MAXIMAL CHAINS OF ISOMORPHIC SUBORDERS OF COUNTABLE ULTRAHOMOGENEOUS PARTIAL ORDERS

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Abstract

We investigate the poset $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$, where $\mathbb{X}$ is a countable ultrahomogeneous partial order and $\mathbb{P}(\mathbb{X})$ the set of suborders of $\mathbb{X}$ isomorphic to $\mathbb{X}$. For $\mathbb{X}$ different from (resp. equal to) a countable antichain the order types of maximal chains in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$ are characterized as the order types of compact (resp. compact and nowhere dense) sets of reals having the minimum non-isolated.
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1 Introduction

The general concept - to explore the relationship between the properties of a relational structure $\mathbb{X}$ and the properties of the poset $\mathbb{P}(\mathbb{X})$ of substructures of $\mathbb{X}$ isomorphic to $\mathbb{X}$ - can be developed in several ways. For example, regarding the forcing theoretic aspect, the poset of copies of each countable non-scattered linear order is forcing equivalent to the two-step iteration of the Sacks forcing and a σ-closed forcing [9], while the posets of copies of countable scattered linear orders have σ-closed forcing equivalents (separative quotients) [10].

Regarding the order-theoretic aspect, one of the extensively investigated order invariants of a poset is the class of order types of its maximal chains [2, 5, 6, 11] and, for the poset of isomorphic suborders of the rational line, $\langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$, this class is characterized in [8]. The main result of the present paper is the following generalization of that result.

Theorem 1.1 If $\mathbb{X}$ is a countable ultrahomogeneous partial order different from a countable antichain, then for each linear order L the following conditions are equivalent:

(a) L is isomorphic to a maximal chain in the poset $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$;

(b) L is an $\mathbb{R}$-embeddable complete linear order with 0_L non-isolated;

(c) L is isomorphic to a compact set $K \subset \mathbb{R}$ such that $0_K \in K'$.

If $\mathbb{X}$ is a countable antichain, then the corresponding characterization is obtained if we replace "complete" by "Boolean" in (b) and "compact" by "compact and nowhere dense" in (c).

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So, for example, there are maximal chains of copies of the random poset isomorphic to $(0, 1]$, to the Cantor set without 0, and to α^*, for each countable limit ordinal α. Although it is not a usual practice, we start with a proof in the introduction. The equivalence of (b) and (c) is a known fact (see, for example, Theorem 6 of [8]) and the implication (a) $\Rightarrow$ (b) for the ultrahomogeneous partial orders different from the countable antichain follows from the general result on ultrahomogeneous structures given in Theorem 2.2 of the present paper. Thus, only the implication (b) $\Rightarrow$ (a) in general, and the implication (a) $\Rightarrow$ (b) in case of the countable antichain remain to be proved. Naturally, we will use the following, well known classification of countable ultrahomogeneous partial orders - the Schmerl list [13]:

Theorem 1.2 (Schmerl) A countable strict partial order is ultrahomogeneous iff it is isomorphic to one of the following partial orders:
$\mathbb{A}_\omega$, a countable antichain (that is, the empty relation on ω);
$\mathbb{B}_n = n \times \mathbb{Q}$, for $1 \leq n \leq \omega$, where $\langle i_1, q_1\rangle < \langle i_2, q_2\rangle \Leftrightarrow i_1 = i_2 \wedge q_1 <_{\mathbb{Q}} q_2$;
$\mathbb{C}_n = n \times \mathbb{Q}$, for $1 \leq n \leq \omega$, where $\langle i_1, q_1\rangle < \langle i_2, q_2\rangle \Leftrightarrow q_1 <_{\mathbb{Q}} q_2$;
$\mathbb{D}$, the unique countable homogeneous universal poset (the random poset).

For the antichain $\mathbb{A}_\omega$ the equivalence (b) $\Leftrightarrow$ (a) follows from Theorem 1.5 and the fact that $\mathbb{P}(\mathbb{A}_\omega) = [\omega]^\omega$ is a positive family. The most difficult part of the proof of (b) $\Rightarrow$ (a) - for the random poset $\mathbb{D}$ - is given in Section 4. In Sections 5 and 6, using the constructions from [8], we prove (b) $\Rightarrow$ (a) for the posets $\mathbb{B}_n$ and $\mathbb{C}_n$.

The rest of this section contains three facts which will be used in the sequel. Before that, we remind the reader that a *partial order* is a structure $\langle P, \leq\rangle$ consisting of a set P and a binary relation $\leq$ on P which is reflexive, antisymmetric and transitive. A set $\mathcal{D} \subset P$ is *dense in a partial order* $\langle P, \leq\rangle$ if for any $p \in P$ there is some $d \in \mathcal{D}$ such that $d \leq p$. To each partial order we can adjoin a *strict partial order* $\langle P, <\rangle$ which is a structure consisting of a set P and a binary relation $<$ on P which is irreflexive and transitive (note that it follows directly from definition that the relation $<$ in a strict partial order is asymmetric).

In a linear order $\langle L, <\rangle$, a *maximum* of L is denoted by 1_L, and a *minimum* of L is denoted by 0_L. A minimum 0_L is *non-isolated* if there is no $x \in L$ such that $\neg\exists y \in L\ (0_L < y < x)$. A set $X \subset L$ is called *dense in a linear order* L iff for any $a, b \in L$ there is $x \in X$ such that $a < x < b$. A pair $\langle \mathcal{A}, \mathcal{B}\rangle$ is a *cut* in a linear order $\langle L, <\rangle$ iff $L = \mathcal{A} \dot\cup \mathcal{B}$, $\mathcal{A}, \mathcal{B} \neq \emptyset$ and $a < b$, for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$. A cut $\langle \mathcal{A}, \mathcal{B}\rangle$ is a *gap* iff neither $\max \mathcal{A}$ nor $\min \mathcal{B}$ exist. A linear order $\langle L, <\rangle$ is called *Dedekind-complete* iff there are no gaps in $\langle L, <\rangle$. Further, a linear order $\langle L, <\rangle$ is called *Boolean* iff it is *complete* (has 0,1 and has no gaps) and *has dense jumps*, which means that for each $x, y \in L$ satisfying $x < y$ there are $a, b \in L$ such that $x \leq a < b \leq y$ and $(a, b)_L = \emptyset$.

For any function $\varphi : X \to Y$ we denote $\operatorname{ran} \varphi := \varphi[X] = \{\varphi(x) : x \in X\}$. If K is a subset of the real line, then K' denotes the set of all *accumulation points* of K, i.e. points x such that x is in the closure of the set $K \setminus \{x\}$ with respect to the natural topology on $\mathbb{R}$.

Fact 1.3 Each countable complete linear order is Boolean.

Lemma 1.4 Let L be an uncountable complete, $\mathbb{R}$-embeddable linear order such that 0_L is non-isolated. Then $L \cong \sum_{x\in[-\infty,\infty]} L_x$, where
(L1) L_x, $x \in [-\infty,\infty]$, are at most countable complete linear orders,
(L2) The set $M = \{x \in [-\infty,\infty] : |L_x| > 1\}$ is at most countable,
(L3) $|L_{-\infty}| = 1$ or $0_{L_{-\infty}}$ is non-isolated.

Proof. $L = \sum_{i\in I} L_i$, where L_i are the equivalence classes corresponding to the condensation relation $\sim$ on L given by: $x \sim y \Leftrightarrow |[\min\{x,y\},\max\{x,y\}]| \leq \omega$ (see [12]). Since the order L is complete and $\mathbb{R}$-embeddable, I is too. Since the cofinalities and coinitialities of the orders L_i are countable, I is a dense linear order. So $I \cong [0,1] \cong [-\infty,\infty]$. Hence, the orders L_i are complete and, since $\min L_i \sim \max L_i$, countable. If $|L_i| > 1$, L_i has a jump (Fact 1.3) so, $L \hookrightarrow \mathbb{R}$ gives $|M| \leq \omega$. □

We recall that a family $\mathcal{P} \subset P(\omega)$ is called a *positive family* iff:
(P1) $\emptyset \notin \mathcal{P}$;
(P2) $\mathcal{P} \ni A \subset B \subset \omega \Rightarrow B \in \mathcal{P}$;
(P3) $A \in \mathcal{P} \wedge |F| < \omega \Rightarrow A \setminus F \in \mathcal{P}$;
(P4) $\exists A \in \mathcal{P}\ \ |\omega \setminus A| = \omega$.

Theorem 1.5 ([7]) If $\mathcal{P} \subset P(\omega)$ is a positive family, then for each linear order L the following conditions are equivalent:
(a) L is isomorphic to a maximal chain in the poset $\langle \mathcal{P} \cup \{\emptyset\}, \subset\rangle$;
(b) L is an $\mathbb{R}$-embeddable Boolean linear order with 0_L non-isolated;
(c) L is isomorphic to a compact nowhere dense set $K \subset \mathbb{R}$ such that $0_K \in K'$.
In addition, (b) implies that there is a maximal chain $\mathcal{L}$ in $\langle \mathcal{P} \cup \{\emptyset\}, \subset\rangle$ satisfying $\bigcap(\mathcal{L} \setminus \{\emptyset\}) = \emptyset$ and isomorphic to L.

2 Copies of countable ultrahomogeneous structures

Let $L = \{R_i : i \in I\}$ be a relational language, where $\mathrm{ar}(R_i) = n_i$, $i \in I$. An L-structure $\mathbb{X} = \langle X, \{\rho_i : i \in I\}\rangle$ is called *countable* iff $|X| = \omega$. If $A \subset X$, then $\langle A, \{(\rho_i)_A : i \in I\}\rangle$ (shortly denoted by $\langle A, \{\rho_i : i \in I\}\rangle$, whenever this abuse of notation does not produce a confusion) is a *substructure* of $\mathbb{X}$, where $(\rho_i)_A = \rho_i \cap A^{n_i}$, $i \in I$. If $\mathbb{Y} = \langle Y, \{\sigma_i : i \in I\}\rangle$ is an L-structure too, a mapping $f : X \to Y$ is an *embedding* (we write $\mathbb{X} \hookrightarrow_f \mathbb{Y}$) iff it is an injection and

$$\forall i \in I\ \ \forall \langle x_1, \ldots x_{n_i}\rangle \in X^{n_i}\ \ (\langle x_1, \ldots, x_{n_i}\rangle \in \rho_i \Leftrightarrow \langle f(x_1), \ldots, f(x_{n_i})\rangle \in \sigma_i).$$

If $\mathbb{X}$ embeds in $\mathbb{Y}$ we write $\mathbb{X} \hookrightarrow \mathbb{Y}$. Let $\mathrm{Emb}(\mathbb{X},\mathbb{Y}) = \{f : \mathbb{X} \hookrightarrow_f \mathbb{Y}\}$ and $\mathrm{Emb}(\mathbb{X}) = \{f : \mathbb{X} \hookrightarrow_f \mathbb{X}\}$. If, in addition, f is a surjection, it is an *isomorphism* (we write $\mathbb{X} \cong_f \mathbb{Y}$) and the structures $\mathbb{X}$ and $\mathbb{Y}$ are *isomorphic*, in notation $\mathbb{X} \cong \mathbb{Y}$.

Each isomorphism between finite substructures of $\mathbb{X}$ is called a *finite isomorphism* of $\mathbb{X}$. A structure $\mathbb{X}$ is *ultrahomogeneous* iff each finite isomorphism of $\mathbb{X}$ can be extended to an automorphism of $\mathbb{X}$. The *age* of $\mathbb{X}$, $\mathrm{Age}\,\mathbb{X}$, is the class of all finite L-structures embeddable in $\mathbb{X}$. We will use the following well known facts from the Fraïssé theory.

Theorem 2.1 (Fraïssé) Let L be an at most countable relational language. Then

(a) A countable L-structure $\mathbb{X}$ is ultrahomogeneous iff for each finite isomorphism φ of $\mathbb{X}$ and each $x \in X \setminus \mathrm{dom}\,\varphi$ there is a finite isomorphism ψ of $\mathbb{X}$ extending φ to x (see [3] p. 389 or [4] p. 326).

