfying:

(S1) $\{v, k\} \in \rho$, for all $k \in K$ and

(S2) $\{v, h\} \notin \rho$, for all $h \in H \setminus K$.

The graphs $\mathbb{H}_n$ can be characterized in the following way.

Fact 4.1 (Henson) A countable graph $\mathbb{G} = \langle G, \rho\rangle$ is isomorphic to $\mathbb{H}_n$ iff it is $\mathbb{K}_n$-free and $G^H_K \neq \emptyset$, for each $\langle H, K\rangle \in C_n(\mathbb{G})$.

Now we prove (I) of Theorem 1.2 for the graphs $\mathbb{H}_n$.

Theorem 4.2 For each $n \geq 3$ and each linear order L the following conditions are equivalent:

(a) L is isomorphic to a maximal chain in the poset $\langle \mathbb{P}(\mathbb{H}_n) \cup \{\emptyset\}, \subset \rangle$;

(b) L is an $\mathbb{R}$-embeddable complete linear order with 0_L non-isolated;

(c) L is isomorphic to a compact set $K \subset \mathbb{R}$ having the minimum non-isolated.

Proof. The equivalence (b) $\Leftrightarrow$ (c) is proved in Theorem 6 of [6] and (a) $\Rightarrow$ (b) follows from Theorem 3.1.

(b) $\Rightarrow$ (a). We intend to use Theorem 3.2. Let $\{J'_n : n \in \omega\}$ be a partition of the set $[0, 1) \cap \mathbb{Q}$ into dense subsets of $[0, 1) \cap \mathbb{Q}$. Let $\mathbb{Z}$ denote the set of integers and let $J_n = \{q + m : q \in J'_n \wedge m \in \mathbb{Z}\}$, for $n \in \omega$. Clearly, $\{J_n : n \in \omega\}$ is a partition of $\mathbb{Q}$ into dense subsets of $\mathbb{Q}$ and conditions (i) and (ii) are satisfied.

Now we construct a copy of $\mathbb{H}_n$ with the domain $\mathbb{Q}$. Let $\mathbb{P}$ be the set of $\mathbb{K}_n$-free graphs $p = \langle G_p, \rho_p \rangle$ such that $G_p \in [\mathbb{Q}]^{<\omega}$ and for each $a, b \in \mathbb{Q}$

(P1) $\{a, b\} \in \rho_p \wedge \{a + 1, b\} \in \rho_p \Rightarrow b > a + 1$,

(P2) $\{a, a - 1\} \notin \rho_p$.

Let the relation $\leq$ on $\mathbb{P}$ be defined by

$$p \leq q \Leftrightarrow G_p \supset G_q \ \wedge \ \rho_p \cap [G_q]^2 = \rho_q. \tag{6}$$

Claim 4.3 $\langle \mathbb{P}, \leq \rangle$ is a partial order.

Proof. It is evident that the relation $\leq$ is reflexive and antisymmetric. If $p \leq q \leq r$, then $G_r \subset G_q \subset G_p$ and $\rho_r = \rho_q \cap [G_r]^2 = \rho_p \cap [G_q]^2 \cap [G_r]^2 = \rho_p \cap [G_r]^2$. $\square$

Claim 4.4 The sets $\mathcal{D}_q = \{p \in \mathbb{P} : q \in G_p\}$, $q \in \mathbb{Q}$, are dense in $\langle \mathbb{P}, \leq \rangle$.

Proof. If $p = \langle G_p, \rho_p \rangle \in \mathbb{P} \setminus \mathcal{D}_q$, then $q \notin G_p$ and, since $\{q, x\} \notin \rho_p$, for all $x \in G_p$, $p_1 = \langle G_p \cup \{q\}, \rho_p \rangle$ is a $\mathbb{K}_n$-free graph and, clearly, satisfies (P1) and (P2). Thus $p_1 \in \mathcal{D}_q$ and $p_1 \leq p$. $\square$

For $H \in [\mathbb{Q}]^{<\omega}$ let $m_H = \max H$.

