Finite homomorphism-homogeneous permutations via edge colourings of chains

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First of all there is Blue. Later there is White, and then there is Black, and before the beginning there is Brown.

Paul Auster: Ghosts (The New York Trilogy)

(Ultra)homogeneity

Let $\mathcal{A}$ be a (countable) first order structure. $\mathcal{A}$ is said to be (ultra)homogeneous if any isomorphism

$$\iota : \mathcal{B} \to \mathcal{B}'$$

between its finitely generated substructures is a restriction of an automorphism α of $\mathcal{A}$: $\iota = \alpha|_B$.

Remark

If we restrict to relational structures, 'finitely generated' becomes simply 'finite'.

Classification programme for countable ultrahomogeneous structures

- finite graphs (Gardiner, 1976)
- posets (Schmerl, 1979)
- undirected graphs (Lachlan & Woodrow, 1980)
- tournaments (Lachlan, 1984)
- directed graphs (Cherlin, 1998 – Memoirs of AMS, 160+ pp.)
- finite groups (Cherlin & Felgner, 2000)
- permutations (???) (Cameron, 2002)
- ...

Fraïssé theory

Fact

For any countably infinite ultrahomogeneous structure $\mathcal{A}$, its age $\mathrm{Age}(\mathcal{A})$ (the class of its finitely generated substructures) has the following properties:

- it has countably many isomorphism types;
- it is closed for taking (copies of) substructures;
- it has the joint embedding property (JEP);
- it has the amalgamation property (AP).

A class of finite(ly generated) structures with such properties is called a Fraïssé class.

Theorem (Fraïssé)

Let $\mathbf{C}$ *be a Fraïssé class. Then there exists a unique countably infinite ultrahomogeneous structure* $\mathcal{F}$ *such that* $\mathrm{Age}(\mathcal{F}) = \mathbf{C}$.

Fraïssé theory (continued)

The structure $\mathcal{F}$ from the previous theorem is called the Fraïssé limit of **C**.

Classical examples:

- finite chains $\longrightarrow$ $(\mathbb{Q}, <)$
- finite undirected graph $\longrightarrow$ the Rado (random) graph R
- finite posets $\longrightarrow$ the random poset
- finite tournaments $\longrightarrow$ the random tournament
- finite metric spaces with rational distances $\longrightarrow$ the rational Urysohn space $\mathbb{U}_{\mathbb{Q}}$
- finite permutations (???) $\longrightarrow$ the random permutation (???!!!)

Fraïssé limits over finite relational languages are ω-categorical, have quantifier elimination, oligomorphic automorphism groups,...

Homomorphism-homogeneity

In 2006, in their seminal paper, P. J. Cameron and J. Nešetřil investigated what happens if one replaces isomorphisms and automorphisms in the classical definition of ultrahomogeneity by other types of morphism.

In particular, a structure $\mathcal{A}$ is said to be homomorphism-homogeneous (HH) if any homomorphism

$$\varphi : \mathcal{B} \to \mathcal{B}'$$

between its finitely generated substructures is a restriction of an endomorphism ψ of $\mathcal{A}$: $\varphi = \psi|_B$.

Homomorphism-homogeneity vs homogeneity

HH is the 'semigroup-theoretical analogue' of ultrahomogeneity!

Theorem (Mašulović & M. Pech, 2011)

A submonoid M of A^A is the endomorphism monoid of a HH structure on A in a residually finite relational language if and only if it is closed (in the pointwise convergence topology) and oligomorphic.

Theorem (M & P, 2011)

A structure $\mathcal{A}$ is HH if and only if $\mathrm{End}(\mathcal{A})$ is oligomorphic (i.e. $\mathcal{A}$ is weakly oligomorphic) and $\mathcal{A}$ admits quatifier elimination for positive formulæ.

