Representing groups by endomorphisms of the random graph

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This talk is dedicated to...

...my very first encounter with alcohol – and beer in particular – almost exactly 30 years ago (on the evening of 30 April 1986, to be exact) in a certain pub/brewery in Prague.
Representation is an important issue
SO, WHO IS THIS RANDOM GRAPH CHARACTER?

I HATE IT ALREADY
Let $\mathcal{A}$ be a (countable) first order structure. $\mathcal{A}$ is said to be \textbf{(ultra)homogeneous} if any isomorphism

$$\iota : \mathcal{B} \rightarrow \mathcal{B}'$$

between its finitely generated substructures is a restriction of an automorphism $\alpha$ of $\mathcal{A}$: $\iota = \alpha|_{\mathcal{B}}$.

**Remark**
If we restrict to relational structures, ‘finitely generated’ becomes simply ‘finite’.
Classification programme for countable ultrahomogeneous structures

- finite graphs (Gardiner, 1976)
- posets (Schmerl, 1979)
- undirected graphs (Lachlan & Woodrow, 1980)
- tournaments (Lachlan, 1984)
- directed graphs (Cherlin, 1998 – Memoirs of AMS, 160+ pp.)
- semilattices (Droste, Kuske, Truss, 1999)
- finite groups (Cherlin & Felgner, 2000)
- permutations (Cameron, 2002)
- multipartite graphs (Jenkinson, Truss, Seidel, 2012)
- coloured multipartite graphs (Lockett, Truss, 2014)
- lattices – ‘unclassifiable’ (Abogatma, Truss, 2015)
- …
Fraïssé theory

**Fact**
For any countably infinite ultrahomogeneous structure $\mathcal{A}$, its age $\text{Age}(\mathcal{A})$ (the class of its finitely generated substructures) has the following properties:

- it has countably many isomorphism types;
- it is closed for taking (copies of) substructures;
- it has the joint embedding property (JEP);
- it has the amalgamation property (AP).

A class of finite(ly generated) structures with such properties is called a **Fraïssé class**.

**Theorem (Fraïssé)**

*Let $\mathcal{C}$ be a Fraïssé class. Then there exists a unique countably infinite ultrahomogeneous structure $\mathcal{F}$ such that $\text{Age}(\mathcal{F}) = \mathcal{C}$.*
Fraïssé theory (continued)

The structure $\mathcal{F}$ from the previous theorem is called the Fraïssé limit of $\mathbf{C}$.

Classical examples:

- finite chains $\rightarrow (\mathbb{Q}, <)$
- finite undirected graphs $\rightarrow$ the random graph $R$
- finite posets $\rightarrow$ the random poset
- finite tournaments $\rightarrow$ the random tournament
- finite metric spaces with rational distances $\rightarrow$ the rational Urysohn space $\mathbb{U}_Q$
- finite permutations $\rightarrow$ the random permutation

Fraïssé limits over finite relational languages are $\omega$-categorical, have quantifier elimination, oligomorphic automorphism groups, . . .
The random graph $R$

$R = \text{the unique countable existentially closed graph}$

\text{You can get anything you want at Alice’s Restaurant}

\text{Arlo Guthrie: Alice’s Restaurant Massacre (1967)}

$= \text{for any disjoint finite sets of vertices } A \text{ and } B \text{ there is a vertex } v \notin A \cup B \text{ adjacent to all vertices from } A \text{ and to none of } B$
Endomorphism of a first-order structure $\mathcal{A} = (A, R^A, F^A, C^A)$ is a transformation $f : A \rightarrow A$ preserving all relations from $R^A$, all operations from $F^A$, and all constants from $C^A$.

All endomorphisms of $\mathcal{A}$ for a monoid (semigroup with 1) under composition of functions: $\text{End}(\mathcal{A})$.