(b) If $\mathbb{X}$ and $\mathbb{Y}$ are countable ultrahomogeneous L-structures and $\mathrm{Age}\,\mathbb{X} = \mathrm{Age}\,\mathbb{Y}$, then $\mathbb{X} \cong \mathbb{Y}$ (see [3] p. 333 or [4] p. 326).

Concerning the order types of maximal chains in the posets of the form $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, where $\mathbb{X} = \langle X, \{\rho_i : i \in I\}\rangle$ is a relational structure and $\mathbb{P}(\mathbb{X})$ the set of the domains of its isomorphic substructures, that is

$$\mathbb{P}(\mathbb{X}) = \{A \subset X : \langle A, \{(\rho_i)_A : i \in I\}\rangle \cong \mathbb{X}\} = \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\}$$

we have the following general statement.

Theorem 2.2 Let $\mathbb{X}$ be a countable ultrahomogeneous structure of an at most countable relational language and $\mathbb{P}(\mathbb{X}) \neq \{X\}$. If $\mathcal{L}$ is a maximal chain in the poset $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$, then

(a) $\mathcal{L}$ is an $\mathbb{R}$-embeddable complete linear order with $0_{\mathcal{L}}(= \emptyset)$ non-isolated;

(b) If there is a positive family $\mathcal{P} \subset \mathbb{P}(\mathbb{X})$, then for each countable linear order L satisfying (a), there is a maximal chain in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$ isomorphic to L.

Proof. (a) First we prove that

$$\bigcup \mathcal{A} \in \mathbb{P}(\mathbb{X}), \text{ for each chain } \mathcal{A} \text{ in the poset } \langle \mathbb{P}(\mathbb{X}), \subset \rangle. \tag{1}$$

Let φ be a finite isomorphism of $\bigcup \mathcal{A}$ and $x \in \bigcup \mathcal{A}$. Since $\mathcal{A}$ is a chain there is $A \in \mathcal{A}$ such that $\mathrm{dom}\,\varphi \cup \mathrm{ran}\,\varphi \cup \{x\} \subset A$. Since $A \cong \mathbb{X}$, by Theorem 2.1(a) there is $y \in A$ such that $\psi = \varphi \cup \{\langle x, y\rangle\}$ is an isomorphism, so ψ is a finite isomorphism of $\bigcup \mathcal{A}$. Thus, by Theorem 2.1(a), the structure $\bigcup \mathcal{A}$ is ultrahomogeneous. Since $\mathbb{X} \cong A \subset \bigcup \mathcal{A} \subset X$ we have $\mathrm{Age}\,\mathbb{X} = \mathrm{Age}\,A \subset \mathrm{Age} \bigcup \mathcal{A} \subset \mathrm{Age}\,\mathbb{X}$, which, by Theorem 2.1(b), implies $\bigcup \mathcal{A} \cong \mathbb{X}$, that is $\bigcup \mathcal{A} \in \mathbb{P}(\mathbb{X})$.

Let $X = \{x_n : n \in \omega\}$ be an enumeration. Since $\mathcal{L} \subset [X]^\omega \cup \{\emptyset\}$, the function $f : \mathcal{L} \to \mathbb{R}$ defined by $f(A) = \sum_{n \in \omega} 2^{-n} \cdot \chi_A(x_n)$ (where $\chi_A : X \to \{0, 1\}$ is the characteristic function of the set $A \subset X$) is an embedding of $\langle \mathcal{L}, \subset \rangle$ into $\langle \mathbb{R}, <_{\mathbb{R}} \rangle$.

Clearly, $\min \mathcal{L} = \emptyset$ and $\max \mathcal{L} = X$. Let $\langle \mathcal{A}, \mathcal{B}\rangle$ be a cut in $\mathcal{L}$. If $\mathcal{A} = \{\emptyset\}$ then $\max \mathcal{A} = \emptyset$. If $\mathcal{A} \neq \{\emptyset\}$, by (1) we have $\bigcup \mathcal{A} \in \mathbb{P}(\mathbb{X})$ and, since $A \subset \bigcup \mathcal{A} \subset B$, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the maximality of $\mathcal{L}$ implies $\bigcup \mathcal{A} \in \mathcal{L}$. So, if $\bigcup \mathcal{A} \in \mathcal{A}$ then $\max \mathcal{A} = \bigcup \mathcal{A}$. Otherwise $\bigcup \mathcal{A} \in \mathcal{B}$ and $\min \mathcal{B} = \bigcup \mathcal{A}$. Thus $\langle \mathcal{L}, \subset \rangle$ is complete.

Suppose that A is the successor of $\emptyset$ in $\mathcal{L}$. Since $\mathbb{P}(\mathbb{X}) \neq \{X\}$ there is $B \in \mathbb{P}(\mathbb{X}) \setminus \{X\}$ and, if $f : \mathbb{X} \hookrightarrow A$, then $f[B] \in \mathbb{P}(\mathbb{X})$, $f[B] \subsetneq A$ and, hence, $\mathcal{L} \cup \{f[B]\}$ is a chain in $\mathbb{P}(\mathbb{X})$. A contradiction to the maximality of $\mathcal{L}$.

(b) By Fact 1.3, L is a Boolean order and, by Theorem 1.5, in the poset $\langle \mathcal{P} \cup \{\emptyset\}, \subset\rangle$ there is a maximal chain $\mathcal{L}$ isomorphic to L and such that $\bigcap(\mathcal{L} \setminus \{\emptyset\}) = \emptyset$. Now, $\mathcal{L}$ is a chain in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset\rangle$ and we check its maximality. Suppose that $\mathcal{L} \cup \{A\}$ is a chain, where $A \in \mathbb{P}(\mathbb{X}) \setminus \mathcal{L}$. Then $A \subsetneq S$ or $S \subsetneq A$, for each $S \in \mathcal{L} \setminus \{\emptyset\}$ and, since $\bigcap(\mathcal{L} \setminus \{\emptyset\}) = \emptyset$, there is $S \in \mathcal{L} \setminus \{\emptyset\}$ such that $S \subset A$, which implies $A \in \mathcal{P}$. But $\mathcal{L} \setminus \{\emptyset\}$ is a maximal chain in $\mathcal{P}$. A contradiction. $\square$

Remark 2.3 Concerning the assumption $\mathbb{P}(\mathbb{X}) \neq \{X\}$ we note that there are countable ultrahomogeneous structures satisfying $\mathbb{P}(\mathbb{X}) = \{X\}$ (see [3], p. 399).

For $1 < n < \omega$ the set $\mathbb{P}(\mathbb{C}_n)$ does not contain a positive family, since (P3) is not satisfied. Namely, if $A \in \mathbb{P}(\mathbb{C}_n)$ and $x \in A$, then $A \setminus \{x\}$ is not a copy of $\mathbb{C}_n$ (one class of incompatible elements is of size $n-1$).

For some ω-saturated, ω-homogeneous-universal relational structures the implication (b) $\Rightarrow$ (a) of Theorem 1.1 is not true. Let L be the language with one binary relational symbol ρ and $\mathcal{T}$ the L-theory of empty relations ($\forall x, y \neg\, x \,\rho\, y$). Then $\mathbb{X} = \langle \omega, \emptyset\rangle$ is the ω-saturated model of $\mathcal{T}$. But $\mathbb{P}(\mathbb{X}) = [\omega]^\omega$ is a positive family and, by Theorem 1.5, maximal chains in $\mathbb{P}(\mathbb{X}) \cup \{\emptyset\}$ are Boolean. Thus, for example, $\mathbb{P}(\mathbb{X}) \cup \{\emptyset\}$ does not contain a maximal chain isomorphic to $[0,1]_{\mathbb{R}}$.

3 Copies of the countable random poset

Let $\mathbb{P} = \langle P, <\rangle$ be a partial order. By $C(\mathbb{P})$ we denote the set of all triples $\langle L, G, U\rangle$ of pairwise disjoint finite subsets of P such that:

(C1) $\forall l \in L\, \forall g \in G \;\; l < g$,

(C2) $\forall u \in U\, \forall l \in L \;\; \neg u < l$ and

(C3) $\forall u \in U\, \forall g \in G \;\; \neg g < u$.

For $\langle L, G, U\rangle \in C(\mathbb{P})$, let $P_{\langle L,G,U\rangle}$ be the set of all $p \in P \setminus (L \cup G \cup U)$ satisfying:

(S1) $\forall l \in L\; p > l$,

(S2) $\forall g \in G\; p < g$ and

(S3) $\forall u \in U\; p \| u$ (where $p\|q$ denotes that $p \neq q \wedge \neg p < q \wedge \neg q < p$).

Fact 3.1 Let $\mathbb{P} = \langle P, <\rangle$ be a partial order and $\emptyset \neq A \subset P$. Then

(a) $C(A, <) = \{\langle L, G, U\rangle \in C(\mathbb{P}) : L, G, U \subset A\}$;

(b) $A_{\langle L,G,U\rangle} = P_{\langle L,G,U\rangle} \cap A$, for each $\langle L, G, U\rangle \in C(A, <)$.

(c) $\langle \emptyset, \emptyset, \emptyset\rangle \in C(\mathbb{P})$ and $P_{\langle \emptyset,\emptyset,\emptyset\rangle} = P$.

Proof. For pairwise disjoint sets $L, G, U \subset A$ we have: $L \times G \subset {<}$ iff $L \times G \subset {<_A}$ and $((U \times L) \cup (G \times U)) \cap {<} = \emptyset$ iff $((U \times L) \cup (G \times U)) \cap {<_A} = \emptyset$. $\square$

Fact 3.2 A countable strict partial order $\mathbb{D} = \langle D, <\rangle$ is a countable random poset iff $D_{\langle L,G,U\rangle} \neq \emptyset$, for each $\langle L, G, U\rangle \in C(\mathbb{D})$ (see [1]).

Lemma 3.3 Let $\mathbb{D} = \langle D, < \rangle$ be a countable random poset. Then
(a) $D_{\langle L,G,U\rangle} \in \mathbb{P}(\mathbb{D})$ and, hence, $|D_{\langle L,G,U\rangle}| = \omega$, for each $\langle L, G, U\rangle \in C(\mathbb{D})$;
(b) $D \setminus F \in \mathbb{P}(\mathbb{D})$, for each finite $F \subset D$;
(c) If $C \subset D$ and $A \not\subset C$ for each $A \in \mathbb{P}(\mathbb{D})$, then $D \setminus C \in \mathbb{P}(\mathbb{D})$;
(d) If $\mathcal{L} \subset \mathbb{P}(\mathbb{D})$ is a chain, then $\bigcup \mathcal{L} \in \mathbb{P}(\mathbb{D})$;
(e) If $D = A \,\dot\cup\, B$, then either A or B contains an element of $\mathbb{P}(\mathbb{D})$.