Claim 4.5 For each $K \subset H \in [\mathbb{Q}]^{<\omega}$ and each $m \in \mathbb{N}$, the set

$$\mathcal{D}^H_{K,m} = \Big\{ p \in \mathbb{P} : H \subset G_p \ \wedge \ \Big(\langle H, K \rangle \in C_n(p) \Rightarrow \exists q \in J_0 \cap (m_H, m_H + \tfrac{1}{m}) \\ \forall k \in K \ (\{q, k\} \in \rho_p) \wedge \forall h \in H \setminus K \ (\{q, h\} \notin \rho_p \ ) \Big) \Big\}$$

is dense in $\mathbb{P}$.

Proof. Let $p_0 \in \mathbb{P}$. By Claim 4.4 there is $p \in \mathbb{P}$ such that $p \leq p_0$ and $H \subset G_p$.

If $\langle H, K \rangle \notin C_n(p)$ then $p \in \mathcal{D}^H_{K,m}$ and we are done.

If $\langle H, K \rangle \in C_n(p)$, we take $q \in J_0 \cap (m_H, m_H + \frac{1}{m}) \setminus \bigcup_{a \in G_p} \{a, a-1, a+1\}$, define

$$p_1 = \langle G_p \cup \{q\}, \rho_p \cup \{\{q, k\} : k \in K\} \rangle. \tag{7}$$

and first prove that $p_1 \in \mathbb{P}$. Clearly $G_{p_1} \in [\mathbb{Q}]^{<\omega}$ and we check that p_1 is $\mathbb{K}_n$-free. Suppose that there is $F \in [G_{p_1}]^n$ such that $[F]^2 \subset \rho_{p_1}$. Since p is $\mathbb{K}_n$-free we have $q \in F$ and there are different $f_1, \dots, f_{n-1} \in G_p \cap F$ such that $\{q, f_i\} \in \rho_{p_1}$, for $i \leq n-1$, which by (7) implies $\{f_1, \dots, f_{n-1}\} \subset K$. Since $[F]^2 \subset \rho_{p_1}$, we have $[\{f_1, \dots, f_{n-1}\}]^2 \subset \rho_p$. But $\langle H, K\rangle \in C_n(p)$ implies that K is $\mathbb{K}_{n-1}$-free. A contradiction.

(P1) Suppose that for some $a, b \in \mathbb{Q}$

$$\{a, b\} \in \rho_{p_1} \ \wedge\ \{a+1, b\} \in \rho_{p_1} \ \wedge b \leq a+1. \tag{8}$$

Then, since $p \in \mathbb{P}$, at least one of the two pairs does not belong to ρ_p and, hence, $q \in \{a, a+1, b\}$. So we have the following three cases.

$q = a$. Then by (8) we have $b \neq q$ and, by (7), $\{q+1, b\} \in \rho_p$ which implies $q+1 \in G_p$. A contradiction to the choice of q.

$q = a+1$. Then by (8) we have $b \neq q$ and, since $a \neq q$, by (7) we have $\{a, b\} \in \rho_p$ which implies $a \in G_p$. A contradiction to the choice of q.

$q = b$. Then by (8) and (7) we have $\{a, q\}, \{a+1, q\} \in \rho_{p_1} \setminus \rho_p$ which implies $a, a+1 \in K$. Since $q > m_H$ and $K \subset H$ we have $q > a+1$ that is $b > a+1$. A contradiction again.

(P2) holds because $p \in \mathbb{P}$ and $q \notin \bigcup_{a \in G_p} \{a, a-1, a+1\}$.

Thus $p_1 \in \mathbb{P}$. Since $H \subset G_p \subset G_{p_1}$ and since, by (7) we have $\{q, k\} \in \rho_{p_1}$, for all $k \in K$, and $\{q, h\} \notin \rho_{p_1}$, for all $h \in H \setminus K$, it follows that $p_1 \in \mathcal{D}^H_{K,m}$.

Since $G_{p_1} \supset G_p$ and $\rho_{p_1} \cap [G_p]^2 = \rho_p$, we have $p_1 \leq p \leq p_0$. $\square$

By Fact 2.1 there is a filter $\mathcal{G}$ in $\langle \mathbb{P}, \leq\rangle$ intersecting all sets $\mathcal{D}_q$, $q \in \mathbb{Q}$, and $\mathcal{D}^H_{K,m}$, for $K \subset H \in [\mathbb{Q}]^{<\omega}$ and $m \in \mathbb{N}$.