There are two ways in which groups can appear within a semigroup $S$:

- **overt**: as maximal subgroups of $S$;
- **covert**: as Schützenberger groups (of $\mathcal{D}$-classes of $S$)
Sometimes, there is a very fine line between overt and covert... 😊
IGOR!

Y U NO SEMIGROUP THEORY?
Green’s relations

The most fundamental tool in studying the structure of semigroups. (Named after J. Alexander “Sandy” Green (1926–2014).)

\[ a \mathbin{R} b \iff aS^1 = bS^1 \iff (\exists x, y \in S^1) ax = b & by = a \]

\[ a \mathbin{L} b \iff S^1a = S^1b \iff (\exists u, v \in S^1) ua = b & vb = a \]

\[ \mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} \]

\[ \mathcal{H} = \mathcal{R} \cap \mathcal{L} \]

\[ a \mathbin{J} b \iff S^1aS^1 = S^1bS^1 \iff (\exists x, y, u, v \in S^1) uax = b & vby = a \]
The eggbox picture of a $\mathcal{D}$-class

Groups (overt): $\mathcal{H}$-classes shaded red (these are all isomorphic)

maximal subgroups of a semigroup = $\mathcal{H}$-classes containing idempotents
Regularity

\( a \in S \) is regular if

\[ a = axa \]

for some \( x \in S \).

**Fact**

For any \( D \)-class \( D \), either all elements of \( D \) are regular or none of them.

Hence, \( a \) is regular \( \iff \ a \preceq e \) for and idempotent \( e \).
A regular $\mathcal{D}$-class
A regular eggbox
A non-regular $\mathcal{D}$-class
A non-regular eggbox
Schützenberger groups – groups the never were

There is a ‘hidden’ / covert group capturing the structure of a (non-regular) $D$-class $D$, called the Schützenberger group of $D$.

Namely, let $H$ be an $H$-class within a $D$-class $D$, and consider $T_H = \{ t \in S^1 : Ht \subseteq H \}$.

Basic results of semigroup theory (Green’s Lemma) show that each $\rho_t : H \rightarrow H \ (t \in T_H)$ defined by

$$h \rho_t = ht$$

is a permutation of $H$.

Hence, $S_H = \{ \rho_t : t \in T_H \}$ is a permutation group on $H$. This is the (right) Schützenberger group of $H$. 
Schützenberger groups – groups the never were

Fact
If both $H_1, H_2$ belong to $D$, then $S_{H_1} \cong S_{H_2}$. Hence the Schützenberger group is really an invariant of a $\mathcal{D}$-class of a semigroup.

Fact
If $H$ is a group (so that $D$ is regular), then $S_H \cong H$. 
A classical example: $\mathcal{T}_X$

Fact

In $\mathcal{T}_X$ we have:

1. $f \mathcal{R} g \iff \ker(f) = \ker(g)$;
2. $f \mathcal{L} g \iff \im(f) = \im(g)$;
3. $f \mathcal{D} g \iff \rank(f) = |\im(f)| = |\im(g)| = \rank(g)$;
4. $\mathcal{I} = \mathcal{D}$;
5. if $e = e^2$ and $\rank(e) = k$, then $H_e \cong S_k$;
6. $\mathcal{T}_X$ is regular.
COULD WE GO
BACK TO ENDOMORPHISM MONOIDS, PLEASE?
Let $A$ be a first-order structure. Since $\text{End}(A) \leq \mathcal{T}_A$, if $f, g \in \text{End}(A)$ are $\mathcal{R}$-/\$\mathcal{L}$-related in $\text{End}(A)$ they are certainly $\mathcal{R}$-/\$\mathcal{L}$-related in $\mathcal{T}_A$. Hence,

\begin{enumerate}[(i)]  
  \item $f \mathcal{R} g \implies \ker(f) = \ker(g)$;
  \item $f \mathcal{L} g \implies \text{im}(f) = \text{im}(g)$.
\end{enumerate}

\textbf{Remark}

We must be careful with the notion of an ‘image’ of an endomorphism if our language contains relational symbols, because besides $\text{im}(f)$ we also have $\langle Af \rangle$, the induced substructure of $A$ on $Af$.