Proof. (a) Let $\langle L, G, U\rangle \in C(\mathbb{D})$. Then L, G and U are disjoint subsets of D,

$$\forall l \in L \;\; \forall g \in G \;\; \forall u \in U \;\; (u \not< l < g \not< u), \tag{2}$$

and $D_{\langle L,G,U\rangle} \cap (L \cup G \cup U) = \emptyset$. Let $\langle L_1, G_1, U_1\rangle \in C(D_{\langle L,G,U\rangle})$. Then L_1, G_1 and U_1 are disjoint subsets of $D_{\langle L,G,U\rangle}$ and, by Fact 3.1, $\langle L_1, G_1, U_1\rangle \in C(\mathbb{D})$ which implies

$$\forall l_1 \in L_1 \;\; \forall g_1 \in G_1 \;\; \forall u_1 \in U_1 \;\; (u_1 \not< l_1 < g_1 \not< u_1). \tag{3}$$

Since $L_1 \cup G_1 \cup U_1 \subset D_{\langle L,G,U\rangle}$, by (S1)-(S3) we have

$$\forall x \in L_1 \cup G_1 \cup U_1 \;\; \forall l \in L \;\; \forall g \in G \;\; \forall u \in U \;\; (l < x < g \wedge x \not< u \wedge u \not< x). \tag{4}$$

First we show that $\langle L \cup L_1, G \cup G_1, U \cup U_1\rangle \in C(\mathbb{D})$. (C1) Let $l' \in L \cup L_1$ and $g' \in G \cup G_1$. Then $l' < g'$ follows from: (2), if $l' \in L$ and $g' \in G$; (3), if $l' \in L_1$ and $g' \in G_1$; (4), if $l' \in L$ and $g' = x \in G_1$ or $l' = x \in L_1$ and $g' \in G$. (C2) Let $l' \in L \cup L_1$ and $u' \in U \cup U_1$. Then $u' \not< l'$ follows from: (2), if $l' \in L$ and $u' \in U$; (3), if $l' \in L_1$ and $u' \in U_1$; (4), if $l' \in L$ and $u' = x \in U_1$ (since $l' < u'$) or $l' = x \in L_1$ and $u' \in U$. In the same way we prove (C3).

So there exists some $x \in D_{\langle L\cup L_1, G\cup G_1, U\cup U_1\rangle}$, which implies $x \in D_{\langle L,G,U\rangle} \cap D_{\langle L_1,G_1,U_1\rangle} = (D_{\langle L,G,U\rangle})_{\langle L_1,G_1,U_1\rangle}$ (Fact 3.1). Thus $D_{\langle L,G,U\rangle}$ is a random poset and, hence a copy of $\mathbb{D}$.

(b) Let $\langle L, G, U\rangle \in C(D \setminus F)$. By Fact 3.1 we have $\langle L, G, U\rangle \in C(\mathbb{D})$ and, by (a), $\emptyset \neq (D \setminus F) \cap D_{\langle L,G,U\rangle} = (D \setminus F)_{\langle L,G,U\rangle}$. Thus $D \setminus F$ is a copy of $\mathbb{D}$.

(c) Let $\langle L, G, U\rangle \in C(D \setminus C)$. Then, by Fact 3.1, $\langle L, G, U\rangle \in C(\mathbb{D})$ and, by (a), $D_{\langle L,G,U\rangle} \in \mathbb{P}(\mathbb{D})$. By the assumption we have $D_{\langle L,G,U\rangle} \cap (D \setminus C) \neq \emptyset$ and, by Fact 3.1, $(D \setminus C)_{\langle L,G,U\rangle} \neq \emptyset$ and $D \setminus C$ is a random poset.

(d) See (1) in the proof of Theorem 2.2.

(e) Follows from (c). □

Lemma 3.4 Let $\mathbb{D} = \langle D, < \rangle$ be a countable random poset, $C \in [D]^\omega$ and $A \not\subset C$ for each $A \in \mathbb{P}(\mathbb{D})$ (for example, C can be an infinite antichain). Then
(a) $\mathcal{P} = \{B \subset D : D \setminus C \subset^* B\} \subset \mathbb{P}(\mathbb{D})$ ($X \subset^* Y$ denotes $|X \setminus Y| < \omega$);
(b) $\mathcal{P}$ is a positive family on D.

Proof. (a) Suppose that $A \subset D \setminus B$, for some $A \in \mathbb{P}(\mathbb{D})$. Since $D \setminus C \subset^* B$ we have $D \setminus B \subset^* C$ and, hence, $A \subset^* C$, that is $|A \setminus C| < \omega$. By Lemma 3.3(b),

$A \cap C = A \setminus (A \setminus C) \in \mathbb{P}(\mathbb{D})$, which is, by our assumption, impossible. So $D \setminus B$ does not contain copies of $\mathbb{D}$ and, by Lemma 3.3(c), $B \in \mathbb{P}(\mathbb{D})$.

(b) Conditions (P1) and (P2) are evident. If $D \setminus C \subset^* B$ and $|F| < \omega$, then, clearly, $D \setminus C \subset^* B \setminus F$ and (P3) is true. Since the set $D \setminus C$ is co-infinite, (P4) holds. $\Box$

Lemma 3.5 Let $A \subset B \subset \omega$ and let L be a complete linear ordering, such that $|B \setminus A| = |L| - 1$ (Notice that we are abusing notation here, namely whenever we write $|X| \pm 1$ in the paper, it will have the obvious meaning in the case $|X| < \omega$, whereas in the case $|X| = \omega$ we will assume $|X| \pm 1 = \omega$). Then there is a chain $\mathcal{L}$ in $[A, B]_{P(B)}$ satisfying $A, B \in \mathcal{L} \cong L$ and such that $\bigcup \mathcal{A}, \bigcap \mathcal{B} \in \mathcal{L}$ and $|\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1$, for each cut $\langle \mathcal{A}, \mathcal{B} \rangle$ in $\mathcal{L}$.

Proof. If $|B \setminus A|$ is a finite set, say $B = A \cup \{a_1, \dots a_n\}$, then $|L| = n + 1$ and $\mathcal{L} = \{A, A \cup \{a_1\}, A \cup \{a_1, a_2\}, \dots, B\}$ is a chain with the desired properties.

If $|B \setminus A| = \omega$, then L is a countable and, hence, $\mathbb{R}$-embeddable complete linear order. It is known that an infinite linear order is isomorphic to a maximal chain in $P(\omega)$ iff it is $\mathbb{R}$-embeddable and Boolean (see, for example, [7]). By Fact 1.3 L is a Boolean order and, thus, there is a maximal chain $\mathcal{L}_1$ in $P(B \setminus A)$ isomorphic to L. Let $\mathcal{L} = \{A \cup C : C \in \mathcal{L}_1\}$. Since $\emptyset, B \setminus A \in \mathcal{L}_1$ we have $A, B \in \mathcal{L}$ and the function $f : \mathcal{L}_1 \to \mathcal{L}$, defined by $f(C) = A \cup C$, witnesses that $\langle \mathcal{L}_1, \subsetneq \rangle \cong \langle \mathcal{L}, \subsetneq \rangle$ so $\mathcal{L}$ is isomorphic to L. For each cut $\langle \mathcal{A}, \mathcal{B} \rangle$ in $\mathcal{L}_1$ we have $\bigcup \mathcal{A} \subset \bigcap \mathcal{B}$ and, by the maximality of $\mathcal{L}_1$, $\bigcup \mathcal{A}, \bigcap \mathcal{B} \in \mathcal{L}_1$ and $|\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1$. Clearly, the same is true for each cut in $\mathcal{L}$. $\Box$

4 Maximal chains of copies of the random poset

In the following theorem we slightly change notation. We denote the linear order by Λ, in order not to be confused with finite sets denoted by L, which appear in triples $\langle L, G, U \rangle$ from the characterization of $\mathbb{D}$ in Fact 3.2.

Theorem 4.1 For each $\mathbb{R}$-embeddable complete linear order Λ with 0_Λ non-isolated there is a maximal chain in $\langle \mathbb{P}(\mathbb{D}) \cup \{\emptyset\}, \subset \rangle$ isomorphic to Λ.

Proof. By Lemma 3.4 and Theorem 2.2 it remains to prove the statement for uncountable Λ's. So let Λ be an uncountable linear order with the given properties. According to the Lemma 1.4, it has a representation $\Lambda \cong \sum_{x \in [-\infty, \infty]} L_x$ satisfying conditions (L1-L3) from Lemma 1.4. For the rest of the proof, we fix this presentation.

Case I: $-\infty \notin M \ni \infty$. First we take the rational line $\langle \mathbb{Q}, <_\mathbb{Q} \rangle$ and construct a set $\lhd \subset \mathbb{Q}^2$ such that $\langle \mathbb{Q}, \lhd \rangle$ is a random poset with additional, convenient properties. Let $\mathbb{P}$ be the set of pairs $p = \langle P_p, \lhd_p \rangle$ satisfying

(i) $P_p \in [\mathbb{Q}]^{<\omega}$,

(ii) $\lhd_p \subset P_p \times P_p$ is a strict partial order on P_p,

(iii) $<_\mathbb{Q}$ extends $\lhd_p$, that is $\forall q_1, q_2 \in P_p \;\; (q_1 \lhd_p q_2 \Rightarrow q_1 <_\mathbb{Q} q_2)$,

and let the relation $\leq$ on $\mathbb{P}$ be defined by:

$$p \leq q \Leftrightarrow P_p \supset P_q \wedge \lhd_p \cap (P_q \times P_q) = \lhd_q . \tag{5}$$

Claim 4.2 $\langle \mathbb{P}, \leq \rangle$ is a partial order.

Proof. The reflexivity of $\leq$ is obvious. If $p \leq q \leq p$, then $P_p = P_q$ and, hence, $\lhd_p = \lhd_p \cap (P_p \times P_p) = \lhd_p \cap (P_q \times P_q) = \lhd_q$ so $p = q$ and $\leq$ is antisymmetric.

If $p \leq q \leq r$, then $P_p \supset P_q \supset P_r$ and, consequently, $\lhd_p \cap (P_r \times P_r) = \lhd_p \cap (P_q \times P_q) \cap (P_r \times P_r) = \lhd_q \cap (P_r \times P_r) = \lhd_r$. Thus $p \leq r$. □

Claim 4.3 The sets $\mathcal{D}_q = \{p \in \mathbb{P} : q \in P_p\}$, $q \in \mathbb{Q}$, are dense in $\mathbb{P}$.

Proof. If $p \in \mathbb{P} \setminus \mathcal{D}_q$, that is $q \notin P_p$, then $\lhd_p$ is an irreflexive and transitive relation on the set P_p and on the set $P_p \cup \{q\}$ as well. Also $\lhd_p \subset <_\mathbb{Q}$ thus $p_1 = \langle P_p \cup \{q\}, \lhd_p \rangle \in \mathbb{P}$. Thus $p_1 \in \mathcal{D}_q$ and, clearly, $p_1 \leq p$. □

Now let $\mathbb{Q} = J \cup \bigcup_{y \in M} J_y$ be a partition of $\mathbb{Q}$ into $|M| + 1$ dense subsets of $\mathbb{Q}$. For $\langle L, G, U \rangle \in ([\mathbb{Q}]^{<\omega})^3 \setminus \{\langle \emptyset, \emptyset, \emptyset \rangle\}$, let $m_{\langle L,G,U \rangle} = \max_{\langle \mathbb{Q}, <_\mathbb{Q} \rangle} (L \cup G \cup U)$.

Claim 4.4 For each $\langle L, G, U \rangle \in ([\mathbb{Q}]^{<\omega})^3 \setminus \{\langle \emptyset, \emptyset, \emptyset \rangle\}$ and each $m \in \mathbb{N}$ the set $\mathcal{D}_{\langle L,G,U \rangle, m}$ is dense in $\mathbb{P}$, where

$$\begin{aligned} \mathcal{D}_{\langle L,G,U \rangle, m} \;\; = \;\; & \Big\{ p \in \mathbb{P} : L \cup G \cup U \subset P_p \;\wedge\; \Big(\langle L, G, U \rangle \notin C(p) \\ & \vee \;\; (G \neq \emptyset \wedge p_{\langle L,G,U \rangle} \cap J \neq \emptyset) \\ & \vee \;\; (G = \emptyset \wedge p_{\langle L,G,U \rangle} \cap (m_{\langle L,G,U \rangle}, m_{\langle L,G,U \rangle} + \tfrac{1}{m}) \cap J \neq \emptyset) \Big) \Big\}. \end{aligned}$$

Proof. Let $p' \in \mathbb{P} \setminus \mathcal{D}_{\langle L,G,U \rangle, m}$. By Claim 4.3 there is $p \in \mathbb{P}$ such that $p \leq p'$ and $L \cup G \cup U \subset P_p$. If $\langle L, G, U \rangle \notin C(p)$ then $p \in \mathcal{D}_{\langle L,G,U \rangle, m}$ and we are done. If

$$\langle L, G, U \rangle \in C(p), \tag{6}$$

then we continue the proof distinguishing the following two cases.

