Facets of the Finite Basis Problem for Finite Involution Semigroups

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Glossary of terms

The equational theory $Eq(A)$ of an algebra $A$

$=$ the set of all identities (over some fixed countably infinite set $X$ of variables, or letters) satisfied by $A$. 
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$\Sigma$, written $\Sigma \models p \approx q$,

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If $\Sigma \subseteq Eq(A)$ is such that every identity from $Eq(A)$ is a consequence of $\Sigma$, then $\Sigma$ is called an (equational) basis of $A$.

A fundamental property that an algebra $A$ may or may not have is that of having a finite basis.
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A fundamental property that an algebra $A$ may or may not have is that of having a finite basis. If there is a finite basis for identities of $A$, then $A$ is said to be finitely based (FB).
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Some classical positive results

Each of the following algebras is FB:

- finite groups (Oates & Powell, 1964)
- commutative semigroups (Perkins, 1968)
- finite lattices and lattice-based algebras (McKenzie, 1970)
- finite (associative) rings (L'vov; Kruse, 1973)
- algebras generating congruence distributive varieties with a finite residual bound (Baker, 1977)
- algebras generating congruence modular varieties with a finite residual bound (McKenzie, 1987)
- algebras generating congruence ∧-semidistributive varieties (Willard, 2000)
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Negative results

Examples of finite NFB algebras:

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\begin{array}{ccc}
0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
2 & 0 & 2 & 2 \\
\end{array}
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(Murskiĭ, 1965)
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▶ a certain 6-element semigroup of matrices (Perkins, 1968)
Negative results

Examples of finite NFB algebras:

- a certain finite pointed group (Bryant, 1982)
- the full transformation semigroup $T_n$ for $n \geq 3$ and the full semigroup of binary relations $R_n$ for $n \geq 2$
- a certain 7-element semiring of binary relations (ID, 2007)
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\textbf{Tarski's Finite Basis Problem: } Is there any algorithmic way to distinguish between finite FB and NFB algebras?
McKenzie’s solution of the Tarski problem

No!

Theorem (McKenzie, 1996)
There is no algorithm to decide whether a finite algebra is FB.
This is exactly why it is so interesting to study the (N)FB property, especially for finite algebras.

The Tarski-Sapir problem: Is there an algorithm to decide whether a finite semigroup is FB?
This problem is still open.


AAA80, Będlewo, June 2010
Igor Dolinka: FBP for Finite Involution Semigroups
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Volkov’s NFB criterion (1989)

Let $A_2$ be the 5-element semigroup given by the presentation

$$\langle a, b : a^2 = a = aba, \ b^2 = 0, \ bab = b \rangle.$$
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**Fact**

$A_2$ is representable by matrices (over any field).
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Theorem (M. V. Volkov, 1989)

Let $S$ be a semigroup and $T$ a subsemigroup of $S$. Assume that there exist a positive integer $d$ and a group $G$ satisfying $x^d \approx e$ such that

If $A^2 \in \text{var } S$, then $S$ is NFB.

Corollary

The following semigroups are NFB:

- the full transformation semigroup $T_n$ ($n \geq 3$)
- the full semigroup of binary relations $B_n$ ($n \geq 2$)
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Unary semigroups

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$= \text{a structure } (S, \cdot, *) \text{ such that } (S, \cdot) \text{ is a semigroup and } * \text{ is a unary operation on } S$
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Involution semigroup

= a unary semigroup satisfying $(xy)^* \approx y^*x^*$ and $(x^*)^* \approx x$
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Examples

- groups
- inverse semigroups
- regular \(^\ast\)-semigroups \((xx^* x \approx x)\)
- matrix semigroups with transposition \(M_n(F) = (M_n(F), \cdot, ^T)\)
‘Unary version’ of Volkov’s Theorem

For a unary semigroup $S$, let $H(S)$ denote the **Hermitian subsemigroup** of $S$, generated by $aa^*$ for all $a \in S$. 
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For a variety $\mathbf{V}$ of unary semigroups, let $H(\mathbf{V})$ be the subvariety of $\mathbf{V}$ generated by all $H(S)$, $S \in \mathbf{V}$. 
For a unary semigroup $S$, let $H(S)$ denote the Hermitian subsemigroup of $S$, generated by $aa^*$ for all $a \in S$.