Claim 4.6 (a) $\bigcup_{p \in \mathcal{G}} G_p = \mathbb{Q}$;

(b) $\langle \mathbb{Q}, \rho\rangle$ is a graph, where $\rho = \bigcup_{p \in \mathcal{G}} \rho_p$, also $\{a, a-1\} \notin \rho$, for all $a \in \mathbb{Q}$;

(c) $\rho \cap [G_p]^2 = \rho_p$, for each $p \in \mathcal{G}$;

(d) If $A \subset \mathbb{Q}$, $\rho_A = \rho \cap [A]^2$, $p \in \mathcal{G}$, and $H \subset A \cap G_p$, then $\rho_A \cap [H]^2 = \rho_p \cap [H]^2$. Thus if, in addition, $\langle H, K\rangle \in C_n(A, \rho_A)$, then $\langle H, K\rangle \in C_n(p)$,

(e) $\langle \mathbb{Q}, \rho\rangle$ is a $\mathbb{K}_n$-free graph.

Proof. (a) For $q \in \mathbb{Q}$ let $p_0 \in \mathcal{G} \cap \mathcal{D}_q$. Then $q \in G_{p_0} \subset \bigcup_{p \in \mathcal{G}} G_p$.

(b) By the definition of $\mathbb{P}$ we have $\{a, a-1\} \notin \rho_p \subset [\mathbb{Q}]^2$, for all $p \in \mathbb{P}$.

(c) The inclusion "$\supset$" is evident. If $\{a, b\} \in \rho \cap [G_p]^2$, then there is $p_1 \in \mathcal{G}$ such that $\{a, b\} \in \rho_{p_1}$ and, since $\mathcal{G}$ is a filter, there is $p_2 \in \mathcal{G}$ such that $p_2 \leq p, p_1$. By the definition of $\leq$ we have $\rho_{p_1} \subset \rho_{p_2}$, which implies $\{a, b\} \in \rho_{p_2}$ and $\{a, b\} \in \rho_{p_2} \cap [G_p]^2 = \rho_p$.

(d) By (c) we have $\rho_A \cap [H]^2 = \rho \cap [A]^2 \cap [H]^2 = \rho \cap [H]^2 = \rho \cap [G_p]^2 \cap [H]^2 = \rho_p \cap [H]^2$. If $\langle H, K\rangle \in C_n(A, \rho_A)$, then K is $\mathbb{K}_{n-1}$-free in $\langle A, \rho_A\rangle$ and, since $\rho_A \cap [K]^2 = \rho_p \cap [K]^2$, K is $\mathbb{K}_{n-1}$-free in p as well. Thus $\langle H, K\rangle \in C_n(p)$.

(e) Suppose that $\langle A, \rho_A\rangle$ is a copy of $\mathbb{K}_n$ and let $p_q \in \mathcal{G} \cap \mathcal{D}_q$, $q \in A$. Since $\mathcal{G}$ is a filter there is $p \in \mathcal{G}$ such that $p \leq p_q$, for all $q \in A$, and, hence, $A \subset G_p$, which by (d) implies $\rho_A = \rho_p \cap [A]^2$. But this is impossible since p is $\mathbb{K}_n$-free. □

Now we show that conditions (iii) and (iv) of Theorem 3.2 are satisfied.

(iii) Let $x \in \mathbb{R} \cup \{\infty\}$ and $J_0 \cap (-\infty, x) \subset A \subset \mathbb{Q} \cap (-\infty, x)$. We show that $\langle A, \rho_A\rangle \cong \mathbb{H}_n$. By Claim 4.6(e) $\langle A, \rho_A\rangle$ is $\mathbb{K}_n$-free. Let $\langle H, K\rangle \in C_n(A, \rho_A)$. Since $m_H \in H \subset A$ we have $m_H < x$ and there is $m \in \mathbb{N}$ satisfying $m_H + \frac{1}{m} < x$. Let $p \in \mathcal{G} \cap \mathcal{D}^H_{K,m}$. Then $K \subset H \subset G_p$ and, by Claim 4.6(d), $\langle H, K\rangle \in C_n(p)$. Thus there is $q \in J_0 \cap (m_H, m_H + \frac{1}{m}) \subset J_0 \cap (-\infty, x) \subset A$ such that $\{q, k\} \in \rho_p \subset \rho$, which implies $\{q, k\} \in \rho_A$, for all $k \in K$, and that $\{q, h\} \notin \rho_p$, which implies $\{q, h\} \notin \rho$, for all $h \in H$. Thus $q \in A^H_K$. By Fact 4.1 we have $\langle A, \rho_A\rangle \cong \mathbb{H}_n$.