Case 1: $G \neq \emptyset$. Let us define $\max_{\langle \mathbb{Q}, <_\mathbb{Q} \rangle} \emptyset = -\infty$. By (6) and (C1) for p, if $L \neq \emptyset$, then $\max_{\langle \mathbb{Q}, <_\mathbb{Q} \rangle} L \lhd_p \min_{\langle \mathbb{Q}, <_\mathbb{Q} \rangle} G$ and, by (iii), $\max_{\langle \mathbb{Q}, <_\mathbb{Q} \rangle} L <_\mathbb{Q} \min_{\langle \mathbb{Q}, <_\mathbb{Q} \rangle} G$. Now, since J is a dense set in $\langle \mathbb{Q}, <_\mathbb{Q} \rangle$ we choose

$$q \in (\max_{\langle \mathbb{Q}, <_\mathbb{Q} \rangle} L, \min_{\langle \mathbb{Q}, <_\mathbb{Q} \rangle} G) \cap J \setminus P_p \tag{7}$$

and define $p_1 = \langle P_p \cup \{q\}, \lhd_{p_1} \rangle$ where

$$\lhd_{p_1} = \lhd_p \cup \{\langle x, q \rangle : \exists l \in L \;\; x \unlhd_p l\} \cup \{\langle q, y \rangle : \exists g \in G \;\; g \unlhd_p y\}. \tag{8}$$

First we prove that $p_1 \in \mathbb{P}$. Clearly, p_1 satisfies condition (i).

(ii) Since $\lhd_p$ is an irreflexive relation and, by (7), $q \not\in P_p$, by (8) the relation $\lhd_{p_1}$ is irreflexive as well.

Suppose that $\lhd_{p_1}$ is not asymmetric. Then, since $\lhd_p$ is asymmetric, there is $t \in P_p$ such that $\langle t, q\rangle, \langle q, t\rangle \in \lhd_{p_1}$ and by (8), $g \unlhd_p t \unlhd_p l$, for some $l \in L$ and $g \in G$ which, by the transitivity of $\unlhd_p$ implies $g \unlhd_p l$. But, by (6) and (C1) we have $l \lhd_p g$. A contradiction.

Let $\langle a, b\rangle, \langle b, c\rangle \in \lhd_{p_1}$. Then, since the relation $\lhd_{p_1}$ is irreflexive and asymmetric, we have $a \neq b \neq c \neq a$. If $q \not\in \{a, b, c\}$, then $\langle a, c\rangle \in \lhd_{p_1}$ by the transitivity of $\lhd_p$. Otherwise we have three possibilities:

$a = q$. Then $\langle b, c\rangle \in \lhd_p$ and there is a $g \in G$ such that $g \unlhd_p b$. Hence $g \lhd_p c$ which, by (8), implies $\langle q, c\rangle \in \lhd_{p_1}$, that is $\langle a, c\rangle \in \lhd_{p_1}$.

$b = q$. Then there are $l \in L$ and $g \in G$ such that $a \unlhd_p l$ and $g \unlhd_p c$. By (C1) we have $l \lhd_p g$ and, by the transitivity of $\lhd_p$, $a \lhd_p c$ and, hence, $\langle a, c\rangle \in \lhd_{p_1}$.

$c = q$. Then $\langle a, b\rangle \in \lhd_p$ and there is an $l \in L$ such that $b \unlhd_p l$. Hence $a \lhd_p l$ which, by (8), implies $\langle a, q\rangle \in \lhd_{p_1}$, that is $\langle a, c\rangle \in \lhd_{p_1}$.

(iii) Since $p \in \mathbb{P}$, we have $\lhd_p \subset <_{\mathbb{Q}}$. If $\langle x, q\rangle \in \lhd_{p_1}$ and $l \in L$, where $x \unlhd_p l$, then, since $\lhd_p$ satisfies (iii), we have $x \leq_{\mathbb{Q}} l$. By (7) we have $l <_{\mathbb{Q}} q$ and, thus, $x <_{\mathbb{Q}} q$. In a similar way we show that $\langle q, y\rangle \in \lhd_{p_1}$ implies $q <_{\mathbb{Q}} y$.

Thus $p_1 \in \mathbb{P}$, $P_{p_1} \supset P_p \supset L \cup G \cup U$ and, by (8), $\lhd_{p_1} \cap (P_p \times P_p) = \lhd_p$, which implies that $p_1 \leq p$ $(\leq p')$. So p is a suborder of p_1 and, by (6) and Fact 3.1, $\langle L, G, U\rangle \in C(p_1)$. Since $G \neq \emptyset$ and $q \in J$, for a proof that $p_1 \in \mathcal{D}_{\langle L,G,U\rangle,m}$ it remains to be shown that $q \in (p_1)_{\langle L,G,U\rangle}$. By (8) $l \lhd_{p_1} q \lhd_{p_1} g$, for each $l \in L$ and $g \in G$, so (S1) and (S2) are true. For $u \in U$, $\langle u, q\rangle \in \lhd_{p_1}$ would give $l \in L$ satisfying $u \unlhd_p l$ and, since $U \cap L = \emptyset$, $u \lhd_p l$, which is impossible by (6) and (C2). Similarly, $\langle q, u\rangle \in \lhd_{p_1}$ is not possible and, thus, $q \parallel_{p_1} u$ and (S3) is satisfied.

Case 2: $G = \emptyset$. Again, since J is a dense set in the linear order $\langle \mathbb{Q}, <_{\mathbb{Q}}\rangle$ we choose

$$q \in (m_{\langle L,G,U\rangle}, m_{\langle L,G,U\rangle} + \tfrac{1}{m}) \cap J \setminus P_p \tag{9}$$

and define $p_1 = \langle P_p \cup \{q\}, \lhd_{p_1}\rangle$, where

$$\lhd_{p_1} = \lhd_p \cup \{\langle x, q\rangle : \exists l \in L \;\; x \unlhd_p l\}. \tag{10}$$

First we prove that $p_1 \in \mathbb{P}$. Clearly, p_1 satisfies condition (i).

(ii) By (9) we have $q \not\in P_p$ so, by (10) the relation $\lhd_{p_1}$ is irreflexive.

Let $\langle a, b\rangle, \langle b, c\rangle \in \lhd_{p_1}$. If $q \not\in \{a, b, c\}$, then $\langle a, c\rangle \in \lhd_{p_1}$ by (10) and the transitivity of $\lhd_p$. Otherwise, by (10) again, $a, b \neq q$ and, thus, $c = q$. Hence there is an $l \in L$ such that $b \unlhd_p l$. Since $a, b \neq q$, by (10) we have $a \lhd_p b$ and, hence $a \lhd_p l$, which implies $\langle a, q\rangle \in \lhd_{p_1}$, that is $\langle a, c\rangle \in \lhd_{p_1}$.

(iii) Since $p \in \mathbb{P}$, we have $\lhd_p \subset <_{\mathbb{Q}}$. If $\langle x, q\rangle \in \lhd_{p_1}$ and $l \in L$, where $x \unlhd_p l$, then, since $\lhd_p$ satisfies (iii), we have $x \leq_{\mathbb{Q}} l$. By (9) we have $l \leq_{\mathbb{Q}} m_{\langle L,G,U\rangle} <_{\mathbb{Q}} q$ and, thus, $x <_{\mathbb{Q}} q$.

Thus $p_1 \in \mathbb{P}$. As in Case 1 we show that $L \cup G \cup U \subset P_{p_1}$, $p_1 \leq p$ $(\leq p')$ and $\langle L, G, U\rangle \in C(p_1)$. By (9) and since $G = \emptyset$, for a proof that $p_1 \in \mathcal{D}_{\langle L,G,U\rangle,m}$ it remains to be shown that $q \in (p_1)_{\langle L,G,U\rangle}$. (S2) is trivial and, by (10), for $l \in L$ we have $\langle l, q\rangle \in \lhd_{p_1}$ thus (S1) holds as well. Suppose that $\neg\ q \parallel_{p_1} u$, for some $u \in U$. Then, by (9) and (10), $\langle u, q\rangle \in \lhd_{p_1}$ and, hence, there is an $l \in L$ satisfying $u \lhd_p l$, which is impossible by (6) and (C2) for p. So (S3) is true. $\Box$

By the Rasiowa-Sikorski theorem there is a filter $\mathcal{G}$ in $\langle \mathbb{P}, \leq\rangle$ intersecting the sets $\mathcal{D}_q$, $q \in \mathbb{Q}$, and $\mathcal{D}_{\langle L,G,U\rangle,m}$, $\langle L, G, U\rangle \in ([\mathbb{Q}]^{<\omega})^3$, $m \in \mathbb{N}$.

Claim 4.5 (a) $\bigcup_{p\in\mathcal{G}} P_p = \mathbb{Q}$;

(b) $\lhd := \bigcup_{p\in\mathcal{G}} \lhd_p$ is a strict partial order on $\mathbb{Q}$;

(c) $\lhd \cap (P_p \times P_p) = \lhd_p$, for each $p \in \mathcal{G}$;

(d) $<_{\mathbb{Q}}$ extends $\lhd$, that is $\forall q_1, q_2 \in \mathbb{Q}\ (q_1 \lhd q_2 \Rightarrow q_1 <_{\mathbb{Q}} q_2)$.

Proof. (a) For $q \in \mathbb{Q}$ let $p_0 \in \mathcal{G} \cap \mathcal{D}_q$. Then $q \in P_{p_0} \subset \bigcup_{p\in\mathcal{G}} P_p$.

(b) The relation $\lhd$ is irreflexive since all the relations $\lhd_p$ are irreflexive.

Let $\langle a, b\rangle, \langle b, c\rangle \in \lhd$, $\langle a, b\rangle \in \lhd_{p_1}$ and $\langle b, c\rangle \in \lhd_{p_2}$, where $p_1, p_2 \in \mathcal{G}$. Since $\mathcal{G}$ is a filter there is a $p \in \mathcal{G}$ such that $p \leq p_1, p_2$, which by (5) implies $\lhd_{p_1}, \lhd_{p_2} \subset \lhd_p$. Thus $\langle a, b\rangle, \langle b, c\rangle \in \lhd_p$ and, by the transitivity of $\lhd_p$, $\langle a, c\rangle \in \lhd_p \subset \lhd$.

(c) The inclusion "$\supset$" follows from (ii) and the definition of $\lhd$. If $\langle a, b\rangle \in \lhd \cap (P_p \times P_p)$, then there is a $p_1 \in \mathcal{G}$ such that $\langle a, b\rangle \in \lhd_{p_1}$ and, since $\mathcal{G}$ is a filter, there is a $p_2 \in \mathcal{G}$ such that $p_2 \leq p, p_1$. By (5) we have $\lhd_{p_1} \subset \lhd_{p_2}$, which implies $\langle a, b\rangle \in \lhd_{p_2}$ and, by (5) again, $\langle a, b\rangle \in \lhd_{p_2} \cap (P_p \times P_p) = \lhd_p$.

(d) If $\langle q_1, q_2\rangle \in \lhd$ and $p \in \mathcal{G}$ where $\langle q_1, q_2\rangle \in \lhd_p$, then by (iii), $q_1 <_{\mathbb{Q}} q_2$. $\Box$

Claim 4.6 (a) $\langle A, \lhd\rangle$ is a random poset, for each $x \in (-\infty, \infty]$ and each set A satisfying

$$(-\infty, x) \cap J \subset A \subset (-\infty, x) \cap \mathbb{Q} \tag{11}$$

(b) If $J \subset A \subset \mathbb{Q}$ then $\langle A, \lhd\rangle$ (in particular, $\langle \mathbb{Q}, \lhd\rangle$) is a random poset.

(c) If $C \subset \mathbb{Q}$ and $\max_{\langle \mathbb{Q}, <_{\mathbb{Q}}\rangle} C$ exists, then $\langle C, \lhd\rangle$ is not a random poset.