For a variety $V$ of unary semigroups, let $H(V)$ be the subvariety of $V$ generated by all $H(S)$, $S \in V$.

Furthermore, let $K_3$ be the 10-element unary Rees matrix semigroup over a trivial group $E = \{e\}$ with the sandwich matrix

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while $(i, e, j)^* = (j, e, i)$ and $0^* = 0$. 
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**Fact**

$K_3$ generates the variety of all strict combinatorial regular $^*$-semigroups (studied by K. Auinger in 1992).
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Theorem (K. Auinger, M. V. Volkov, cca. 1991/92)

Let $S$ be a unary semigroup such that $\mathcal{V} = \text{var } S$ contains $K_3$. If there exist a group $G$ which belongs to $\mathcal{V}$ but not to $H(\mathcal{V})$, then $S$ is NFB.
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Corollary

The following unary semigroups are NFB:

- the full involution semigroup of binary relations $R \lor n$ ($n \geq 2$), endowed with relational converse
- matrix semigroups with transposition $M_n(F)$, where $F$ is a finite field, $|F| \geq 3$
- matrix semigroups $(M_2(F), \cdot, \dagger)$, where $F$ is either a finite field such that $|F| \equiv 3 \pmod{4}$, or a subfield of $\mathbb{C}$ closed under complex conjugation, and $\dagger$ is the unary operation of taking the Moore-Penrose inverse.
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*Exactly which of the involution semigroups $M_n(F)$ are NFB, $n \geq 2$, $F$ is a finite field?*
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Also, the following open problem was both intriguing and inviting.

**Problem**
*Do finite INFB involution semigroups exist at all?*
An algebra $A$ is inherently nonfinitely based (INFB) if:

- $A$ generates a locally finite variety, and
- any locally finite variety $\mathbf{V}$ containing $A$ is NFB.

Said otherwise, for any finite set of identities $\Sigma$ satisfied by $A$, the variety defined by $\Sigma$ is not locally finite. Therefore, problems concerning INFB algebras are in fact Burnside-type problems.

INFB algebras are a powerful tool for proving the NFB property; namely, the INFB property is “contagious”: if $\text{var} A$ is locally finite and contains an INFB algebra $B$, then $A$ is NFB. In particular, $B$ is NFB.
INFB...(?)

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Said otherwise, for any finite set of identities $\Sigma$ satisfied by $A$, the variety defined by $\Sigma$ is not locally finite.
An algebra $A$ is inherently nonfinitely based (INFB) if:

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INFB algebras are a powerful tool for proving the NFB property; namely, the INFB property is “contagious”:

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In particular, $B$ is NFB.
Finite INFB semigroups: a success story

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Zimin words: \( Z_1 = x_1 \) and \( Z_{n+1} = Z_n x_{n+1} Z_n \) for \( n \geq 1 \).
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Theorem (Sapir, 1987)

Let $S$ be a finite semigroup. Then

$$S \text{ is INFB } \iff S \not\cong Z_n \approx W$$

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Sapir also found an effective structural description of finite INFB semigroups, thus proving

Theorem (Sapir, 1987)

It is decidable whether a finite semigroup is INFB or not.
Examples of finite INFB semigroups

The example: the 6-element Brandt inverse monoid

\[ B^1_2 = \langle a, b : a^2 = b^2 = 0, \ aba = a, \ bab = b \rangle \cup \{1\}. \]
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\[ B_2^1 = \langle a, b : a^2 = b^2 = 0, aba = a, bab = b \rangle \cup \{1\}. \]

\[ B_2^1 \] is representable by matrices (over any field):

\[
\begin{pmatrix}
0 & 0 \\
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\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
0 & 0
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\( B_2^1 \) is obtained by adjoining an identity element to the Rees matrix semigroup over the trivial group \( E = \{e\} \) with the sandwich matrix

\[
\begin{pmatrix}
e & 0 \\
0 & e
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Since $B_2^1 \in \text{var } A_2^1$, where $A_2$ is the 5-element semigroup from Volkov’s theorem, we have that $A_2^1$ is (I)NFB as well.
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The same argument applies to \( T_n \ (n \geq 3),\ R_n \ (n \geq 2),\ PT_n \ (n \geq 2)\,...\)
What a difference an involution makes? Well...

