APPROXIMATE GENERALIZED SOLUTIONS AND MEASURE VALUED SOLUTIONS TO CONSERVATION LAWS

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Abstract. We relate the concept of measure valued solutions to conservation laws, introduced by DiPerna, to the concept of generalized function solutions arising in a differential algebra containing the distributions and having the algebra of smooth functions as a subalgebra. As an example, following results of DiPerna and Majda on measure valued solutions, we construct generalized solutions to the Euler equations.

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1. Introduction

In this paper, we relate generalized function solutions and measure valued solutions to

$$\operatorname{div} f(u) = (\operatorname{div} f^1(u), \dots, \operatorname{div} f^m(u)) = 0, \quad u : \Omega \to \mathbb{R}^m, \tag{1}$$

where Ω is an open subset of $\mathbb{R}^n$, $u = (u^1, \dots, u^m) : \Omega \to \mathbb{R}^m$ and $f = (f^1, \dots, f^m)$, $f^k = (f_1^k, \dots, f_n^k) : \mathbb{R}^m \to \mathbb{R}^n$, $k = 1, \dots, m$, is assumed to be smooth and of at most polynomial growth, together with all derivatives, as well as the perturbed system

$$\operatorname{div} f(u) = \varepsilon L(u), \quad \varepsilon \in (0,1), \tag{2}$$

where $L = (L^1, \dots, L^m)$ is a linear partial differential operator with smooth coefficients.

The concept of measure valued solutions to (1) has been introduced by DiPerna [4], having the motivating background from the work of Tartar and Murat on compensated compactness, [14, 15]. Measure valued solutions have been designed to handle the convergence question arising with problem (2) and to capture concentration phenomena. One has the following basic properties [4, 9, 11]: (i) if $u \in L^p(\Omega)$, $1 \le p \le \infty$ is a weak solution to (1), then the family of Dirac measures $\delta_{u(x)}$, $x \in \Omega$, is a measure valued solution; (ii) if $\{u_\varepsilon\}_{\varepsilon \in (0,1)}$ is a bounded sequence of solutions to (2) in $L^p(\Omega)$, $1 \le p \le \infty$, and f satisfies appropriate growth conditions, then the associated Young measure is a measure valued solution to (1).

On the other hand, we work with the algebra of generalized functions $\mathcal{G}_s(\Omega; \mathbb{R}^m)$, which was introduced by the second author. The elements of this algebra are equivalence classes of nets of smooth functions on Ω. In this setting, the system (1) is understood as

$$\left(\sum_{j=1}^n \nabla f_j^k(u) \bullet \frac{\partial u}{\partial x_j} \right)_{k=1,\dots,m} = 0 \iff \left\{ \left(\sum_{j=1}^n \nabla f_j^k(u_\varepsilon) \bullet \frac{\partial u_\varepsilon}{\partial x_j} \right)_{k=1,\dots,m} \right\}_\varepsilon \in \mathcal{N}_s(\Omega; \mathbb{R}^m), \tag{3}$$

where $\mathcal{N}_s(\Omega;\mathbb{R}^m)$ is the set of negligible functions (see Section 2), and $\bullet$ stands for scalar product. We call a solution $u \in \mathcal{G}_s(\Omega;\mathbb{R}^m)$ to (3) a strong solution. However, it turns out that this strong formulation does not allow discontinuous solutions. For this reason, we shall use the concept of association (see Section 2 for the definition of $\approx$). An element u of $\mathcal{G}_s(\Omega;\mathbb{R}^m)$ is called an approximate generalized solution to (1) if

$$\sum_{j=1}^{n} \nabla f_j^k(u) \bullet \frac{\partial u}{\partial x_j} \approx 0 \quad \iff \quad \sum_{j=1}^{n} \int_\Omega \nabla f_j^k(u_\varepsilon) \bullet \frac{\partial u_\varepsilon}{\partial x_j} \phi^k(x)\, dx \to 0,\ \varepsilon \to 0, \tag{4}$$

holds for all $\phi = (\phi^1,\dots,\phi^m) \in \mathcal{D}(\Omega;\mathbb{R}^m)$ and all $k = 1,\dots,m$. For the concept of approximate and generalized solutions to (1) we refer to [1, 2, 7, 12].

In many cases, equation (2) has a classical solution u_ε for each $\varepsilon > 0$, and by means of some maximum principle, one can often prove that the family $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is a bounded subset of $L^p(\Omega;\mathbb{R}^m)$, $1 \le p \le \infty$. Those uniformly bounded families are in the background of both approximate and measure valued solutions. We will show in Lemma 1 how to construct a solution $u \in \mathcal{G}_s(\Omega;\mathbb{R}^m)$ to (4) from such a sequence. Theorems 2 and 4 are related to $\mathcal{Y}^\infty(\Omega;\mathbb{R}^m)$-measure valued solutions and approximate generalized solutions of bounded type to (1), while Theorems 6 and 7 are related to $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$-measure valued solutions and approximate generalized solutions of p-bounded type to (1).