Proof. (a) By Claim 4.5(b), $\langle A, \lhd\rangle$ is a strict partial order. Let $\langle L, G, U\rangle \in C(A, \lhd)$. Then

$$L \cup G \cup U \subset A \ \wedge\ L \cap G = G \cap U = U \cap A = \emptyset, \tag{12}$$

$$\forall l \in L\ \ \forall g \in G\ \ \forall u \in U\ \ (\langle l, g\rangle \in \lhd\ \wedge\ \langle u, l\rangle \notin \lhd\ \wedge\ \langle g, u\rangle \notin \lhd). \tag{13}$$

We show that $\langle A, \lhd\rangle_{\langle L,G,U\rangle} \neq \emptyset$. For $\langle L, G, U\rangle \neq \langle \emptyset, \emptyset, \emptyset\rangle$ we have two cases.

Case 1: $G \neq \emptyset$. Let $p \in \mathcal{G} \cap \mathcal{D}_{\langle L,G,U\rangle,1}$. Then

$$L \cup G \cup U \subset P_p. \tag{14}$$

First we show that $\langle L, G, U\rangle \in C(p)$. Let $l \in L$, $g \in G$ and $u \in U$. By (13), (14) and Claim 4.5(c) we have $\langle l, g\rangle \in \lhd_p$ and (C1) is true. Since $\lhd_p \subset \lhd$ by (13) we have $\langle u, l\rangle \notin \lhd_p$ and $\langle g, u\rangle \notin \lhd_p$ and (C2) and (C3) are true as well.

Since $p \in \mathcal{D}_{\langle L,G,U\rangle,1}$ there is a $q \in p_{\langle L,G,U\rangle} \cap J$. We prove that $q \in \langle A, \lhd\rangle_{\langle L,G,U\rangle}$. For a $g \in G$ we have $q \lhd_p g$ and, by (iii), $q <_{\mathbb{Q}} g$. By (11) and (12) we have $g \in G \subset A \subset (-\infty, x)$ and, hence, $q <_{\mathbb{Q}} g <_{\mathbb{R}} x$, thus $q \in (-\infty, x) \cap J \subset A$. Let $l \in L$, $g \in G$ and $u \in U$. Since $q \in p_{\langle L,G,U\rangle}$ we have $l \lhd_p q \lhd_p g$ and $\lhd_p \subset \lhd$ implies $l \lhd q \lhd g$. Thus (S1) and (S2) are true. Suppose that $\neg\, q \parallel_{\langle A,\lhd\rangle} u$. Since $q \notin U$ we have $q \neq u$ and, hence, $q \lhd u$ or $u \lhd q$. But then, since $u, q \in P_p$, by Claim 4.5(c) we would have $q \lhd_p u$ or $u \lhd_p q$, which is impossible because $q \in p_{\langle L,G,U\rangle}$. So (S3) is true as well.

Case 2: $G = \emptyset$. By (11) and (12) we have $L \cup G \cup U \subset (-\infty, x)$, which implies $m_{\langle L,G,U\rangle} < x$ and, hence, there is an $m \in \mathbb{N}$ such that

$$m_{\langle L,G,U\rangle} + \tfrac{1}{m} < x. \tag{15}$$

Let $p \in \mathcal{G} \cap \mathcal{D}_{\langle L,G,U\rangle,m}$. Then (14) holds again and exactly like in Case 1 we show that $\langle L, G, U\rangle \in C(p)$. Thus, since $p \in \mathcal{D}_{\langle L,G,U\rangle,m}$ there is $q \in p_{\langle L,G,U\rangle} \cap (m_{\langle L,G,U\rangle}, m_{\langle L,G,U\rangle} + \frac{1}{m}) \cap J$ and, by (15), $q \in J \cap (-\infty, x)$. Thus, by (11), $q \in A$ and exactly like in Case 1 we prove that $q \in \langle A, \lhd\rangle_{\langle L,G,U\rangle}$.

(b) Follows from (a) for $x = \infty$.

(c) Suppose that $\max_{\langle \mathbb{Q},<_{\mathbb{Q}}\rangle} C = q$ and that $\langle C, \lhd\rangle$ is a random poset. Then $C_{\langle \{q\},\emptyset,\emptyset\rangle} \neq \emptyset$ and, by (S1), there is a $q_1 \in C$ such that $q \lhd q_1$, which, by Claim 4.5(d) implies $q <_{\mathbb{Q}} q_1$. A contradiction with the maximality of q. □

For $y \in M$ let us take $I_y \in [J_y \cap (-\infty, y)]^{|L_y|-1}$ and define $A_{-\infty} = \emptyset$ and

$$A_x = (J \cap (-\infty, x)) \cup \bigcup\nolimits_{y \in M \cap (-\infty,x)} I_y, \quad \text{for } x \in (-\infty, \infty];$$
$$A_x^+ = A_x \cup I_x, \quad \text{for } x \in M.$$

Since $J \subset A_\infty^+ \subset \mathbb{Q}$, by Claim 4.6(b) $\langle A_\infty^+, \lhd\rangle$ is a random poset and we construct a maximal chain $\mathcal{L}$ in $\langle \mathbb{P}(A_\infty^+, \lhd), \subset\rangle$, such that $\mathcal{L} \cong \Lambda$.

Claim 4.7 The sets A_x, $x \in [-\infty, \infty]$ and A_x^+, $x \in M$ are subsets of the set A_∞^+ and of $\mathbb{Q}$. In addition, for each $x, x_1, x_2 \in [-\infty, \infty]$ we have

(a) $A_x \subset (-\infty, x)$;
(b) $A_x^+ \subset (-\infty, x)$, if $x \in M$;
(c) $x_1 < x_2 \Rightarrow A_{x_1} \subsetneq A_{x_2}$;
(d) $M \ni x_1 < x_2 \Rightarrow A_{x_1}^+ \subsetneq A_{x_2}$;
(e) $|A_x^+ \setminus A_x| = |L_x| - 1$, if $x \in M$;
(f) $A_x \in \mathbb{P}(A_\infty^+)$, for each $x \in (-\infty, \infty]$;
(g) $A_x^+ \in \mathbb{P}(A_\infty^+)$ and $[A_x, A_x^+]_{\mathbb{P}(A_\infty^+)} = [A_x, A_x^+]_{P(A_x^+)}$, for each $x \in M$.

Proof. Statements (c) and (d) are true since J is a dense subset of $\mathbb{Q}$; (a), (b) and (e) follow from the definitions of A_x and A_x^+ and the choice of the sets I_y. Since $J \cap (-\infty, x) \subset A_x \subset A_x^+ \subset (-\infty, x) \cap \mathbb{Q}$, (f) and (g) follow from Claim 4.6(a). □

Now, for $x \in [-\infty, \infty]$ we define chains $\mathcal{L}_x \subset \mathbb{P}(A_\infty^+) \cup \{\emptyset\}$ in the following way.

For $x \notin M$ we define $\mathcal{L}_x = \{A_x\}$. In particular, $\mathcal{L}_{-\infty} = \{\emptyset\}$.

For $x \in M$, using Claim 4.7 and Lemma 3.5 we obtain a set $\mathcal{L}_x \subset [A_x, A_x^+]_{P(A_x^+)}$ such that $\langle \mathcal{L}_x, \subsetneq \rangle \cong \langle L_x, <_x \rangle$ and

$$A_x, A_x^+ \in \mathcal{L}_x \subset [A_x, A_x^+]_{\mathbb{P}(A_\infty^+)}, \tag{16}$$

$$\bigcup \mathcal{A}, \bigcap \mathcal{B} \in \mathcal{L}_x \quad \text{and} \quad |\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1, \text{ for each cut } \langle \mathcal{A}, \mathcal{B} \rangle \text{ in } \mathcal{L}_x. \tag{17}$$

For $\mathcal{A}, \mathcal{B} \subset \mathbb{P}(A_\infty^+)$ we will write $\mathcal{A} \prec \mathcal{B}$ iff $A \subsetneq B$, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Claim 4.8 Let $\mathcal{L} = \bigcup_{x \in [-\infty, \infty]} \mathcal{L}_x$. Then
(a) If $-\infty \leq x_1 < x_2 \leq \infty$, then $\mathcal{L}_{x_1} \prec \mathcal{L}_{x_2}$ and $\bigcup \mathcal{L}_{x_1} \subset A_{x_2} \subset \bigcup \mathcal{L}_{x_2}$.
(b) $\mathcal{L}$ is a chain in $\langle \mathbb{P}(A_\infty^+) \cup \{\emptyset\}, \subset \rangle$ isomorphic to $\Lambda = \sum_{x \in [-\infty, \infty]} L_x$.
(c) $\mathcal{L}$ is a maximal chain in $\langle \mathbb{P}(A_\infty^+) \cup \{\emptyset\}, \subset \rangle$.

Proof. (a) Let $A \in \mathcal{L}_{x_1}$ and $B \in \mathcal{L}_{x_2}$. If $x_1 \in (-\infty, \infty] \setminus M$, then, by (16) and Claim 4.7(c) we have $A = A_{x_1} \subsetneq A_{x_2} \subset B$. If $x_1 \in M$, then, by (16) and Claim 4.7(d), $A \subset A_{x_1}^+ \subsetneq A_{x_2} \subset B$. The second statement follows from $A_{x_2} \in \mathcal{L}_{x_2}$.

(b) By (a), $\langle [-\infty, \infty], < \rangle \cong \langle \{\mathcal{L}_x : x \in [-\infty, \infty]\}, \prec \rangle$. Since $\mathcal{L}_x \cong L_x$, for $x \in [-\infty, \infty]$, we have $\langle \mathcal{L}, \subsetneq \rangle \cong \sum_{x \in [-\infty, \infty]} \langle \mathcal{L}_x, \subsetneq \rangle \cong \sum_{x \in [-\infty, \infty]} L_x = \Lambda$.

(c) Suppose that $C \in \mathbb{P}(A_\infty^+) \cup \{\emptyset\}$ witnesses that $\mathcal{L}$ is not maximal. Clearly $\mathcal{L} = \mathcal{A} \dot\cup \mathcal{B}$ and $\mathcal{A} \prec \mathcal{B}$, where $\mathcal{A} = \{A \in \mathcal{L} : A \subsetneq C\}$ and $\mathcal{B} = \{B \in \mathcal{L} : C \subsetneq B\}$. Now $\emptyset \in \mathcal{L}_{-\infty}$ and, since $\infty \in M$, by (16) we have $A_\infty^+ \in \mathcal{L}_\infty$. Thus $\emptyset, A_\infty^+ \in \mathcal{L}$, which implies $\mathcal{A}, \mathcal{B} \neq \emptyset$ and, hence, $\langle \mathcal{A}, \mathcal{B} \rangle$ is a cut in $\langle \mathcal{L}, \subsetneq \rangle$. By (16) we have $\{A_x : x \in (-\infty, \infty]\} \subset \mathcal{L} \setminus \{\emptyset\}$ and, by Claim 4.7(a), $\bigcap(\mathcal{L} \setminus \{\emptyset\}) \subset \bigcap_{x \in (-\infty, \infty]} A_x \subset \bigcap_{x \in (-\infty, \infty]} (-\infty, x) = \emptyset$, which implies $\mathcal{A} \neq \{\emptyset\}$. Clearly,

$$\bigcup \mathcal{A} \subset C \subset \bigcap \mathcal{B}. \tag{18}$$

Case 1: $\mathcal{A} \cap \mathcal{L}_{x_0} \neq \emptyset$ and $\mathcal{B} \cap \mathcal{L}_{x_0} \neq \emptyset$, for some $x_0 \in (-\infty, \infty]$. Then $|\mathcal{L}_{x_0}| > 1$, $x_0 \in M$ and $\langle \mathcal{A} \cap \mathcal{L}_{x_0}, \mathcal{B} \cap \mathcal{L}_{x_0} \rangle$ is a cut in $\mathcal{L}_{x_0}$ satisfying (17). By (a), $\mathcal{A} = \bigcup_{x < x_0} \mathcal{L}_x \cup (\mathcal{A} \cap \mathcal{L}_{x_0})$ and, consequently, $\bigcup \mathcal{A} = \bigcup (\mathcal{A} \cap \mathcal{L}_{x_0}) \in \mathcal{L}$. Similarly, $\bigcap \mathcal{B} = \bigcap (\mathcal{B} \cap \mathcal{L}_{x_0}) \in \mathcal{L}$ and, since $|\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1$, by (18) we have $C \in \mathcal{L}$. A contradiction.