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For example, an involution $*$ can be defined on $B_2^1$ by $a^* = b$, $b^* = a$, the remaining 4 elements (which are idempotents: 0, 1, $ab$, $ba$) being fixed.
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For example, an involution $^*$ can be defined on $B_2^1$ by $a^* = b$, $b^* = a$, the remaining 4 elements (which are idempotents: 0, 1, $ab$, $ba$) being fixed. This turns $B_2^1$ into an inverse semigroup.

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For example, an involution $\ast$ can be defined on $B_2^1$ by $a^* = b$, $b^* = a$, the remaining 4 elements (which are idempotents: $0, 1, ab, ba$) being fixed. This turns $B_2^1$ into an inverse semigroup.

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$B_2^1$ is not INFB as an inverse semigroup.
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Still, the inverse semigroup $B_2^1$ is NFB (Kleiman, 1979).
What a difference an involution makes? Well...

How on Earth is the case of unary semigroups different?

For example, an involution $*$ can be defined on $B^1_2$ by $a^* = b$, $b^* = a$, the remaining 4 elements (which are idempotents: $0, 1, ab, ba$) being fixed. This turns $B^1_2$ into an inverse semigroup.

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Still, the inverse semigroup $B^1_2$ is NFB (Kleiman, 1979).

So, once again:

**Problem**

> Do finite INFB involution semigroups exist at all?
An INFB criterion for involution semigroups

Yes!
Yes!

**Theorem (ID, cca. 2007/08)**

Let $S$ be an involution semigroup such that $\text{var } S$ is locally finite. If $S$ fails to satisfy any nontrivial identity of the form

$$Z_n \approx W,$$

where $W$ is an involutorial word (a word over the ‘doubled’ alphabet $X \cup X^*$), then $S$ is INFB.
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How about a (finite) example?
‘C’mon baby, let’s do the twist…!’

**Rescue:** Luckily, $B_2^1$ admits one more involution aside from the inverse one: define the nilpotents $a, b$ (and, of course, 0, 1) to be fixed by $^*$, which results in $(ab)^* = ba$ and $(ba)^* = ab$. 
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In this way we obtain the **twisted Brandt monoid** $TB_2^1$. 

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$TB_2^1$ fails to satisfy a nontrivial identity of the form $Z_n \approx W$. Hence, it is INFB.
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$TB_2^1$ fails to satisfy a nontrivial identity of the form $Z_n \cong W$. Hence, it is INFB.

Similarly to $B_2^1$, this little guy is quite powerful.
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Proposition

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Remark

Analogously, one can also define $TA_2^1$, the “involutorial version” of $A_2^1$, which is also INFB.
Examples of finite INFB involution semigroups

- $R_n^\vee$, the involution semigroup of binary relations, is (I)NFB for all $n \geq 2$, 

Reason: $TB_{12}$ embeds into $R_{2}$.

Reason: This is precisely the case when $-1$ has a square root in $F$, which is sufficient and necessary for $TB_{12}$ to embed into $M_2(F)$.

Reason: $TB_{12}$ embeds into $M_n(F)$ as a consequence of the Chevalley-Warning theorem from algebraic number theory.