As a special case we will consider the Euler equations and construct approximate generalized solutions to the Euler equations arising from a sequence of weak solutions with L^2-uniform bound. The Euler equations for an incompressible homogeneous fluid in n space dimensions are given by

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p\,, \tag{5}$$

where $t > 0,\ x \in \mathbb{R}^n,\ v = (v_1,\dots,v_n)^\top : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, is the fluid velocity, $\operatorname{div} v = 0,\ v(x;0) = v_0(x)$, and p is the scalar pressure. These equations are the limiting case for the Navier-Stokes equations, with Reynold's numbers $1/\varepsilon$,

$$\frac{\partial v^\varepsilon}{\partial t} + v^\varepsilon \cdot \nabla v^\varepsilon = -\nabla p^\varepsilon + \varepsilon \triangle v^\varepsilon, \quad \varepsilon > 0. \tag{6}$$

Recall, [6]: If v_0 is a smooth divergence-free velocity field in $L^2(\mathbb{R}^3)$ and v_ε are weak solutions of the Navier-Stokes equations (6) with initial data v_0, then a subsequence $\{v_{\varepsilon_k}\}_k$ has a limit that defines a measure valued solution to the Euler equations (5). Approximate generalized solutions to (5) arise from the same subsequence of weak solutions to the Navier-Stokes equations. That construction is given in Proposition 8.

2. Generalized functions as approximate solutions

2.1. **The algebra $\mathcal{G}_s(\Omega;\mathbb{R}^m)$.** We briefly recall the definition of $\mathcal{G}_s(\Omega;\mathbb{R}^m)$, where Ω is an open subset of $\mathbb{R}^n$. A net $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \in (\mathcal{C}^\infty(\Omega;\mathbb{R}^m))^{(0,1)}$ is called *moderate* if it has the property

$$(\forall K \subset\subset \Omega)\,(\forall \alpha \in \mathbb{N}_0^n)\,(\exists N > 0)\ \|\partial^\alpha u_\varepsilon\|_{L^\infty(K;\mathbb{R}^m)} = \mathcal{O}(\varepsilon^{-N}) \ \text{ as } \varepsilon \to 0.$$

The set of moderate nets is denoted by $\mathcal{E}_{M,s}(\Omega)$. A net $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is called *null* or *negligible* if

$$(\forall K \subset\subset \Omega)\,(\forall \alpha \in \mathbb{N}_0^n)\,(\forall M > 0)\ \|\partial^\alpha u_\varepsilon\|_{L^\infty(K;\mathbb{R}^m)} = \mathcal{O}(\varepsilon^{M})\ , \varepsilon \to 0.$$

This subset is denoted by $\mathcal{N}_s(\Omega;\mathbb{R}^n)$. It is clear that $\mathcal{E}_{M,s}(\Omega;\mathbb{R}^m)$ is an algebra with partial derivatives, where operations are defined componentwise, while $\mathcal{N}_s(\Omega;\mathbb{R}^n)$ is an ideal therein, closed under differentiation. The special algebra is defined as the factor algebra $\mathcal{G}_s(\Omega;\mathbb{R}^m) = \mathcal{E}_{M,s}(\Omega;\mathbb{R}^m)/\mathcal{N}_s(\Omega;\mathbb{R}^m)$. With the operations defined on representatives, we have that $\mathcal{G}_s(\Omega;\mathbb{R}^m)$ is an algebra with partial derivatives. All spaces of functions and distributions we work with (e.g. the space of distributions with compact support $\mathcal{E}'$, smooth functions with compact support C_0^∞, $\mathcal{D}'$, L^∞_{loc}, L^p and so on) are embedded in $\mathcal{G}_s(\Omega;\mathbb{R}^m)$. For those embeddings we refer to [7]. If f is smooth and of at most polynomial growth, together with all derivatives, then the composition $f(u)$ with $u \in \mathcal{G}_s(\Omega;\mathbb{R}^m)$ is well-defined by $f(u) = [\{f(u_\varepsilon)\}_{\varepsilon\in(0,1)}]$. Two elements $u, v \in \mathcal{G}_s(\Omega;\mathbb{R}^m)$ are called associated, $u \approx v$, if $u_\varepsilon - v_\varepsilon \to 0$ in $\mathcal{D}'(\Omega;\mathbb{R}^m)$, as $\varepsilon \to 0$, for some, and hence all, representatives $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$. We shall say that an element $u \in \mathcal{G}_s(\Omega;\mathbb{R}^m)$ is of bounded type, resp. of p-bounded type, if it has a representing net $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \in \mathcal{E}_{M,s}(\Omega;\mathbb{R}^m)$, which is a bounded subset of $L^\infty(\Omega;\mathbb{R}^m)$, resp. $L^p(\Omega;\mathbb{R}^m)$.