Case 2: $\neg$ Case 1. Then for each $x \in (-\infty, \infty]$ we have $\mathcal{L}_x \subset \mathcal{A}$ or $\mathcal{L}_x \subset \mathcal{B}$. Since $\mathcal{L} = \mathcal{A} \dot\cup \mathcal{B}$, $\mathcal{A} \neq \{\emptyset\}$ and $\mathcal{A}, \mathcal{B} \neq \emptyset$, the sets $\mathcal{A}' = \{x \in (-\infty, \infty] : \mathcal{L}_x \subset \mathcal{A}\}$ and $\mathcal{B}' = \{x \in (-\infty, \infty] : \mathcal{L}_x \subset \mathcal{B}\}$ are non-empty and $(-\infty, \infty] = \mathcal{A}' \dot\cup \mathcal{B}'$. Since $\mathcal{A} \prec \mathcal{B}$, for $x_1 \in \mathcal{A}'$ and $x_2 \in \mathcal{B}'$ we have $\mathcal{L}_{x_1} \prec \mathcal{L}_{x_2}$ so, by (a), $x_1 < x_2$. Thus $\langle \mathcal{A}', \mathcal{B}' \rangle$ is a cut in $(-\infty, \infty]$ and, consequently, there is $x_0 \in (-\infty, \infty]$ such that $x_0 = \max \mathcal{A}'$ or $x_0 = \min \mathcal{B}'$.

Subcase 2.1: $x_0 = \max \mathcal{A}'$. Then $x_0 < \infty$ because $\mathcal{B} \neq \emptyset$ and $\mathcal{A} = \bigcup_{x \leq x_0} \mathcal{L}_x$ so, by (a), $\bigcup \mathcal{A} = \bigcup_{x \leq x_0} \bigcup \mathcal{L}_x = \bigcup_{x < x_0} \bigcup \mathcal{L}_x \cup \bigcup \mathcal{L}_{x_0} = \bigcup \mathcal{L}_{x_0}$ which, together with (16) implies

$$\bigcup \mathcal{A} = \begin{cases} A_{x_0} & \text{if } x_0 \notin M, \\ A_{x_0}^+ & \text{if } x_0 \in M. \end{cases} \tag{19}$$

Since $\mathcal{B} = \bigcup_{x\in(x_0,\infty]} \mathcal{L}_x$, we have $\bigcap \mathcal{B} = \bigcap_{x\in(x_0,\infty]} \bigcap \mathcal{L}_x$. By (16) $\bigcap \mathcal{L}_x = A_x$, so we have $\bigcap \mathcal{B} = (\bigcap_{x\in(x_0,\infty]}(-\infty,x) \cap J) \cup (\bigcap_{x\in(x_0,\infty]} \bigcup_{y\in M\cap(-\infty,x)} I_y) = ((-\infty,x_0] \cap J) \cup \bigcup_{y\in M\cap(-\infty,x_0]} I_y = A_{x_0} \cup (\{x_0\} \cap J) \cup \bigcup_{y\in M\cap\{x_0\}} I_y$, so

$$\bigcap \mathcal{B} = \begin{cases} A_{x_0} & \text{if} \quad x_0 \notin J \quad \wedge \quad x_0 \notin M, \\ A_{x_0} \cup \{x_0\} & \text{if} \quad x_0 \in J \quad \wedge \quad x_0 \notin M, \\ A^+_{x_0} & \text{if} \quad x_0 \notin J \quad \wedge \quad x_0 \in M, \\ A^+_{x_0} \cup \{x_0\} & \text{if} \quad x_0 \in J \quad \wedge \quad x_0 \in M. \end{cases} \tag{20}$$

If $x_0 \notin J$, then, by (18), (19) and (20), we have $\bigcup \mathcal{A} = \bigcap \mathcal{B} = C \in \mathcal{L}$. A contradiction.

If $x_0 \in J$ and $x_0 \notin M$, then $\bigcup \mathcal{A} = A_{x_0}$ and $\bigcap \mathcal{B} = A_{x_0} \cup \{x_0\}$. So, by (18) and since $C \notin \mathcal{L}$ we have $C = \bigcap \mathcal{B}$. But, by Claim 4.7(a), $x_0 = \max \bigcap \mathcal{B}$ so, by Claim 4.6(c), $C \notin \mathbb{P}(A^+_\infty)$. A contradiction.

If $x_0 \in J$ and $x_0 \in M$, then $\bigcup \mathcal{A} = A^+_{x_0}$ and $\bigcap \mathcal{B} = A^+_{x_0} \cup \{x_0\}$. Again, by (18) and since $C \notin \mathcal{L}$ we have $C = \bigcap \mathcal{B}$. By Claim 4.7(b), $x_0 = \max \bigcap \mathcal{B}$ so, by Claim 4.6(c), $C \notin \mathbb{P}(A^+_\infty)$. A contradiction.

Subcase 2.2: $x_0 = \min \mathcal{B}'$. Then, by (16), $A_{x_0} \in \mathcal{L}_{x_0} \subset \mathcal{B}$ which, by (a), implies $\bigcap \mathcal{B} = A_{x_0}$. Since $A_x \in \mathcal{L}_x$, for $x \in (-\infty,\infty]$ and $\mathcal{A} = \bigcup_{x<x_0} \mathcal{L}_x$ we have $\bigcup \mathcal{A} = \bigcup_{x<x_0} \bigcup \mathcal{L}_x \supset \bigcup_{x<x_0} A_x = \bigcup_{x<x_0} ((-\infty,x) \cap J) \cup \bigcup_{x<x_0} \bigcup_{y\in M\cap(-\infty,x)} I_y$ $= ((-\infty,x_0) \cap J) \cup \bigcup_{y\in M\cap(-\infty,x_0)} I_y = A_{x_0}$ so $A_{x_0} \subset \bigcup \mathcal{A} \subset \bigcap \mathcal{B} = A_{x_0}$, which implies $C = A_{x_0} \in \mathcal{L}$. A contradiction. $\square$

Case II: $-\infty \notin M \not\ni \infty$. Then $L_\infty = \{\max \Lambda\}$ and the sum $\Lambda + 1$ belongs to Case I. So, there are a maximal chain $\mathcal{L}$ in $\langle \mathbb{P}(\mathbb{D}) \cup \{\emptyset\}, \subset \rangle$ and an isomorphism $f : \langle \Lambda + 1, < \rangle \to \langle \mathcal{L}, \subset \rangle$. Then $A = f(\max \Lambda) \in \mathbb{P}(\mathbb{D})$ and $\mathcal{L}' = f[\Lambda] \cong \Lambda$. By the maximality of $\mathcal{L}$, $\mathcal{L}'$ is a maximal chain in $\langle \mathbb{P}(A) \cup \{\emptyset\}, \subset \rangle$.

Case III: $-\infty \in M$. Then $\Lambda = \sum_{x\in[-\infty,\infty]} L_x$, (L1) and (L2) of Lemma 1.4 hold and

(L3$'$) $L_{-\infty}$ is a countable complete linear order with $0_{L_{-\infty}}$ non-isolated.

Clearly $\Lambda = L_{-\infty} + \Lambda^+$, where $\Lambda^+ = \sum_{x\in(-\infty,\infty]} L_x = \sum_{y\in(0,\infty]} L_{\ln y}$ (here $\ln \infty = \infty$). Let L'_y, $y \in [-\infty,\infty]$, be disjoint linear orders such that $L'_y \cong 1$, for $y \in [-\infty, 0]$, and $L'_y \cong L_{\ln y}$, for $y \in (0,\infty]$. Now $\sum_{y\in[-\infty,\infty]} L'_y \cong [-\infty,0] + \Lambda^+$ belongs to Case I or Case II and we obtain a maximal chain $\mathcal{L}$ in $\mathbb{P}(\mathbb{D}) \cup \{\emptyset\}$ and an isomorphism $f : \langle [-\infty,0] + \Lambda^+, < \rangle \to \langle \mathcal{L}, \subset \rangle$. Clearly, for $A_0 = f(0)$ and $\mathcal{L}^+ = f[\Lambda^+]$ we have $A_0 \in \mathcal{L}$ and $\mathcal{L}^+ \cong \Lambda^+$.

By (L3$'$) and the fact that (b) $\Rightarrow$ (a) for countable Λ's, $\mathbb{P}(A_0) \cup \{\emptyset\}$ contains a maximal chain $\mathcal{L}_{-\infty} \cong L_{-\infty}$. Clearly, $A_0 \in \mathcal{L}_{-\infty}$ and $\mathcal{L}_{-\infty} \cup \mathcal{L}^+ \cong L_{-\infty} + \Lambda^+ = \Lambda$. Suppose that B witnesses that $\mathcal{L}_{-\infty} \cup \mathcal{L}^+$ is not a maximal chain in $\mathbb{P}(\mathbb{D}) \cup \{\emptyset\}$. Then either $A_0 \subsetneq B$, which is impossible since $\mathcal{L}$ is maximal in $\mathbb{P}(\mathbb{D}) \cup \{\emptyset\}$, or $B \subsetneq A_0$, which is impossible since $\mathcal{L}_{-\infty}$ is maximal in $\mathbb{P}(A_0) \cup \{\emptyset\}$. $\square$

5 Maximal chains in $\mathbb{P}(\mathbb{B}_n)$

Theorem 5.1 For $n \in \mathbb{N}$ and each $\mathbb{R}$-embeddable complete linear order L with 0_L non-isolated there is a maximal chain in $\langle \mathbb{P}(\mathbb{B}_n) \cup \{\emptyset\}, \subset\rangle$ isomorphic to L.

Proof. Let the order on $\mathbb{B}_n = \bigcup_{i<n} \mathbb{Q}_i = \bigcup_{i<n}\{i\} \times \mathbb{Q}$ be given by

$$\langle i_1, q_1\rangle < \langle i_2, q_2\rangle \Leftrightarrow i_1 = i_2 \ \wedge \ q_1 <_{\mathbb{Q}} q_2.$$

Clearly, $\langle \mathbb{Q}, <_{\mathbb{Q}}\rangle \cong_{f_i} \langle \mathbb{Q}_i, <\rangle$, where $f_i(q) = \langle i, q\rangle$, for all $q \in \mathbb{Q}$ and, hence, $\mathbb{P}(\mathbb{Q}_i) = \{\{i\} \times C : C \in \mathbb{P}(\mathbb{Q})\}$. If $f : \mathbb{B}_n \hookrightarrow \mathbb{B}_n$, then for each $i < n$ the restriction $f|\mathbb{Q}_i$ is an isomorphism, thus there is a $j_i < n$ such that $f[\mathbb{Q}_i] \subset \mathbb{Q}_{j_i}$ and, moreover, $f[\mathbb{Q}_i] \in \mathbb{P}(\mathbb{Q}_{j_i})$. Clearly, $i_1 \neq i_2$ implies $j_{i_1} \neq j_{i_2}$ and, thus, we have

$$\mathbb{P}(\mathbb{B}_n) = \{ \textstyle\bigcup_{i<n}\{i\} \times C_i : \forall i < n \ \ C_i \in \mathbb{P}(\mathbb{Q})\}. \tag{21}$$

Now, by Theorem 6 of [8], there is a maximal chain $\mathcal{L}$ in $\langle \mathbb{P}(\mathbb{Q}) \cup \{\emptyset\}, \subset\rangle$ isomorphic to L. For $A \in \mathcal{L} \setminus \{\emptyset\}$ let

$$A^* = (\{0\} \times A) \cup \textstyle\bigcup_{0<i<n}\{i\} \times \mathbb{Q}. \tag{22}$$