Classification of (countable) HH structures

- finite groups ('quasi-injective', Bertholf & Walls, 1979)
- some classes of infinite groups (Tomkinson, 1988)
- posets – of arbitrary cardinality! (Mašulović, 2007)
- finite tournaments with loops (Ilić, Mašulović & Rajković, 2008)
- lattices and some classes of semilattices (ID & Mašulović, 2011)
- some classes of finite (point-line) geometries (Mašulović, 2013)
- mono-unary algebras (Jungábel & Mašulović, 2013)
- Fraïssé limits (ID, 2014) – the 'one-point homomorphism extension property'

Classification of (countable) HH structures

WARNING!

co-NP-complete classes of finite HH structures:

- finite undirected graphs with loops (Rusinov & Schweitzer, 2010)
- finite algebras of a (fixed) similarity type containing either a symbol of arity ≥ 2, or at least two unary symbols (Mašulović, 2013)
- ...

Few questions

So, what about finite HH permutations?

How, on Earth, is a permutation considered in the role of a structure???

What is, in fact, a permutation?

- To an algebraist: an element of the symmetric group $\mathrm{Sym}(X)$, a bijection $\pi : X \to X$, e.g.

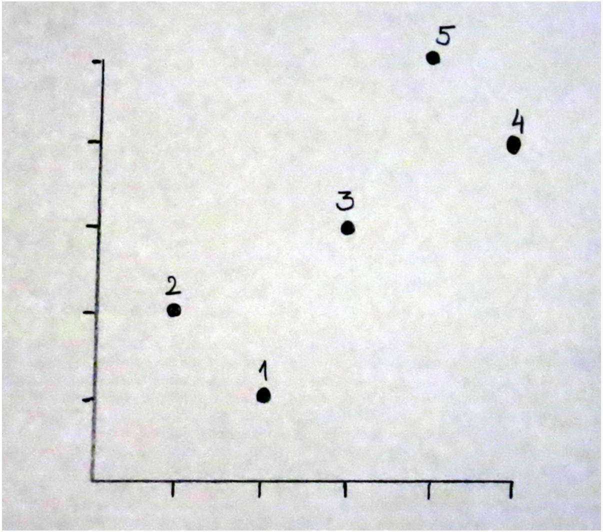
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix}$$

  Has nothing to do with $|X|$.
- To a combinatorialist: a sequence $a_1 a_2 \ldots$ over X in which each element occurs exactly once, e.g.

$$21354$$

  Also, can be represented by 'plots'. Runs into trouble when X is infinite.

What is, in fact, a permutation?



What is, in fact, a permutation?

- To a model theorist: a structure $(X, \leq_1, \leq_2)$, where the set X is equipped by two linear orders, e.g.

  $$1 <_1 2 <_1 3 <_1 4 <_1 5 \quad \text{and} \quad 2 <_2 1 <_2 3 <_2 5 <_2 4.$$

  Very suitable for infinite generalisations.

$\oplus$ and $\ominus$

Let π and σ be permutations of $[1, p]$ and $[1, s]$, respectively.

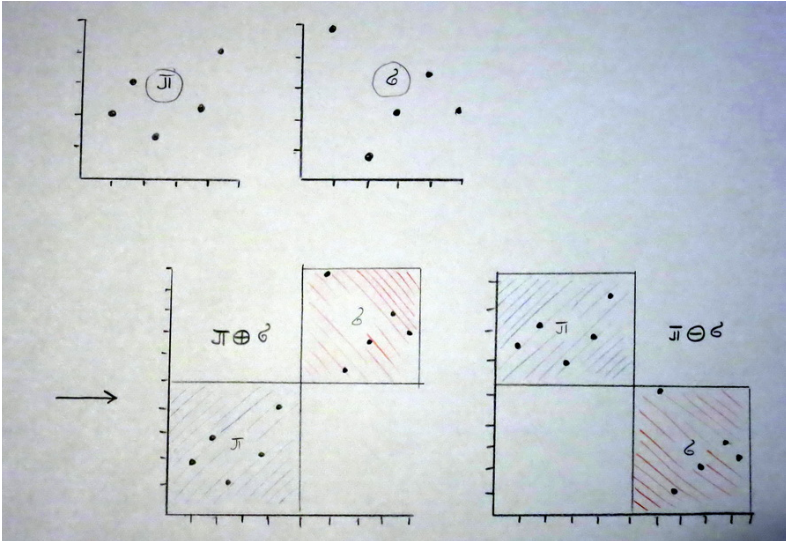
$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{for } 1 \leq i \leq p, \\ \sigma(i-p)+p & \text{for } p+1 \leq i \leq p+s, \end{cases}$$