2.2. **Conservation laws in $\mathcal{G}_s(\Omega;\mathbb{R}^m)$.** From now on we make the assumption. For all $k = 1,\dots,m$ and $j = 1,\dots,n$,

$$f_j^k \text{ are smooth and of polynomial growth, together with all derivatives.} \tag{7}$$

Lemma 1. *Assume that f_j^k satisfy (7).*
(a) If there exists a sequence $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega;\mathbb{R}^m)$, bounded in $L^\infty(\Omega;\mathbb{R}^m)$, such that $\operatorname{div} f^k(v_\varepsilon) \to 0$ *in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0$, $k = 1,\dots,m$, then there also exists a solution $u \in \mathcal{G}_s(\Omega;\mathbb{R}^m)$ of bounded type to* $\operatorname{div} f(u) \approx 0$.
(b) Let $1 \le p < \infty$ and f additionally satisfy

$$|\nabla f_j^k(\lambda)| \le (1+|\lambda|)^{p-1}, \quad \lambda \in \mathbb{R}^m,\ k = 1,...,m,\ j = 1,...,n. \tag{8}$$

If there exists a sequence $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$, bounded in $L^p(\Omega;\mathbb{R}^m)$, such that $\operatorname{div} f^k(v_\varepsilon) \to 0$ *in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0$, then there exists a solution $u \in \mathcal{G}_s(\Omega;\mathbb{R}^m)$ of p-bounded type to* $\operatorname{div} f(u) \approx 0$.

Proof. Let ϕ be an element of $\mathcal{C}_0^\infty(\mathbb{R}^n)$ with integral one and $\phi_\delta(\cdot) = \delta^{-n}\phi(\cdot/\delta)$, $\delta > 0$. For fixed $\varepsilon \in (0,1)$, we have that

$$v_\varepsilon^i * \phi_\delta \to v_\varepsilon^i, \text{ as } \delta \to 0, \text{ for } i = 1,...,m \tag{9}$$

in $L^1_{\mathrm{loc}}(\Omega)$ provided that $v_\varepsilon \in L^\infty(\Omega;\mathbb{R}^m)$ (part *a)*). If $v_\varepsilon \in L^p(\Omega;\mathbb{R}^m)$ with $p < \infty$, then the convergence (9) holds in $L^p(\Omega)$ (part *b)*). Moreover, $\{v_\varepsilon^i * \phi_\delta : \varepsilon \in (0,1),\ \delta \in (0,1)\}$ is bounded independently of δ and ε in $L^\infty(\Omega)$, resp. $L^p(\Omega)$ (for the convolution here we extend v_ε^i to be zero outside Ω and after convolving with ϕ_δ, we take the restriction to Ω). Letting $(K_m)_{m\ge1}$ be an exhausting sequence of compact subsets of Ω (we need it just for the $L^1_{\mathrm{loc}}(\Omega)$-convergence in (9)), we can find a strictly decreasing zero sequence $(\delta_m)_{m\ge1}$ of positive numbers such that $\|v_{1/m}^i * \phi_\delta - v_{1/m}^i\|_{L^1(K_m)} \le 1/m$, for $0 < \delta \le \delta_m$. In the case of $L^p(\Omega;\mathbb{R}^m)$-convergence in (9) we determine $(\delta_m)_{m\ge1}$ so that $\|v_{1/m}^i * \phi_\delta - v_{1/m}^i\|_{L^p(\Omega)} \le 1/m$ holds for $0 < \delta \le \delta_m$ and $i = 1,...,m$. Define an increasing, piecewise constant function $\eta : (0,1) \to (0,1]$ by $\eta(\varepsilon) = 1/m$ for $\delta_{m+1} < \varepsilon \le \delta_m, m \ge 1$ and $\eta(\varepsilon) = 1$ for $\varepsilon \ge \delta_1$ and let $u_\varepsilon^i = v_{\eta(\varepsilon)}^i * \phi_\varepsilon$, $\varepsilon \in (0,1)$. We note that u_ε is smooth, and that the family $\{v_{\eta(\varepsilon)}\}_{\varepsilon\in(0,1)}$ has the same bound in L^∞, resp. L^p, as $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$. Finally, we see that $\partial^\alpha u^i_\varepsilon = v^i_{\eta(\varepsilon)} * \varepsilon^{-|\alpha|}(\partial^\alpha\phi)_\varepsilon$ where, using the notation introduced above, $(\partial^\alpha\phi)_\varepsilon(x) = \varepsilon^{-n}(\partial^\alpha\phi)(x/\varepsilon)$. It follows that $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is moderate. Let u be its class in $\mathcal{G}_s(\Omega;\mathbb{R}^m)$. Clearly, u is of bounded resp. p-bounded type. By definition, we have that $\|u_\varepsilon - v_{\eta(\varepsilon)}\|_{L^1(K_m;\mathbb{R}^m)} \le 1/m$, resp. $\|u_\varepsilon - v_{\eta(\varepsilon)}\|_{L^p(\Omega;\mathbb{R}^m)} \le 1/m$, for $\delta_{m+1} < \varepsilon \le \delta_m, m\in\mathbb{N}$, so $u_\varepsilon - v_{\eta(\varepsilon)} \to 0$, $\varepsilon\to 0$, in $L^1_{\text{loc}}(\Omega;\mathbb{R}^m)$, resp. in $L^p(\Omega;\mathbb{R}^m)$. Now we will prove parts *a)* and *b)* separately.