By (21) we have $\mathcal{L}^* = \{A^* : A \in \mathcal{L} \setminus \{\emptyset\}\} \cup \{\emptyset\} \subset \mathbb{P}(\mathbb{B}_n) \cup \{\emptyset\}$ and, clearly, $\langle \mathcal{L}^*, \subset\rangle$ is a chain in $\langle \mathbb{P}(\mathbb{B}_n) \cup \{\emptyset\}, \subset\rangle$ isomorphic to $\langle \mathcal{L}, \subset\rangle$ and, hence, to L. Suppose that some $C = \bigcup_{i<n}\{i\} \times C_i \in \mathbb{P}(\mathbb{B}_n)$ witnesses that $\mathcal{L}^*$ is not a maximal chain. By (21) and (22) $C \subset \bigcap_{A\in\mathcal{L}\setminus\{\emptyset\}} A^*$ would imply $\mathbb{P}(\mathbb{Q}) \ni C_0 \subset \bigcap(\mathcal{L}\setminus\{\emptyset\})$, which is impossible ($\mathcal{L}$ is a maximal chain in $\mathbb{P}(\mathbb{Q}) \cup \{\emptyset\}$ and $C_0 \setminus F \in \mathbb{P}(\mathbb{Q})$ for each finite $F \subset C_0$). Thus there is an $A \in \mathcal{L} \setminus \{\emptyset\}$ such that $A^* \subset C$ and, by (22),

$$C = \{0\} \times C_0 \cup \textstyle\bigcup_{0<i<n}\{i\} \times \mathbb{Q}. \tag{23}$$

Since $\mathcal{L}^* \cup \{C\}$ is a chain, for each $A \in \mathcal{L} \setminus \{\emptyset\}$ we have $A^* \subsetneq C \ \vee \ C \subsetneq A^*$ which together with (22) and (23) implies $A \subsetneq C_0$ or $C_0 \subsetneq A$. A contradiction to the maximality of $\mathcal{L}$. □

Theorem 5.2 For each $\mathbb{R}$-embeddable complete linear order L with 0_L non-isolated there is a maximal chain in $\langle \mathbb{P}(\mathbb{B}_\omega) \cup \{\emptyset\}, \subset\rangle$ isomorphic to L.

Proof. Let $x_0 = \infty$, let $\langle x_n : n \in \mathbb{N}\rangle$ be a descending sequence in $\mathbb{R} \setminus \mathbb{Q}$ without a lower bound and let $\mathbb{B}_\omega = \langle \mathbb{Q}, <_\omega\rangle = \bigcup_{i\in\omega}\langle (x_{i+1}, x_i) \cap \mathbb{Q}, <_i\rangle$ where

$$q_1 <_\omega q_2 \Leftrightarrow \exists i \in \omega \ \ (q_1, q_2 \in (x_{i+1}, x_i) \ \wedge \ q_1 <_{\mathbb{Q}} q_2).$$

Then for the sets $\mathbb{Q}_i = (x_{i+1}, x_i) \cap \mathbb{Q}$, $i \in \omega$, we have $\langle \mathbb{Q}_i, <_i\rangle \cong \langle \mathbb{Q}, <_{\mathbb{Q}}\rangle$, which implies $\mathbb{P}(\mathbb{Q}_i, <_i) \cong \mathbb{P}(\mathbb{Q}, <_{\mathbb{Q}})$. As in the proof of Theorem 5.1 we obtain

$$\mathbb{P}(\mathbb{B}_\omega) = \{\textstyle\bigcup_{i\in S} C_i : S \in [\omega]^\omega \ \wedge \ \forall i \in S \ \ C_i \in \mathbb{P}(\mathbb{Q}_i)\}. \tag{24}$$

Let L be a linear order with the given properties and, first, let $|L| = \omega$. Clearly the family $\mathrm{Dense}(\mathbb{Q}_i)$ of dense subsets of $\mathbb{Q}_i$ is a subset of $\mathbb{P}(\mathbb{Q}_i)$ and by (24) we have $\mathcal{P} = \{\bigcup_{i\in\omega} C_i : \forall i \in \omega \;\; C_i \in \mathrm{Dense}(\mathbb{Q}_i)\} \subset \mathbb{P}(\mathbb{B}_\omega)$. It is easy to check that $\mathcal{P}$ is a positive family on $\mathbb{Q}$ so, by Theorem 2.2(b), there is a maximal chain in $\langle \mathbb{P}(\mathbb{B}_\omega) \cup \{\emptyset\}, \subset\rangle$ isomorphic to L.

Now, let $|L| > \omega$. Then, by Lemma 1.4, we can assume that $L = \sum_{x\in[-\infty,\infty]} L_x$, where conditions (L1-L3) from Lemma 1.4 are satisfied. We distinguish two cases.

Case 1: $-\infty \notin M$. Then, by the construction from [8] (if $(0, 1]$ is replaced by $(-\infty, \infty]$ and A_1^+ by $\mathbb{Q}$), there is a maximal chain $\mathcal{L}$ in $\langle \mathbb{P}(\mathbb{Q}) \cup \{\emptyset\}, \subset\rangle$ such that

$$\forall A \in \mathcal{L} \setminus \{\emptyset\} \; \exists x \in (-\infty, \infty] \; (A \subset (-\infty, x) \;\wedge\; A \text{ is dense in } (-\infty, x)) \tag{25}$$

and $\mathcal{L} \cong L$. Now we prove

$$\mathcal{L} \setminus \{\emptyset\} \subset \mathbb{P}(\mathbb{B}_\omega) \subset \mathbb{P}(\mathbb{Q}, <_\mathbb{Q}). \tag{26}$$

Let $A \in \mathcal{L} \setminus \{\emptyset\}$, let x be the real corresponding to A in the sense of (25) and let $i_0 = \min\{i \in \omega : (-\infty, x) \cap (x_{i+1}, x_i) \neq \emptyset\}$. Then $x_{i_0+1} < x \leq x_{i_0}$ and, by (25) the set $C_{i_0} = A \cap (x_{i_0+1}, x)$ is dense in (x_{i_0+1}, x) and, hence, $C_{i_0} \in \mathbb{P}(\mathbb{Q}_{i_0})$. Similarly, $C_i = A \cap (x_{i+1}, x_i) \in \mathbb{P}(\mathbb{Q}_i)$, for all $i > i_0$. Since $A \subset \mathbb{Q}$, we have $A = \bigcup_{i\geq i_0} C_i$ and, by (24), $A \in \mathbb{P}(\mathbb{B}_\omega)$. So the first inclusion of (26) is proved.

Let $C = \bigcup_{i\in S} C_i \in \mathbb{P}(\mathbb{B}_\omega)$. By (24) for each $i \in S$ we have $C_i \cong \mathbb{Q}_i \cong \mathbb{Q}$ and, hence, $C \cong \sum_{\omega^*} \mathbb{Q} \cong \mathbb{Q}$. The second inclusion of (26) is proved as well.

By (26) we have $\mathcal{L} \subset \mathbb{P}(\mathbb{B}_\omega) \cup \{\emptyset\} \subset \mathbb{P}(\mathbb{Q}, <_\mathbb{Q}) \cup \{\emptyset\}$ and, clearly, $\mathcal{L}$ is a chain in $\mathbb{P}(\mathbb{B}_\omega) \cup \{\emptyset\}$. Suppose that $\mathcal{L} \cup \{C\}$ is a chain, for some $C \in (\mathbb{P}(\mathbb{B}_\omega) \cup \{\emptyset\}) \setminus \mathcal{L}$. Then, by (26), $C \in \mathbb{P}(\mathbb{Q}, <_\mathbb{Q})$ and $\mathcal{L}$ would not be a maximal chain in the poset $\langle \mathbb{P}(\mathbb{Q}, <_\mathbb{Q}) \cup \{\emptyset\}, \subset\rangle$. So $\mathcal{L}$ is a maximal chain in $\langle \mathbb{P}(\mathbb{B}_\omega) \cup \{\emptyset\}, \subset\rangle$ and $\mathcal{L} \cong L$.

Case 2: $-\infty \in M$. Then we proceed as in (III) of the proof of Theorem 4.1. □

6 Maximal chains in $\mathbb{P}(\mathbb{C}_n)$

Theorem 6.1 For all $n \in \mathbb{N}$ and each $\mathbb{R}$-embeddable complete linear order L with 0_L non-isolated there is a maximal chain in $\langle \mathbb{P}(\mathbb{C}_n) \cup \{\emptyset\}, \subset\rangle$ isomorphic to L.

Proof. Let the order $<$ on $\mathbb{C}_n = \mathbb{Q} \times n$ be given by $\langle q_1, i_1\rangle < \langle q_2, i_2\rangle \Leftrightarrow q_1 <_\mathbb{Q} q_2$. Clearly, the incomparability relation $a \| b \Leftrightarrow a \not< b \;\wedge\; b \not< a$ on $\mathbb{C}_n$ is an equivalence relation with the equivalence classes $\{q\} \times n$, $q \in \mathbb{Q}$, of size n and the corresponding quotient, $\mathbb{C}_n/\|$, is isomorphic to $\langle \mathbb{Q}, <_\mathbb{Q}\rangle$. Since each element of $\mathbb{P}(\mathbb{C}_n)$ has such classes we have $\mathbb{P}(\mathbb{C}_n) = \{A \times n : A \in \mathbb{P}(\mathbb{Q}, <_\mathbb{Q})\}$. It is easy to see that the mapping $f : \mathbb{P}(\mathbb{Q}, <_\mathbb{Q}) \cup \{\emptyset\} \to \mathbb{P}(\mathbb{C}_n) \cup \{\emptyset\}$, given by $f(A) = A \times n$, is an isomorphism between the partial orders $\langle \mathbb{P}(\mathbb{Q}, <_\mathbb{Q}) \cup \{\emptyset\}, \subset\rangle$ and $\langle \mathbb{P}(\mathbb{C}_n) \cup \{\emptyset\}, \subset\rangle$. Hence, the statement follows from Theorem 6 of [8]. □

Theorem 6.2 For each $\mathbb{R}$-embeddable complete linear order L with 0_L non-isolated there is a maximal chain in $\langle \mathbb{P}(\mathbb{C}_\omega) \cup \{\emptyset\}, \subset\rangle$ isomorphic to L.

Proof. Let the strict order $<$ on $\mathbb{C}_\omega = \mathbb{Q}\times\omega = \bigcup_{q\in\mathbb{Q}}\{q\}\times\omega = \bigcup_{q\in\mathbb{Q}}\omega_q$ be given by $\langle q_1,i_1\rangle < \langle q_2,i_2\rangle \Leftrightarrow q_1 <_{\mathbb{Q}} q_2$. For a set $X\subset\mathbb{C}_\omega$ let us define $\operatorname{supp} X = \{q\in\mathbb{Q} : X\cap\omega_q\neq\emptyset\}$. Now the incomparability classes ω_q are infinite and, again, the corresponding quotient, $\mathbb{C}_\omega/\|$, is isomorphic to the rational line $\langle\mathbb{Q},<_{\mathbb{Q}}\rangle$. Since the same holds for the copies of $\mathbb{C}_\omega$ it is easy to check that

$$\mathbb{P}(\mathbb{C}_\omega) = \{\textstyle\bigcup_{q\in A}\{q\}\times C_q : A\in\mathbb{P}(\mathbb{Q},<_{\mathbb{Q}}) \ \wedge\ \forall q\in A\ (C_q\in[\omega]^\omega)\}. \tag{27}$$

$$X\subset\mathbb{C}_\omega \ \wedge\ \max\operatorname{supp} X \text{ exists} \quad\Rightarrow\quad X\notin\mathbb{P}(\mathbb{C}_\omega). \tag{28}$$

By (27), $\mathcal{P} = \{\bigcup_{q\in\mathbb{Q}}\{q\}\times C_q : \forall q\in\mathbb{Q}\ (C_q\in[\omega]^\omega)\}\subset\mathbb{P}(\mathbb{C}_\omega)$ and, clearly, $\mathcal{P}$ is a positive family, so for a countable L the statement follows from Theorem 2.2(b).