$$(\pi \ominus \sigma)(i) = \begin{cases} \pi(i)+s & \text{for } 1 \leq i \leq p, \\ \sigma(i-p) & \text{for } p+1 \leq i \leq p+s. \end{cases}$$

This is particularly convenient to explain on the plots.

Easily generalises to infinite permutations.

$\oplus$ and $\ominus$



Countable ultrahomogeneous permutations

Theorem (Cameron, 2002)

The countable ultrahomogeneous permutations are precisely the following:

1. *the trivial permutation on a singleton set;*
2. $\mathbb{Q}^+ = (\mathbb{Q}, \leq, \leq)$, *where* $\leq$ *is the usual order of the rationals;*
3. $\mathbb{Q}^- = (\mathbb{Q}, \leq, \geq)$*;*
4. $\cdots \ominus \mathbb{Q}^+ \ominus \mathbb{Q}^+ \ominus \mathbb{Q}^+ \ominus \mathbb{Q}^+ \ominus \dots$*;*
5. $\cdots \oplus \mathbb{Q}^- \oplus \mathbb{Q}^- \oplus \mathbb{Q}^- \oplus \mathbb{Q}^- \oplus \dots$*;*
6. *the random permutation* Π = *the Fraïsssé limit of all finite permutations.*

A model for Π: an everywhere dense and independent subset of $\mathbb{Q} \times \mathbb{Q}$.

Changing the view

For the task of characterising HH permutations, yet another approach is needed. Let $(A, \leq_1, \leq_2)$ be a permutation.

Consider now two posets on A: the agreement poset

$$\sqsubseteq_1 = \leq_1 \cap \leq_2$$

and the disagreement (inversion) poset

$$\sqsubseteq_2 = \leq_1 \cap \geq_2 .$$

Now we have $\sqsubseteq_1 \cup \sqsubseteq_2 = \leq_1$ and $\sqsubseteq_1 \cap \sqsubseteq_2 = \Delta_A$. So, in fact, we have a colouring of the non-loop edges of the graph of $(A, \leq_1)$ into two colours: blue and red, such that each coloured component induces a poset on A.

Changing the view

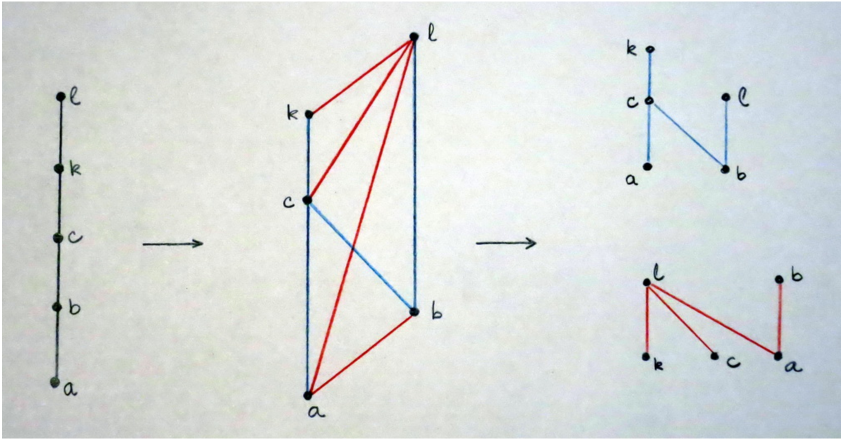
Let us now call a permutation a structure of the form $(A, \leq, \sqsubseteq_1, \sqsubseteq_2)$, where

- $\leq$ is a linear order of A, and
- $(\sqsubseteq_1, \sqsubseteq_2)$ is a partition of $\leq$ into two partial orders on A, in the sense that $\sqsubseteq_1 \cup \sqsubseteq_2 = \leq$ and $\sqsubseteq_1 \cap \sqsubseteq_2 = \Delta_A$ (so all loops are violet).