a) We have that $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{v_{\eta(\varepsilon)}\}_{\varepsilon\in(0,1)}$ are bounded nets in $L^\infty(\Omega;\mathbb{R}^m)$. It follows immediately that for $k = 1,...,m$ and $j = 1,...,n$,

$$f^k_j(u_\varepsilon) - f^k_j(v_{\eta(\varepsilon)}) = (u_\varepsilon - v_{\eta(\varepsilon)}) \bullet \int_0^1 \nabla f^k_j\big(\tau u_\varepsilon + (1-\tau)v_{\eta(\varepsilon)}\big)\, d\tau$$

converges to zero in $L^1_{\text{loc}}(\Omega)$. So,

$$\operatorname{div} f^k(u_\varepsilon) - \operatorname{div} f^k(v_{\eta(\varepsilon)}) \to 0 \text{ in } \mathcal{D}'(\Omega). \tag{10}$$

Using the fact that $\operatorname{div} f^k(v_\varepsilon)\to 0$ and that $v_{\eta(\varepsilon)}$ is a subsequence of v_ε, we obtain from (10) that $\operatorname{div} f^k(u_\varepsilon)\to 0$, $\varepsilon\to 0$, in $\mathcal{D}'(\Omega)$. This proves that $\operatorname{div} f(u)\approx 0$.

b) In this subcase, the families $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{v_{\eta(\varepsilon)}\}_{\varepsilon\in(0,1)}$ are bounded in $L^p(\Omega;\mathbb{R}^m)$. In order to show that u is an approximate generalized solution it is enough to show (10), i.e. that for all test functions $\theta\in\mathcal{D}(\Omega)$,

$$-\sum_{j=1}^n \int_\Omega \Big(f^k_j(u_\varepsilon(x)) - f^k_j(v_{\eta(\varepsilon)}(x))\Big)\cdot\frac{\partial\theta(x)}{\partial x_j}\,dx \to 0, \text{ as } \varepsilon\to 0,$$

for all $k = 1,\dots,m$. But (8), the mean value theorem and Hölder's inequality give

$$\Big|\int_\Omega \Big(f^k_j\big(u_\varepsilon(x)\big) - f^k_j\big(v_{\eta(\varepsilon)}(x)\big)\Big)\cdot\frac{\partial\theta(x)}{\partial x_j}\,dx\Big| \le c\,\|u_\varepsilon - v_{\eta(\varepsilon)}\|_{L^p}\Big(1 + \|u_\varepsilon\|^{\frac{p}{q}}_{L^p} + \|v_{\eta(\varepsilon)}\|^{\frac{p}{q}}_{L^p}\Big)$$

where for $p\ne 1$, $q = \frac{p}{p-1}$, so the desired convergence result follows. Note that for $p = 1$, from (8) we have a L^∞-bound for gradients ∇f^k_j. □