\textbf{Lemma}

$f \mathcal{D} g \implies \langle Af \rangle \cong \langle Ag \rangle$. 
Regular elements in $\text{End}(\mathcal{A})$

**Proposition (Magill, Subbiah, 1974)**

If $f \in \text{End}(\mathcal{A})$ is regular, then $\text{im}(f) = \langle Af \rangle$.

**Lemma (Magill, Subbiah, 1974)**

Let $f, g \in \text{End}(\mathcal{A})$ be regular. Then:

(i) $f \mathcal{R} g \iff \ker(f) = \ker(g)$;
(ii) $f \mathcal{L} g \iff \text{im}(f) = \text{im}(g)$;
(iii) $f \mathcal{D} g \iff \text{im}(f) \cong \text{im}(g)$;
(iv) if $e$ is idempotent, then $H_e \cong \text{Aut}(\text{im}(e)) \cong \text{Aut}(\text{im}(f))$ for any $f \in D_e$. 
Proposition

Let $f \in \text{End}(\mathcal{A})$ and $H = H_f$.

(i) If $t \in T_H$, then $t|_{Af}$ is an automorphism of both $\langle Af \rangle$ and $\text{im}(f)$;

(ii) the mapping $\phi : \rho_t \mapsto t|_{Af}$ is an embedding of $S_H$ into $\text{Aut}(\langle Af \rangle) \cap \text{Aut(\text{im}(f))}$. 

Schützenberger groups in $\text{End}(\mathcal{A})$
So, what the heck are the images of (idempotent) endomorphisms of Fraïssé limits?

Call a Fraïssé class $\mathcal{C}$ neat if it consists of finite structures, and for each $n \geq 1$ the number of isomorphism types of $n$-generated structures in $\mathcal{C}$ is finite.

Examples:
- relational structures
- Fraïssé classes of algebras contained in locally finite varieties

**Theorem (ID, 2012)**

Let $\mathcal{C}$ be a neat Fraïssé class enjoying the strict AP and the 1PHEP. Then there exists and (idempotent) endomorphism $f$ of $F$, the Fraïssé limit of $\mathcal{C}$, such that $A \cong \text{im}(f)$ if and only if $A$ is algebraically closed in $\overline{\mathcal{C}}$. 
ALGEBRAICALLY CLO...

WAIT, WHAT?
An $L$-formula $\Phi(x)$ is **primitive** if it is of the form

$$(\exists y) \bigwedge_{i<k} \Psi_i(x, y)$$

where each $\Psi_i$ is a **literal**: an atomic formula or its negation. No negation $\rightarrow$ primitive positive formula.

Let $K$ be a class of $L$-structures. An $L$-structure $\mathcal{A}$ is **existentially (algebraically) closed** (in $K$) if for any primitive (positive) formula $\Phi(x)$ and any tuple $a$ from $\mathcal{A}$ we have already $\mathcal{A} \models \Phi(a)$ whenever there is an extension $\mathcal{A}' \in K$ of $\mathcal{A}$ such that $\mathcal{A}' \models \Phi(a)$. 
Graphs

Countable e.c. graphs: $R$ (Alice’s Restaurant property)
Countable a.c. graphs: any finite set of vertices has a common neighbour ($\Rightarrow$ infinitely many of them)

In the rest of this talk we will be concerned with simple graphs and $\text{End}(R)$. However, all these results can be adapted for:

▶ the random digraph,
▶ the random bipartite graph,
▶ the random (non-strict) poset,
▶ ...