(iv) Let $q \in J_0$ and $J_0 \cap (-\infty, q] \subset C \subset \mathbb{Q} \cap (-\infty, q]$. We prove that $\langle C, \rho_C\rangle \not\cong \mathbb{H}_n$. Since $q \in J_0$ by the construction of J_0 we have $q - 1 \in J_0$ and, by the assumption, $H = \{q - 1, q\} \subset C$. By Claim 4.6(b) we have $\{q - 1, q\} \notin \rho$ which implies that H is $\mathbb{K}_{n-1}$-free and, hence, $\langle H, H\rangle \in C_n(C, \rho_C)$. Suppose that $b \in C^H_H$. Then $\{q - 1, b\}, \{q, b\} \in \rho$ and, since $\mathcal{G}$ is a filter, $\{q - 1, b\}, \{q, b\} \in \rho_p$, for some $p \in \mathcal{G}$. By (P1) we have $b > q$, which is impossible since $q = \max C$. Thus $C^H_H = \emptyset$ and by Fact 4.1 we have $\langle C, \rho_C\rangle \not\cong \mathbb{H}_n$.

Claim 4.7 The family $\mathcal{P} = \left\{\mathbb{Q} \setminus \bigcup_{n\in\mathbb{Z}} F_n : \forall n \in \mathbb{Z}\ \ F_n \in \left[[n, n+1) \cap \mathbb{Q}\right]^{<\omega}\right\}$ is a positive family in $P(\mathbb{Q})$ satisfying $\bigcap \mathcal{P} = \emptyset$ and $\mathcal{P} \subset \mathbb{P}(\mathbb{Q}, \rho)$.

Proof. It is easy to check (P1)-(P4). Since $\mathbb{Q} \setminus \{q\} \in \mathcal{P}$, for each $q \in \mathbb{Q}$, we have $\bigcap \mathcal{P} = \emptyset$. Let $A = \mathbb{Q} \setminus \bigcup_{n\in\mathbb{Z}} F_n \in \mathcal{P}$, $\langle H, K\rangle \in C_n(A, \rho_A)$ and $m_H = \max H \in [n_0, n_0 + 1) \cap \mathbb{Q}$. Since $|F_{n_0}| < \omega$ and $m_H \in A \subset \mathbb{Q} \setminus F_{n_0}$ there is $m \in \mathbb{N}$ such that $(m_H, m_H + \frac{1}{m}) \cap \mathbb{Q} \subset A$. Let $p \in \mathcal{G} \cap \mathcal{D}^H_{K,m}$. Then $H \subset G_p$ and, by Claim 4.6(d), $\langle H, K\rangle \in C_n(p)$. Hence there is $q \in J_0 \cap (m_H, m_H + \frac{1}{m}) \subset A$ such that

- for each $k \in K$ we have $\{q, k\} \in \rho_p$ which, since $\{q, k\} \subset A \cap G_p$, by Claim 4.6(d) implies $\{q, k\} \in \rho_A$;
- for each $h \in H \setminus K$ we have $\{q, h\} \notin \rho_p$, which by Claim 4.6(c) implies $\{q, h\} \notin \rho$ and, hence, $\{q, h\} \notin \rho_A$.