3. Young measures as measure valued solutions and equivalence

Let Ω be an open subset of $\mathbb{R}^n$ and denote by $\mathcal{M}(\mathbb{R}^m)$ the space of regular Borel measures on $\mathbb{R}^m$ with finite total mass; $\mathcal{M}(\mathbb{R}^m)$ can be represented as the dual of the space $C_0(\mathbb{R}^m)$ of the continuous functions vanishing at infinity endowed with the sup norm. A Young measure is a weakly measurable mapping $x\mapsto\nu_x\in\mathcal{M}(\mathbb{R}^m)$, $x\in\Omega$, such that each ν_x is a probability measure. Weak measurability means that for all $v\in C_0(\mathbb{R}^m)$, the mappings $x\mapsto\langle\,\nu_x,\,v\,\rangle = \int_{\mathbb{R}^m} v(\lambda)\nu_x(d\lambda)\in\mathbb{R}$, $x\in\Omega$, are measurable. We denote the set of Young measures by $\mathcal{Y}(\Omega;\mathbb{R}^m)$. Let $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ be a sequence of $L^\infty(\Omega;\mathbb{R}^m)$-functions, whose L^∞-norms are bounded by some positive constant, independently of ε. The basic Young measure theorem, see for example [4], says that there exists a subsequence $\{u_k\}_k$ and a Young measure $\nu = \{\nu_x\}_{x\in\Omega}$, such that for every $F\in\mathcal{C}(\mathbb{R}^m)$,

$$w^* - \lim_{k\to\infty} F(u_k) = \overline{F} \tag{11}$$

in L^∞, where $\overline{F}(x) := \int_{\mathbb{R}^m} F(\lambda)\,d\nu_x(\lambda)$, for almost all $x\in\Omega$. In addition, almost all ν_x are supported in the same compact set. Such Young measures are referred to as L^∞-Young measures and denoted by $\mathcal{Y}^\infty(\Omega;\mathbb{R}^m)$. In fact, the support condition characterizes $\mathcal{Y}^\infty(\Omega;\mathbb{R}^m)$: a Young measure ν arises from a sequence of $L^\infty(\Omega;\mathbb{R}^m)$-uniformly bounded functions if and only if there exists $K\subset\subset\mathbb{R}^m$

such that $\operatorname{supp}\nu_x \subset K$, for almost all $x \in \Omega$. On the other hand, Schonbek, [13], introduced L^p-Young measures, $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$, for $1 < p < \infty$, as Young measures arising from sequences of $L^p(\Omega,\mathbb{R}^m)$-bounded functions, and proved a similar result, i.e., given a sequence of $L^p(\Omega,\mathbb{R}^m)$-bounded functions, there exists a subsequence $\{u_k\}_k$ and a Young measure $\nu = \{\nu_x\}_{x\in\Omega}$, such that for every $F \in \mathcal{C}_p(\mathbb{R}^m) = \{v \in \mathcal{C}(\mathbb{R}^m) : v(\lambda) = o(|\lambda|^p),\ |\lambda| \to \infty\}$, $F \circ u_k \to \overline{F}$ as $k \to \infty$, weakly in $L^1(\Omega;\mathbb{R}^m)$. A characterization of $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$ was given by Kružík and Roubíček, [9], i.e. a Young measure $\nu = \{\nu_x\}_x$ belongs to $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$ if and only if the function $x \mapsto \int_{\mathbb{R}^m} |\lambda|^p\, d\nu_x(\lambda)$ belongs to $L^1(\Omega)$. The result actually holds for $p = 1$ as well.

Following DiPerna [4] who defined the notion of a measure valued solution to (1) we say that a Young measure $\{\nu_x\}_{x\in\Omega}$ is a $\mathcal{Y}^\infty(\Omega;\mathbb{R}^m)$, resp. $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$-measure valued solution to (1) if:

1. $\operatorname{supp}\nu_x \subset K$, for some $K \subset\subset \mathbb{R}^m$, resp. $\left[x \mapsto \int_{\mathbb{R}^m} |\lambda|^p\, d\nu_x(\lambda)\right] \in L^1(\Omega)$,
2. for all $g \in \mathcal{C}(\mathbb{R}^m)$, resp. $g \in \mathcal{C}_p(\mathbb{R}^m)$, $x \mapsto \int_{\mathbb{R}^m} g\, d\nu_x$ is measurable,
3. for all $k = 1, ..., m$, $\sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\int_{\mathbb{R}^m} f_j^k\, d\nu_x\right) = 0$, in $\mathcal{D}'(\Omega)$.

We are now in the position to relate measure valued solutions and approximate generalized solutions.

Theorem 2. *a) Let $u \in \mathcal{G}_s(\Omega;\mathbb{R}^m)$ be of bounded type and solve* $\operatorname{div} f(u) \approx 0$. *Then each Young measure $\{\mu_x\}_{x\in\Omega}$, arising from a subsequence of any bounded representing sequence of u is a $\mathcal{Y}^\infty(\Omega;\mathbb{R}^m)$-measure valued solution to (1).*
b) If $u \approx \iota(v)$, for some $v \in L^\infty(\Omega;\mathbb{R}^m)$, then $v(x) = \int_{\mathbb{R}^m} \lambda d\mu_x(\lambda)$ for almost all $x \in \Omega$.