Proposition

A countable graph $(V, E)$ is a.c. if and only if there exists $E' \subseteq E$ such that $(V, E') \cong R$ (that is, it is e.c.). Consequently, for any a.c. graph $\Gamma$ there is a bijective homomorphism $R \rightarrow \Gamma$. 
WOW, COOL!
Frucht’s Theorem (1939)

Any finite group is $\cong \text{Aut}(\Gamma)$ for a finite graph $\Gamma$.

de Groot / Sabidussi (1959/60) $\Rightarrow$ automorphism groups of countable graphs include all countable groups.

**Name of the game:** Strengthen this for countable a.c. graphs.
The team

Point Guard: Martyn Quick

Forward: “Baby” James Mitchell

Center: Jillian “Jay” McPhee

Shooting Guard: Robert “Bob” Gray

Power Forward: Dr. D
Automorphism groups of countable a.c. graphs

Theorem
Let $\Gamma$ be a countable graph. Then there exist $2^{\aleph_0}$ pairwise non-isomorphic countable a.c. graphs whose automorphism group is $\cong \text{Aut}(\Gamma)$.

Proof. For a (simple) graph $\Delta$, let $\Delta^\dagger$ denote its complement.

- $\text{Aut}(\Delta^\dagger) = \text{Aut}(\Delta)$.
- $\Delta$ any graph, $\Lambda$ infinite locally finite graph $\Rightarrow (\Delta \cup \Lambda)^\dagger$ is a.c.
- The central idea – consider l.f. graphs $L_S$ for $S \subseteq \mathbb{N} \setminus \{0, 1\}$:
Automorphism groups of countable a.c. graphs

Proof (cont’d).

- Properties of $L_S$ ($S, T \subseteq \mathbb{N} \setminus \{0, 1\}$):
  - Each $L_S$ is rigid ($\text{Aut}(L_S) = 1$).
  - $L_S \cong L_T \iff S = T$.

- If $L_S$ is isomorphic to no connected component of $\Gamma$ (and this excludes only countably many choices of $S$), then
  \[ \text{Aut}(\Gamma \uplus L_S)^\dagger = \text{Aut}(\Gamma \uplus L_S) \cong \text{Aut}(\Gamma) \times \text{Aut}(L_S) \cong \text{Aut}(\Gamma). \]

- $S_1 \neq S_2$ yield non-isomorphic a.c. graphs.
Images of idempotent endomorphisms

**Theorem (Bonato, Delić, 2000; ID, 2012)**

Let $\Gamma$ be a countable graph. There exists an idempotent $f \in \text{End}(R)$ such that $\text{im}(f) \cong \Gamma$ if and only if $\Gamma$ is a.c.

**Theorem**

If $\Gamma$ is a countable a.c. graph, then there exists an (induced) subgraph $\Gamma' \cong \Gamma$ of $R$ such that there are $2^{\aleph_0}$ idempotent endomorphisms $f$ of $R$ such that $\text{im}(f) = \Gamma'$. 
The number of regular $\mathcal{D}$-classes with a given group $\mathcal{H}$-class

Theorem

(i) Let $\Gamma$ be a countable graph. Then there exist $2^\aleph_0$ distinct regular $\mathcal{D}$-classes of $\text{End}(R)$ whose group $\mathcal{H}$-classes are $\cong \text{Aut}(\Gamma)$.

(ii) Every regular $\mathcal{D}$-class contains $2^\aleph_0$ distinct group $\mathcal{H}$-classes.

Corollary

$\text{End}(R)$ has $2^\aleph_0$ regular $\mathcal{D}$-classes. (You know, the ones with eggs...)
The size of a regular eggbox

Theorem
Every regular $\mathcal{D}$-class of $\text{End}(R)$ contains $2^{\aleph_0}$ many $\mathcal{R}$- and $\mathcal{L}$-classes.

Proof. Let $e$ be an idempotent endomorphism of $R$, and let $\Gamma = \text{im}(e)$ (a.c.).