So, what about $M_2(F)$ if $|F| \equiv 3 \pmod{4}$? (We already know it is NFB.)
Examples of finite INFB involution semigroups

- $\mathcal{R}_n^\vee$, the involution semigroup of binary relations, is (I)NFB for all $n \geq 2$,
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- $\mathcal{M}_n(F)$ for all $n \geq 3$ and all finite fields $F$.
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- $R_n^\triangledown$, the involution semigroup of binary relations, is (I)NFB for all $n \geq 2$,
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So, what about $\mathcal{M}_2(\mathbb{F})$ if $|\mathbb{F}| \equiv 3$ (mod 4)?
(We already know it is NFB.)
Non-INFB results

Theorem (ID, 2010)

Let $S$ be a finite involution semigroup satisfying a nontrivial identity of the form $Z_n \approx W$ such that $B_2^1 \not\in \var S$. Then $S$ is not INFB.

Proof idea: Either $W$ is an ordinary semigroup word, or for any $*$-fixed idempotent $e$ of $S$, $\var eS$ consists of involution semilattices of Archimedean semigroups.
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Let $S$ be a finite semigroup satisfying an identity of the form $Z_n \approx Z_n W$. Then $S$ is not INFB.

Proof idea: Stretching the approach of Margolis & Sapir (1995) developed for finitely generated quasivarieties of semigroups to what seems to be the final limits of that method: certain semigroup quasiidentities can be “encoded” into unary semigroup identities.
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Corollary

No finite regular \( * \)-semigroup is INFB.
(Namely, \( x \approx x(x^*x) \) holds.)
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For any finite group \(G\), the involution semigroup of subsets \(\mathcal{P}_G^* = (\mathcal{P}(G), \cdot, *)\) is not INFB.
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Remark

The ordinary power semigroup \(\mathcal{P}_G = (\mathcal{P}(G), \cdot)\) is INFB if and only if \(G\) is not Dedekind.
Non-INFB results

Proposition (Crvenković, 1982)

If a finite involution semigroup $S$ admits a Moore-Penrose inverse $†$, then the inverse is term-definable in $S$. 

In particular, such a semigroup satisfies $x \approx x \cdot w(x, x^*) \cdot x$ for some $w = \Rightarrow$ it is not INFB.

Proposition

The involution semigroup of $2 \times 2$ matrices over a finite field $F$ with transposition admits a Moore-Penrose inverse if and only if $|F| \equiv 3 \pmod{4}$.

This completes our classification!
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Solution to the (I)NFB problem for matrix involution semigroups

Theorem (Auinger, ID, Volkov, 2008-10)

Let \( n \geq 2 \) and \( \mathbb{F} \) be a finite field. Then

1. \( \mathcal{M}_n(\mathbb{F}) \) is not finitely based;
2. \( \mathcal{M}_n(\mathbb{F}) \) is INFB if and only if either \( n \geq 3 \), or \( n = 2 \) and \( |\mathbb{F}| \not\equiv 3 \pmod{4} \).
Unfortunately, we have not yet accomplished a full classification of finite involution semigroups with respect to the INFB property.
The gap

Unfortunately, we have not yet accomplished a full classification of finite involution semigroups with respect to the INFB property. We don’t know what to do with finite involution semigroups (if they exist) such that:

(a) \( B_1^2 \in \text{var} \ S \),
(b) \( S \) satisfies a nontrivial identity of the form \( Z_n \approx W \),
(c) \( S \), however, fails to satisfy an identity of the form \( Z_n \approx Z_n W' \).

This “gap” does not occur for ordinary semigroups, as (b) renders (a) impossible. But this is no longer the case for involution semigroups!
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This “gap” does not occur for ordinary semigroups, as (b) renders (a) impossible. But this is no longer the case for involution semigroups!
The gap

Unfortunately, we have not yet accomplished a full classification of finite involution semigroups with respect to the INFB property. We don’t know what to do with finite involution semigroups (if they exist) such that:

(a) $B_2^1 \in \text{var } S$,
(b) $S$ satisfies a nontrivial identity of the form $Z_n \approx W$,
(c) $S$, however, fails to satisfy an identity of the form $Z_n \approx Z_n W'$.

This “gap” does not occur for ordinary semigroups, as (b) renders (a) impossible. But this is no longer the case for involution semigroups!

Test-Example

Is $xyxzxyx \approx xyxx^*xzxyx$ implying the non-INFB property?
THANK YOU!

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http://sites.dmi.rs/personal/dolinkai