Proof. Part a) follows from the Young measure theorem and the fact that u is an approximate generalized solution, while part b) follows from the definition of association and the fact that $u_{\varepsilon_k} \to \int_{\mathbb{R}^m} \lambda\, d\mu_x(\lambda)$, $k \to \infty$, what we obtain from (11), by taking F equal to the identity mapping, $F(\lambda) \equiv \lambda$. □

Remark 3. Even if $\iota(v) \approx u$ for some $v \in L^\infty(\Omega;\mathbb{R}^m)$, in general, different sequences lead to different Young measures, since $f(u)$ is not generally associated with $f(\iota(v))$. In this sense, $u \in \mathcal{G}_s(\Omega;\mathbb{R}^m)$ could be viewed as representing a collection of Young measures having the same first moment, i.e. for all Young measures $\int_{\mathbb{R}^m} \lambda d\mu_x(\lambda)$ is the same for almost all $x \in \Omega$.

Theorem 4. *a) Let $\{\mu_x\}_{x\in\Omega}$ be a $\mathcal{Y}^\infty(\Omega;\mathbb{R}^m)$-measure valued solution to (1). Then there is a sequence $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \in \mathcal{E}_{M,s}(\Omega;\mathbb{R}^m)$, whose class u is of bounded type and solves* $\operatorname{div} f(u) \approx 0$.

b) Moreover, this sequence admits a unique Young measure coinciding with $\{\mu_x\}_{x\in\Omega}$ for almost all $x \in \Omega$, and $u \approx \iota(v)$, where $v(x) = \int_{\mathbb{R}^m} \lambda d\mu_x(\lambda)$.

Proof. a) According to the characterization of $\mathcal{Y}^\infty(\Omega;\mathbb{R}^m)$, there is a uniformly bounded sequence $\{v_p\}_p \subset L^\infty(\Omega;\mathbb{R}^m)$ having $\{\mu_x\}_{x\in\Omega}$ as its Young measure, that is for every $g \in \mathcal{C}(\mathbb{R}^m)$, $g(v_p) \to \bar{g}$, weak* in L^∞, as $p \to \infty$, where $\bar{g}(x) = \int_{\mathbb{R}^m} g\, d\mu_x$. It follows that, as $p \to \infty$,

$$\sum_{j=1}^n \frac{\partial}{\partial x_j} f_j^k(v_p) \to \sum_{j=1}^n \frac{\partial}{\partial x_j} \int_{\mathbb{R}^m} f_j^k\, d\mu_x, \quad \text{in } \mathcal{D}'(\Omega),\ k = 1, ..., m. \tag{12}$$

The right hand side of (12) equals zero because $\{\mu_x\}_{x\in\Omega}$ is a measure valued solution to (1). Applying Lemma 1 to the sequence $\{v_p\}_p$, we obtain an element $u \in \mathcal{G}_s(\Omega;\mathbb{R}^m)$ of bounded type solving (4).

b) Since we have that for all $g \in \mathcal{C}(\mathbb{R}^m)$, $g(v_p) \to \bar{g}$, weak* in L^∞, as $k \to \infty$, every subsequence of v_p leads to the same Young measure. We replace $v_{1/m}$ appearing in the proof of Lemma 1 by v_m, construct $\eta(\varepsilon)$ accordingly, so that $v_{\eta(\varepsilon)} = v_m$, for $0 < \delta \le \delta_m,\ m \ge 1$, and let $u^i_\varepsilon = v^i_{\eta(\varepsilon)} * \phi_\varepsilon$, with ϕ as in Lemma 1, and $u = [\{u_\varepsilon\}_{\varepsilon\in(0,1)}] \in \mathcal{G}_s(\Omega;\mathbb{R}^m)$. Since $u_\varepsilon - v_{\eta(\varepsilon)} \to 0$, in $L^1_{\rm loc}(\Omega;\mathbb{R}^m)$, $\varepsilon \to 0$, it follows that $g(u_\varepsilon) - g(v_{\eta(\varepsilon)}) \to 0$, in $L^1_{\rm loc}(\Omega;\mathbb{R}^m)$, $\varepsilon \to 0$, for every $g \in \mathcal{C}(\mathbb{R}^m)$ and that $\{g(u_\varepsilon)\}_\varepsilon$ and $\{g(v_{\eta(\varepsilon)})\}_\varepsilon$ have the same weak*-limits in $L^\infty(\Omega;\mathbb{R}^m)$. Thus $\{u_\varepsilon\}_\varepsilon$ and $\{v_{\eta(\varepsilon)}\}_\varepsilon$ give rise to the same unique Young measure. It is then clear that $u \approx \iota(v) = \iota(\lim_{k\to\infty} v_k) = \iota(v)$, where $v(x) = \int_{\mathbb{R}^m} \lambda\, d\mu_x(\lambda)$. □