Now, let L be an uncountable linear order. Then, by Lemma 1.4, we can assume that $L=\sum_{x\in[-\infty,\infty]}L_x$, where conditions (L1-L3) from Lemma 1.4 are satisfied.

Case I: $-\infty\notin M\ni\infty$. Let $\mathbb{Q}=\bigcup_{y\in M}J_y$ be a partition of $\mathbb{Q}$ into $|M|$ disjoint dense sets and, for $y\in M$, let $I_y\in[J_y\cap(-\infty,y)]^{|L_y|-1}$. Let $(-\infty,x)_{\mathbb{Q}} = (-\infty,x)\cap\mathbb{Q}$ and $\omega^+=\omega\setminus\{0\}$. Let us define $A_{-\infty}=\emptyset$ and, for $x\in(-\infty,\infty]$,

$$A_x = ((-\infty,x)_{\mathbb{Q}}\times\omega^+)\cup\textstyle\bigcup_{y\in M\cap(-\infty,x)} I_y\times\{0\},$$
$$A_x^+ = A_x\cup(I_x\times\{0\}),\quad \text{for } x\in M.$$

By (27), $A^+_\infty\cong\mathbb{C}_\omega$ and we will construct a maximal chain $\mathcal{L}\cong L$ in the poset $\langle\mathbb{P}(A^+_\infty)\cup\{\emptyset\},\subset\rangle$. By (27), for each $x\in(-\infty,\infty]$ and each set $A\subset\mathbb{C}_\omega$ we have

$$(-\infty,x)_{\mathbb{Q}}\times\omega^+\subset A\subset(-\infty,x)_{\mathbb{Q}}\times\omega \quad\Rightarrow\quad A\in\mathbb{P}(\mathbb{C}_\omega). \tag{29}$$

Claim 6.3 The sets A_x, $x\in[-\infty,\infty]$ and A_x^+, $x\in M$ are subsets of the set A^+_∞. In addition, for each $x,x_1,x_2\in[-\infty,\infty]$ we have

(a) $A_x\subset(-\infty,x)_{\mathbb{Q}}\times\omega$;
(b) $A_x^+\subset(-\infty,x)_{\mathbb{Q}}\times\omega$, if $x\in M$;
(c) $x_1<x_2\Rightarrow A_{x_1}\subsetneq A_{x_2}$;
(d) $M\ni x_1<x_2\Rightarrow A^+_{x_1}\subsetneq A_{x_2}$;
(e) $|A_x^+\setminus A_x| = |L_x|-1$, if $x\in M$;
(f) $A_x\in\mathbb{P}(A^+_\infty)$, for each $x\in(-\infty,\infty]$.
(g) $A_x^+\in\mathbb{P}(A^+_\infty)$ and $[A_x,A_x^+]_{\mathbb{P}(A^+_\infty)} = [A_x,A_x^+]_{P(A_x^+)}$, for each $x\in M$.

Proof. Statements (c) and (d) are true since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$; (a), (b) and (e) follow from the definitions of A_x and A_x^+ and the choice of the sets I_y. Since $(-\infty,x)_{\mathbb{Q}}\times\omega^+\subset A_x\subset A_x^+\subset(-\infty,x)_{\mathbb{Q}}\times\omega$, (f) and (g) follow from (29). □

Now, for $x\in[-\infty,\infty]$ we define chains $\mathcal{L}_x\subset\mathbb{P}(A^+_\infty)\cup\{\emptyset\}$ in the following way.

For $x\notin M$ we define $\mathcal{L}_x=\{A_x\}$. In particular, $\mathcal{L}_{-\infty}=\{\emptyset\}$.

For $x\in M$, by Claim 6.3 and Lemma 3.5 there is a set $\mathcal{L}_x\subset[A_x,A_x^+]_{P(A_x^+)}$ such that $\langle\mathcal{L}_x,\subsetneq\rangle\cong\langle L_x,<_x\rangle$ and

$$A_x, A_x^+\in\mathcal{L}_x\subset[A_x,A_x^+]_{\mathbb{P}(A^+_\infty)}, \tag{30}$$

$$\bigcup \mathcal{A}, \bigcap \mathcal{B} \in \mathcal{L}_x \quad \text{and} \quad |\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1, \text{ for each cut } \langle \mathcal{A}, \mathcal{B} \rangle \text{ in } \mathcal{L}_x. \tag{31}$$

For $\mathcal{A}, \mathcal{B} \subset \mathbb{P}(A^+_\infty)$ we will write $\mathcal{A} \prec \mathcal{B}$ iff $A \subsetneq B$, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Claim 6.4 Let $\mathcal{L} = \bigcup_{x \in [-\infty,\infty]} \mathcal{L}_x$. Then
(a) If $-\infty \leq x_1 < x_2 \leq \infty$, then $\mathcal{L}_{x_1} \prec \mathcal{L}_{x_2}$ and $\bigcup \mathcal{L}_{x_1} \subset A_{x_2} \subset \bigcup \mathcal{L}_{x_2}$.
(b) $\mathcal{L}$ is a chain in $\langle \mathbb{P}(A^+_\infty) \cup \{\emptyset\}, \subset \rangle$ isomorphic to $L = \sum_{x \in [-\infty,\infty]} L_x$.
(c) $\mathcal{L}$ is a maximal chain in $\langle \mathbb{P}(A^+_\infty) \cup \{\emptyset\}, \subset \rangle$.

Proof. The proof of (a) and (b) is a copy of the proof of (a) and (b) of Claim 4.8, if we replace (16) and Claim 4.7 by (30) and Claim 6.3.

(c) Suppose that $C \in \mathbb{P}(A^+_\infty) \cup \{\emptyset\}$ witnesses that $\mathcal{L}$ is not maximal. Using (30) and Claim 6.3, as in the proof of Claim 4.8(c) for $\mathcal{A} = \{A \in \mathcal{L} : A \subsetneq C\}$ and $\mathcal{B} = \{B \in \mathcal{L} : C \subsetneq B\}$ we show that $\langle \mathcal{A}, \mathcal{B} \rangle$ is a cut in $\langle \mathcal{L}, \subsetneq \rangle$, $\mathcal{A} \neq \{\emptyset\}$ and

$$\bigcup \mathcal{A} \subset C \subset \bigcap \mathcal{B}. \tag{32}$$

Case 1: $\mathcal{A} \cap \mathcal{L}_{x_0} \neq \emptyset$ and $\mathcal{B} \cap \mathcal{L}_{x_0} \neq \emptyset$, for some $x_0 \in (-\infty, \infty]$. Then we obtain a contradiction exactly like in Claim 4.8.

Case 2: $\neg$ Case 1. Then like in Claim 4.8 for $\mathcal{A}' = \{x \in (-\infty,\infty] : \mathcal{L}_x \subset \mathcal{A}\}$ and $\mathcal{B}' = \{x \in (-\infty,\infty] : \mathcal{L}_x \subset \mathcal{B}\}$ we show that $\langle \mathcal{A}', \mathcal{B}' \rangle$ is a cut in $(-\infty,\infty]$. Thus, there is $x_0 \in (-\infty,\infty]$ such that $x_0 = \max \mathcal{A}'$ or $x_0 = \min \mathcal{B}'$.

Subcase 2.1: $x_0 = \max \mathcal{A}'$. Then like in Claim 4.8 we prove

$$\bigcup \mathcal{A} = \begin{cases} A_{x_0} & \text{if } x_0 \notin M, \\ A^+_{x_0} & \text{if } x_0 \in M. \end{cases} \tag{33}$$

Since $\mathcal{B} = \bigcup_{x \in (x_0,\infty]} \mathcal{L}_x$, we have $\bigcap \mathcal{B} = \bigcap_{x \in (x_0,\infty]} \bigcap \mathcal{L}_x$. By (30) $\bigcap \mathcal{L}_x = A_x$, so $\bigcap \mathcal{B} = (\bigcap_{x \in (x_0,\infty]} (-\infty,x)_\mathbb{Q} \times \omega^+) \cup (\bigcap_{x \in (x_0,\infty]} \bigcup_{y \in M \cap (-\infty,x)} I_y \times \{0\}) = ((-\infty,x_0]_\mathbb{Q} \times \omega^+) \cup \bigcup_{y \in M \cap (-\infty,x_0]} I_y \times \{0\} = A_{x_0} \cup ((\{x_0\} \cap \mathbb{Q}) \times \omega^+) \cup \bigcup_{y \in M \cap \{x_0\}} I_y \times \{0\}$, so

$$\bigcap \mathcal{B} = \begin{cases} A_{x_0} & \text{if } x_0 \notin \mathbb{Q} \wedge x_0 \notin M, \\ A_{x_0} \cup (\{x_0\} \times \omega^+) & \text{if } x_0 \in \mathbb{Q} \wedge x_0 \notin M, \\ A^+_{x_0} & \text{if } x_0 \notin \mathbb{Q} \wedge x_0 \in M, \\ A^+_{x_0} \cup (\{x_0\} \times \omega^+) & \text{if } x_0 \in \mathbb{Q} \wedge x_0 \in M. \end{cases} \tag{34}$$

If $x_0 \notin \mathbb{Q}$ then by (32-34), we have $\bigcup \mathcal{A} = \bigcap \mathcal{B} = C \in \mathcal{L}$, a contradiction.

If $x_0 \in \mathbb{Q}$ and $x_0 \notin M$, then $\bigcup \mathcal{A} = A_{x_0}$ and $\bigcap \mathcal{B} = A_{x_0} \cup (\{x_0\} \times \omega^+)$. So, by (32) and since $C \notin \mathcal{L}$ we have $C = A_{x_0} \cup S$, where $\emptyset \neq S \subset \{x_0\} \times \omega^+$. By Claim 6.3(a), $x_0 = \max \operatorname{supp} C$, so by (28), $C \notin \mathbb{P}(A^+_\infty)$. This is a contradiction.

If $x_0 \in \mathbb{Q}$ and $x_0 \in M$, then $\bigcup \mathcal{A} = A^+_{x_0}$ and $\bigcap \mathcal{B} = A^+_{x_0} \cup (\{x_0\} \times \omega^+)$. Again, by (32) and since $C \notin \mathcal{L}$ we have $C = A_{x_0} \cup S$, where $\emptyset \neq S \subset \{x_0\} \times \omega^+$. By Claim 6.3(b), $x_0 = \max \operatorname{supp} C$ so, by (28), $C \notin \mathbb{P}(A^+_\infty)$, a contradiction.

Subcase 2.2: $x_0 = \min \mathcal{B}'$. Then, by (30), $A_{x_0} \in \mathcal{L}_{x_0} \subset \mathcal{B}$ which, by (a), implies $\bigcap \mathcal{B} = A_{x_0}$. Since $A_x \in \mathcal{L}_x$, for all $x \in (-\infty, \infty]$ and $\mathcal{A} = \bigcup_{x<x_0} \mathcal{L}_x$ we have $\bigcup \mathcal{A} \supset \bigcup_{x<x_0} A_x = \bigcup_{x<x_0}((-\infty, x)_{\mathbb{Q}} \times \omega^+) \cup \bigcup_{x<x_0} \bigcup_{y \in M \cap (-\infty,x)} I_y \times \{0\} = ((-\infty, x_0)_{\mathbb{Q}} \times \omega^+) \cup \bigcup_{y \in M \cap (-\infty,x_0)} I_y \times \{0\} = A_{x_0}$ so $A_{x_0} \subset \bigcup \mathcal{A} \subset \bigcap \mathcal{B} = A_{x_0}$, which implies $C = A_{x_0} \in \mathcal{L}$. This is a contradiction. □

Case II: $-\infty \notin M \not\ni \infty$ or $-\infty \in M$. Then we proceed like in Cases II and III of Theorem 4.1. □

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