Changing the view

Example

Permutation *black* (of the set $\{a, b, c, k, l\}$)



Changing the view

We have a categorical equivalence between two ways to represent a permutation as a structure. In particular, the following holds.

Lemma

A permutation $\pi = (A, \leq_1, \leq_2)$ is (homomorphism-)homogeneous if and only if it adjoined 'permutation' $\mathcal{P}_\pi = (A, \leq_1, \sqsubseteq_1, \sqsubseteq_2)$ is (homomorphism-)homogeneous.

The result

Let ι_k denote the identical permutation on $[1, k]$, and let $\overline{\pi}$ be the dual permutation of π, obtained by reversing the second order.

Theorem (ID & É. Jungábel)

Let π be a permutation of $[1, n]$. Then π is HH if and only if either $\pi = \overline{\iota_{r_1}} \oplus \cdots \oplus \overline{\iota_{r_m}}$, or $\pi = \iota_{r_1} \ominus \cdots \ominus \iota_{r_m}$, where the sequence $(r_1, \ldots, r_m)$ satisfies one of the following conditions:

(i) $m = n$ *and* $r_1 = \cdots = r_n = 1$;

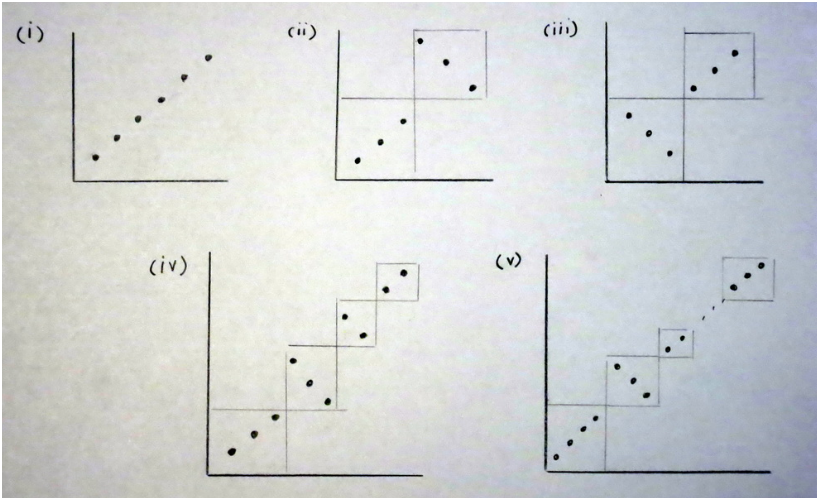
(ii) $m \geq 2$, $r_1 = \cdots = r_{m-1} = 1$ *and* $r_m > 1$;

(iii) $m \geq 2$, $r_1 > 1$ *and* $r_2 = \cdots = r_m = 1$;

(iv) $m \geq 4$ *and there exists an index* j *such that* $2 \leq j \leq m - 2$, $r_j, r_{j+1} > 1$, $r_1 = \cdots = r_{j-1} = 1$ *and* $r_{j+2} = \cdots = r_m = 1$;

(v) $m \geq 3$, $r_1 = r_m = 1$, *and for any pair of indices* j, k *such that* $1 < j < k < m$ *and* $r_j, r_k > 1$ *there exists an index* q *such that* $j < q < k$ *and* $r_q = 1$.

The result



The key step

For a permutation $\mathcal{P}_\pi = (A, \leq, \sqsubseteq_1, \sqsubseteq_2)$ let $\mathsf{B}_\pi = (A, \sqsubseteq_1)$ be the 'blue poset' (agreement), while $\mathsf{R}_\pi = (A, \sqsubseteq_2)$ is the 'red poset' (inversion).

Proposition

If $\mathcal{P}_\pi$ is a HH permutation (of arbitrary cardinality!), then both B_π and R_π are HH posets.