Remark 5. Again, the generalized function u constructed in the proof of the Theorem 4 is not unique. In fact, let $\{w_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathcal{E}_{M,s}(\Omega)$ be any other uniformly bounded sequence, all whose subsequences give rise to the same Young measure, and let w be its class in $\mathcal{G}_s(\Omega;\mathbb{R}^m)$. Then clearly we must have that $u \approx w$. But it is not necessarily true that $u = w$ in $\mathcal{G}_s(\Omega;\mathbb{R}^m)$. For example, it suffices to take $\{w_\varepsilon\}_{\varepsilon\in(0,1)}$ in such a way that $u_\varepsilon - w_\varepsilon$ converges to zero in $L^1_{\rm loc}$, but $\{u_\varepsilon - w_\varepsilon\}_{\varepsilon\in(0,1)}$ does not belong to $\mathcal{N}_s(\Omega;\mathbb{R}^m)$.

This time nonuniqueness comes from the fact that $\mathcal{N}_s(\Omega;\mathbb{R}^m)$ is strictly smaller than the space of $L^1_{\rm loc}$-zero sequences. To put it more explicitly, let $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{w_\varepsilon\}_{\varepsilon\in(0,1)}$ be two uniformly bounded sequences in $\mathcal{E}_{M,s}(\Omega)$. Then $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{w_\varepsilon\}_{\varepsilon\in(0,1)}$ are equal in $\mathcal{G}_s(\Omega;\mathbb{R}^m)$, if their difference belongs to $\mathcal{N}_s(\Omega;\mathbb{R}^m)$; they produce the same Young measure, if their difference converges to zero in $L^1_{\rm loc}(\Omega;\mathbb{R}^m)$; they are associated in $\mathcal{G}_s(\Omega;\mathbb{R}^m)$, if their difference converges to zero in $\mathcal{D}'(\Omega;\mathbb{R}^m)$. Thus the various occurrences of nonuniqueness can be attributed to differing stability properties inherent in the respective solution concepts.

For $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$-measure valued solution and approximate generalized solutions of p-bounded type, results analogous to Theorems 2 and 4 are valid. In the following two theorems, in addition to (7) and (8), we assume that $f^k_j \in C_p(\mathbb{R}^m)$.

Theorem 6. *a) Let $u \in \mathcal{G}_s(\Omega;\mathbb{R}^m)$ be of p-bounded type and solve* $\operatorname{div} f(u) \approx 0$. *Then each Young measure $\{\mu_x\}_{x\in\Omega}$, arising from a subsequence of any p-bounded representing sequence of u is a $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$-measure valued solution to (1).*

b) If $u \approx \iota(v)$, for some $v \in L^p(\Omega;\mathbb{R}^m)$, then $v(x) = \int_{\mathbb{R}^m} \lambda d\mu_x(\lambda)$ for almost all $x \in \Omega$.

Theorem 7. *a) Let $\{\mu_x\}_{x\in\Omega}$ be a $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$-measure valued solution to (1). Then there is a sequence $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \in \mathcal{E}_{M,s}(\Omega;\mathbb{R}^m)$, whose class u is of p-bounded type and solves* $\operatorname{div} f(u) \approx 0$.

b) Moreover, this sequence admits a unique Young measure coinciding with $\{\mu_x\}_{x\in\Omega}$ for almost all $x \in \Omega$, and $\iota(v) \approx u$, where $v(x) = \int_{\mathbb{R}^m} \lambda d\mu_x(\lambda)$.

Proof. a) According to the characterization of $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$, there is a sequence $\{u_p\}$ having $\{\mu_x\}_{x\in\Omega}$ as its Young measure. The sequence $\{u_p\}$ is uniformly bounded in L^p and for all $g \in C_p$ we have weak convergence in L^1 of the composition $g(u_p)$. Applying that to $f^k = (f^k_1, \dots, f^k_n)$, we obtain $\operatorname{div} f^k(u_p) \to \int_{\mathbb{R}^m} \operatorname{div} f^k(\lambda)\, d\mu_x(\lambda) = 0$ in $\mathcal{D}'(\Omega)$, $k = 1, ..., m$, because $\{\mu_x\}_{x\in\Omega}$ is a measure valued solution to $\operatorname{div} f(u) = 0$, and that is why $\{u_p\}$ fulfills the conditions of Lemma 1b). Now, by the construction from the proof of Lemma 1, we obtain a solution to $\operatorname{div} f(u) \approx 0$ of p-bounded type. The proof of the part b) is similar to the proof of Theorem 4b). $\square$