$\mathcal{R}$-classes: Assume $R$ is constructed as $R_\Gamma$. (Start with $\Gamma$; at each stage, for any finite subset $S$ of vertices of the existing graph, add a new vertex adjacent to $S$ and nothing else; iterate this $\aleph_0$ times.)

We already know that the identity mapping on $\Gamma$ can be extended to $f \in \text{End}(R)$ in $2^{\aleph_0}$ ways such that $\text{im}(f) = \text{im}(e)$.

All such $f$ are idempotents, and $f \mathcal{D} e$, moreover, $f \mathcal{L} e$.

However, all these idempotents are not $\mathcal{R}$-related.
The size of a regular eggbox

**L-classes:** Key idea – construct the graph $\Gamma^\#$ from $\Gamma$ by replacing each edge by the following gadget:

Construct $R$ around $\Gamma^\#$, so that $R = R_{\Gamma^\#}$.

$\Gamma$ a.c. $\longrightarrow$ $\Gamma^\#$ a.c. Hence, the identity map on $\Gamma^\#$ can be extended to an endomorphism $g : R \rightarrow \Gamma^\#$. 
The size of a regular eggbox

For each binary sequence $b = (b_i)_{i \in \mathbb{N}}$ define a map $\psi_b$ on $\Gamma^\#$ by

$$v_{i,r} \psi_b = v_{i,b_i}$$

for all $i \in \mathbb{N}$ and $r \in \{0, 1\}$. Easy: $\psi_b \in \text{End}(\Gamma^\#)$ and $\text{im}(\psi_b) \cong \Gamma$ is induced by $\{v_{i,b_i} : i \in \mathbb{N}\}$.

$g \psi_b \in \text{End}(R)$ are idempotents, $\text{im}(g \psi_b) \cong \Gamma \Rightarrow$ all these idempotents are $\mathcal{D}$-related to $e$.

Different images $\Rightarrow$ they are not $\mathcal{L}$-related.
Non-regular eggboxes

Theorem
Let $\Gamma \not\sim R$ be a countable a.c. graph. Then there exists a non-regular endomorphism of $R$ such that $\text{im}(f) \cong \Gamma$ and $D_f$ contains $2^{\aleph_0}$ many $R$- and $L$-classes.

The proof is a variation of the idea of $\Gamma^\#$ and binary sequences.

Theorem
There are $2^{\aleph_0}$ non-regular $D$-classes in $\text{End}(R)$.

Open Problem
Are there any non-regular eggboxes of some other size?
Let $\Gamma = (V_0, E_0)$ be a countable a.c. graph. Then, as we already know, there is a subset $F \subseteq E_0$ such that $(V_0, F) \cong R$. Now build $R_\Gamma \cong R$ around $\Gamma$, and let $f : R_\Gamma \rightarrow (V_0, F)$ be an isomorphism. Then $f$ is an injective endomorphism of $R$; if $F \neq E_0$ then $f$ is non-regular.

**Proposition**

Let $f$ be an injective endomorphism of $R = (V, E)$ as described above, with $Vf = V_0$. Then

$$S_{H_f} \cong \text{Aut}(\langle V_0 \rangle) \cap \text{Aut}(\text{im}(f))$$
Schützenberger groups in \( \text{End}(R) \)

So, to show a universality result for Schützenberger groups in \( \text{End}(R) \), one needs to extend the Frucht-de Groot-Sabidussi Theorem to countable a.c. graphs with 2-coloured edges (blue and red, say) where the ‘red graph’ is \( \cong R \).

This is what we did via an involved construction that again uses the rigid graphs \( L_S \) (for a particular countable family of sets \( S \)).

**Theorem**

Let \( \Gamma \) be any countable graph. There are \( 2^\aleph_0 \) non-regular \( \mathcal{D} \)-classes of \( \text{End}(R) \) such that the Schützenberger groups of the \( \mathcal{H} \)-classes within them are \( \cong \text{Aut}(\Gamma) \).

THANK YOU!

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Further information may be found at:

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