Thus $q \in A^H_K$. By Fact 4.1 we have $\langle A, \rho_A\rangle \cong \mathbb{H}_n \cong \langle \mathbb{Q}, \rho\rangle$ and, hence, $A \in \mathbb{P}(\mathbb{Q}, \rho)$. □

Now (b) ⇒ (a) of Theorem 4.2 for countable L follows from Claim 4.7 and Theorem 3.6(b). Thus condition (v) of Theorem 3.2 is satisfied and, by (B) of Theorem 3.2, (b) ⇒ (a) of Theorem 4.2 is true for uncountable L. □

5 Maximal chains of copies of $\mathbb{G}_{\mu\nu}$

Theorem 5.1 If μ and ν are cardinals satisfying $\mu\nu = \omega$, then for each linear order L the following conditions are equivalent:

(a) L is isomorphic to a maximal chain in the poset $\langle \mathbb{P}(\mathbb{G}_{\mu\nu}) \cup \{\emptyset\}, \subset\rangle$;

(b) L is an $\mathbb{R}$-embeddable Boolean linear order with 0_L non-isolated;

(c) L is isomorphic to a compact nowhere dense set $K \subset \mathbb{R}$ having the minimum non-isolated.

Proof. Clearly, concerning the values of μ and ν we have three cases.

I. $\mathbb{G}_{\omega n} = \bigcup_{i\in\omega} \mathbb{G}_i$, where $n \in \mathbb{N}$ and $\mathbb{G}_i = \langle G_i, [G_i]^2\rangle$, $i \in \omega$, are disjoint copies of $\mathbb{K}_n$. Then, clearly $\mathbb{P}(\mathbb{G}_{\omega n}) = \{\bigcup_{i\in A} G_i : A \in [\omega]^\omega\}$ and, hence, $\langle \mathbb{P}(\mathbb{G}_{\omega n}) \cup \{\emptyset\}, \subset\rangle \cong \langle [\omega]^\omega \cup \{\emptyset\}, \subset\rangle$. Since $[\omega]^\omega$ is a positive family in $P(\omega)$ the statement follows from Fact 2.3.

II. $\mathbb{G}_{m\omega} = \bigcup_{i<m} \mathbb{G}_i$, where $m \in \mathbb{N}$ and $\mathbb{G}_i = \langle G_i, [G_i]^2\rangle$, $i < m$, are disjoint copies of $\mathbb{K}_\omega$. Then, since each copy of $\mathbb{G}_{m\omega}$ must have m components of size ω, we have $\mathbb{P}(\mathbb{G}_{m\omega}) = \{\bigcup_{i<m} A_i : \forall i < m\ A_i \in [G_i]^\omega\}$ and it is easy to see that $\mathbb{P}(\mathbb{G}_{m\omega})$ is a positive family in $P(G)$ so we apply Fact 2.3 again.

III. $\mathbb{G}_{\omega\omega} = \bigcup_{i<\omega} \mathbb{G}_i$, where $\mathbb{G}_i = \langle G_i, [G_i]^2\rangle$, $i < \omega$, are disjoint copies of $\mathbb{K}_\omega$. The equivalence (b) $\Leftrightarrow$ (c) is a part of Fact 2.3

(a) $\Rightarrow$ (b). If $\mathcal{L}$ is a maximal chain in $\langle \mathbb{P}(\mathbb{G}_{\omega\omega}) \cup \{\emptyset\}, \subset\rangle$, then, by Theorem 3.1, it is an $\mathbb{R}$-embeddable complete linear order with $0_\mathcal{L}$ non-isolated and we prove that it has dense jumps. Let $G = \bigcup_{i<\omega} G_i$. Since each copy of $\mathbb{G}_{\omega\omega}$ must have ω components of size ω, we have

$$\mathbb{P}(\mathbb{G}_{\omega\omega}) = \{\bigcup_{i\in S} A_i : S \in [\omega]^\omega \wedge \forall i \in S\ A_i \in [G_i]^\omega\} \tag{9}$$

and, for $A = \bigcup_{i\in S} A_i \in \mathbb{P}(\mathbb{G}_{\omega\omega})$ we will write $S = \operatorname{supp} A$.

Let $A, B \in \mathcal{L} \setminus \{\emptyset\}$, where $A \subsetneq B$.

Claim 5.2 There is $C \in \mathcal{L}$ satisfying $A \subset C \subset B$ and such that $C \cap G_i \subsetneq B \cap G_i$, for some $i \in \operatorname{supp} C$.