Theorem (Mašulović, 2007)

A partially ordered set $(A, \preceq)$ is HH if and only if one of the following condition holds:

(1) *each connected component of $(A, \preceq)$ is a chain;*

(2) *$(A, \preceq)$ is a tree;*

(3) *$(A, \preceq)$ is a dual tree;*

(4) *$(A, \preceq)$ splits into a tree and a dual tree;*

(5) *$(A, \preceq)$ is locally bounded and $\mathcal{X}_5$-dense (A finite $\Rightarrow$ lattice).*

The key step (continued)

Corollary

If $\mathcal{P}_\pi = (A, \leq, \sqsubseteq_1, \sqsubseteq_2)$ is a finite HH permutation and $|A| > 1$, then at least one of the posets B_π and R_π are disconnected and thus a free sum of at least two chains.

Therefore, by duality of blue and red, w.l.o.g. we may assume that B_π is a free sum of chains.

Hence,

$$\pi = \iota_{r_1} \ominus \cdots \ominus \iota_{r_m}$$

for some positive integers $(r_1, \ldots, r_m)$ such that $r_1 + \cdots + r_m = n$; these are the lengths of maximal blue chains.

The cases

Case 1: R_π is a free sum of chains $\Longrightarrow$ (i)

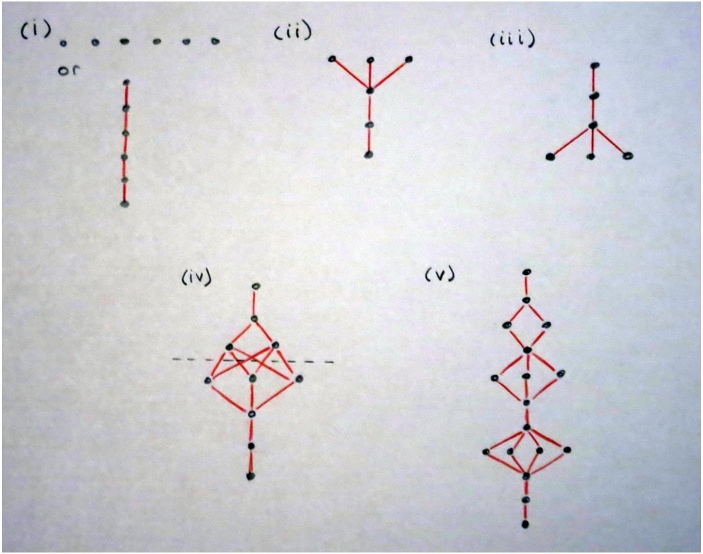
Case 2: R_π is a tree $\Longrightarrow$ (ii)

Case 3: R_π is a dual tree $\Longrightarrow$ (iii)

Case 4: R_π splits into a tree and a dual tree $\Longrightarrow$ (iv) or (v)

Case 5: R_π is a lattice $\Longrightarrow$ (v)

The cases



The converse...

...consists in verifying that each permutation of the type (i)–(v) is indeed HH.

This is quite a technical proof (which, however, has its hidden beauties) involving combinatorics of finite posets and partial order-preserving transformations.

The most complicated case is (v) – its proof exceeds in length the other four combined.

For details, see

I. Dolinka, É. Jungábel, Finite homomorphism-homogeneous permutations via edge colourings of chains, *Electronic Journal of Combinatorics* **19(4)** (2012), #P17, 15 pp.

Problems

Open Problem

Describe countably infinite homomorphism-homogeneous permutations.

Open Problem

Describe the finite homomorphism-homogeneous structures with n independent linear orders, $n \geq 3$.

On certain nights, when it is clear to Blue that Black will not be going anywhere, he slips out to a bar not far away for a beer or two, enjoying the conversations he sometimes has with the bartender, whose name is Red, and who bears an uncanny resemblance to Green, the bartender from the Gray Case so long ago.

Paul Auster: Ghosts (The New York Trilogy)

THANK YOU!

Questions and comments to:

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http://people.dmi.uns.ac.rs/∼dockie