4. Approximate generalized solutions to the Euler equations

The conservative form of the Euler equations is

$$\frac{\partial v}{\partial t} + \operatorname{div}(v \otimes v) + \nabla p = 0, \quad \operatorname{div} v = 0,$$

where $\operatorname{div} v \otimes v = \big(\operatorname{div}(v_1 v_1, \dots, v_n v_1), \dots, \operatorname{div}(v_1 v_n, \dots, v_n v_n)\big)^\top$. Thus we deal with a system of n equations of the form $\operatorname{div}_{n+1} f_j(v) \approx 0$, $j = 1, \dots, n$, where $f_j(v) = (v_j, v_1 v_j, v_2 v_j, \dots, v_n v_j)$, $f_j : \mathbb{R}^n \to \mathbb{R}^{n+1}$, $j = 1, \dots, n$ and the divergence is taken with respect to the variables $(t, x_1, x_2, \dots, x_n)$.

Following DiPerna, Majda [6], who constructed measure-valued solutions to the Euler equations, we have that, under the same assumptions as in [6, Proposition 5.1], there exists an approximate generalized solution to the Euler equations.

Proposition 8. *Assume that* $\{v_\varepsilon\}_{\varepsilon \in (0,1)}$ *is a sequence of functions satisfying* $\operatorname{div} v_\varepsilon = 0$, *and the following conditions:*

a) Weak stability: For any $\Omega \subset \mathbb{R}^n \times \mathbb{R}^+$, *there exists a constant* $C = C_\Omega$ *such that*

$$\int_\Omega |v_\varepsilon(x;t)|^2 \, dxdt \le C.$$

b) Weak consistency: For all test functions $\phi \in C_0^\infty(\Omega; \mathbb{R}^n)$,

$$\lim_{\varepsilon \to 0} \int_\Omega \phi_t \cdot v_\varepsilon + \nabla \phi : v_\varepsilon \otimes v_\varepsilon \, dxdt = 0.$$

Then there exists $u \in \mathcal{G}_s(\Omega; \mathbb{R}^n)$, *which is an approximate generalized solution of* 2*-bounded type to the Euler equations.*

Proof. Weak consistency gives that $\operatorname{div}_{n+1} f_j(v_\varepsilon) \to 0$ in $\mathcal{D}'(\Omega)$, for all $j = 1, \dots, n$, so we can just apply Lemma 1, to obtain an approximate generalized solution of 2-bounded type, because the condition (8) is fulfilled for f_j defined above and $p = 2$. Further, we can infer that

$$\lim_{\varepsilon \to 0} \int_\Omega \big(u_\varepsilon^j - v_{\eta(\varepsilon)}^j\big)\phi_t^j + \sum_{i=1}^n \big(u_\varepsilon^j u_\varepsilon^i - v_{\eta(\varepsilon)}^j v_{\eta(\varepsilon)}^i\big)\phi_{x_i}^j \, dxdt = 0,$$

for all test functions $\phi^j \in C_0^\infty(\Omega)$, directly from the following estimate:

$$\begin{aligned}
&\left| \int_\Omega \big(u_\varepsilon^j - v_{\eta(\varepsilon)}^j\big)\phi_t^j + \sum_{i=1}^n \big(u_\varepsilon^j u_\varepsilon^i - v_{\eta(\varepsilon)}^j v_{\eta(\varepsilon)}^i\big)\phi_{x_i}^j \, dxdt \right| \\
= & \left| \int_\Omega \big(u_\varepsilon^j - v_{\eta(\varepsilon)}^j\big)\phi_t^j + \sum_{i=1}^n \big(u_\varepsilon^j u_\varepsilon^i \pm v_{\eta(\varepsilon)}^j u_\varepsilon^i - v_{\eta(\varepsilon)}^j v_{\eta(\varepsilon)}^i\big)\phi_{x_i}^j \, dxdt \right| \\
\le & \|u_\varepsilon^j - v_{\eta(\varepsilon)}^j\|_{L^2} \|\phi_t^j\|_{L^2} + \sum_{i=1}^n \sup |\phi_{x_i}^j| \Big(\|u_\varepsilon^j - v_{\eta(\varepsilon)}^j\|_{L^2} \|u_\varepsilon^i\|_{L^2} + \|v_{\eta(\varepsilon)}^j\|_{L^2} \|u_\varepsilon^i - v_{\eta(\varepsilon)}^i\|_{L^2} \Big) \\
\le & c_1 \|u_\varepsilon^j - v_{\eta(\varepsilon)}^j\|_{L^2} + \sum_{i=1}^n \Big(c_2 \|u_\varepsilon^j - v_{\eta(\varepsilon)}^j\|_{L^2} + c_3 \|u_\varepsilon^i - v_{\eta(\varepsilon)}^i\|_{L^2} \Big), \ j = 1, ..., n.
\end{aligned}$$

$\square$

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