Proof. Suppose that for each $C \in \mathcal{L} \cap [A, B]$ we have: $C \cap G_i = B \cap G_i$, for all $i \in \operatorname{supp} C$. Then, since $A \subsetneq B$, we have $\operatorname{supp} A \subsetneq \operatorname{supp} B$ and we choose $i \in \operatorname{supp} B \setminus \operatorname{supp} A$. Clearly, for the sets $\mathcal{L}^- = \{C \in \mathcal{L} : i \notin \operatorname{supp} C\}$ and $\mathcal{L}^+ = \{C \in \mathcal{L} : i \in \operatorname{supp} C\}$ we have $\mathcal{L} = \mathcal{L}^- \cup \mathcal{L}^+$ and $C_1 \subsetneq C_2$, for each $C_1 \in \mathcal{L}^-$ and $C_2 \in \mathcal{L}^+$. By Theorem 3.1 we have $C^- = \bigcup \mathcal{L}^- \in \mathbb{P}(\mathbb{G}_{\omega\omega})$ and, since $\mathcal{L}^- \lhd \mathcal{L}^+$, by the maximality of $\mathcal{L}$ we have $C^- \in \mathcal{L}$. Clearly $i \notin \operatorname{supp} C^-$, which implies $C^- = \max \mathcal{L}^-$. Let $C^+ = C^- \cup (B \cap G_i)$. By (9) we have $C^+ \in \mathbb{P}(\mathbb{G}_{\omega\omega})$. For $C \in \mathcal{L}^+$ we have $i \in \operatorname{supp} C$ and, by the assumption, $C \cap G_i = B \cap G_i$, which implies $C^+ \subset C$. Thus, by the maximality of $\mathcal{L}$, $C^+ \in \mathcal{L}$, and, moreover, $C^+ = \min \mathcal{L}^+$. Let $a \in B \cap G_i$. Then $C = C^- \cup (B \cap G_i \setminus \{a\}) \in \mathbb{P}(\mathbb{G}_{\omega\omega})$ and $C^- \subsetneq C \subsetneq C^+$, which implies that $\mathcal{L}$ is not a maximal chain in $\mathbb{P}(\mathbb{G}_{\omega\omega})$. A contradiction. $\Box$

Let $C_0 \in \mathcal{L}$ and $i_0 \in \operatorname{supp} C_0$ be the objects provided by Claim 5.2. Let $a \in (B \setminus C_0) \cap G_{i_0}$, $\mathcal{L}^- = \{C \in \mathcal{L} : a \notin C\}$ and $\mathcal{L}^+ = \{C \in \mathcal{L} : a \in C\}$. Then we have $\mathcal{L} = \mathcal{L}^- \cup \mathcal{L}^+$, $C_0 \in \mathcal{L}^-$ and $C_1 \subsetneq C_2$, for each $C_1 \in \mathcal{L}^-$ and $C_2 \in \mathcal{L}^+$. By Theorem 3.1 we have $C^- = \bigcup \mathcal{L}^- \in \mathbb{P}(\mathbb{G}_{\omega\omega})$ and, by the maximality of $\mathcal{L}$,

$C^- \in \mathcal{L}$. Since $a \notin C^-$ we have $C^- = \max \mathcal{L}^-$, which implies $C_0 \subset C^-$ and, hence, $i_0 \in \operatorname{supp} C^-$. Thus, by (9), $C^+ = C^- \cup \{a\} \in \mathbb{P}(\mathbb{G}_{\omega\omega})$. For $C \in \mathcal{L}^+$ we have $C^+ \subset C$ and, by the maximality of $\mathcal{L}$, $C^+ \in \mathcal{L}$, in fact $C^+ = \min \mathcal{L}^+$. Clearly the pair $\langle C^-, C^+ \rangle$ is a jump in $\mathcal{L}$. Since $A \subset C_0$ and $B \in \mathcal{L}^+$ we have $A \subset C^- \subset C^+ \subset B$. Thus, $\mathcal{L} \setminus \{\emptyset\}$ has dense jumps and, since $0_{\mathcal{L}}$ is non-isolated, the same holds for $\mathcal{L}$.

(b) $\Rightarrow$ (a). Clearly, $\mathcal{P} = \{\bigcup_{i \in \omega} A_i : \forall i \in \omega \ A_i \in [G_i]^\omega\}$ is a positive family contained in $\mathbb{P}(\mathbb{G}_{\omega\omega})$ and $\bigcap \mathcal{P} = \emptyset$. Now the statement follows from Theorem 3.6(a). □

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