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On a scalar conservation law with nonlinear diffusion and linear dispersion in heterogeneous media

Jelena Aleksić[1], Darko Mitrović[2], Stevan Pilipović[1]

Abstract

We obtain the strong pre-compactness of a family of solutions to a suitable regularization of multidimensional scalar conservation law via vanishing nonlinear diffusion and linear dispersion. We consider the flux which depends on the time and space variables, and obtain condition $\delta = O(\varepsilon^2)$, $\varepsilon \to 0$ for the existence of a weak entropy solution. In comparison to known results for heterogeneous media (cf. [4]), our condition is weaker, thus more general.

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1 Introduction

The subject of the paper is the following Cauchy problem for multidimensional scalar conservation law:

$$\partial_t u(t,x) + \mathrm{div}_x f(t,x,u) = 0, \quad u(0,x) = u_0(x), \qquad x \in \mathbb{R}^d,\ t \in \mathbb{R}^+. \tag{1}$$

We study the behavior of a family of solutions to a regularization of (1),

$$\partial_t u^{\varepsilon,\delta} + \mathrm{div}_x f(t,x,u^{\varepsilon,\delta}) = \varepsilon \sum_{j=1}^{d} \partial_{x_j} b_j(\nabla_x u^{\varepsilon,\delta}) + \delta \sum_{j=1}^{d} \partial_{x_j x_j x_j} u^{\varepsilon,\delta}, \tag{2}$$

$$u(0,x) = u_0^{\varepsilon}(x), \quad x \in \mathbb{R}^d,\ \varepsilon,\delta \in (0,1),\ \delta = \delta(\varepsilon), \tag{3}$$

where initial data u_0^ε from (3) converge to u_0 from (1) strongly in $L^2(\mathbb{R}^d)$ as $\varepsilon \to 0$ (all terms from (1), (2) and (3) are precisely described in the next section). In order to obtain a weak entropy solution to (1) as a limit of a subsequence of the solutions to (2)-(3), we study the pre-compactness properties of the family $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$. The goal is to obtain optimal balance of the two parameters ε and δ.

Let us briefly recall already obtained results concerning diffusion-dispersion limits for (1). In [10], using compensated compactness arguments, the author

[1]Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Serbia e-mail: jelena.aleksic@im.ns.ac.yu, stevan.pilipovic@im.ns.ac.yu

[2]Norwegian University of Science and Technology, Department of Mathematical Sciences, Alfred Getz vei 1, NO-7491 Trondheim, Norway e-mail: matematika@cg.yu

proved that a family of solutions to KdV-Burgers equation converges to a weak solution to Burgers equation if diffusion and dispersion parameter are balanced in the sense that $\delta = \mathcal{O}(\varepsilon^2)$, as $\varepsilon \to 0$. Using the same methodology (in one dimensional case) with the same balance result, in [7] the diffusion-dispersion problem is solved in the case when the flux has a general homogeneous form, $f = f(u)$. Multidimensional case ($x \in \mathbb{R}^d$) is solved in [6], but with the stronger balance, $\delta = o(\varepsilon^2)$, as $\varepsilon \to 0$. More general result concerning the relative size of diffusion and dispersion parameters is made in [5]. Using the kinetic approach [9] and the averaging lemma [8, 9, 11], the author obtained in [5] the diffusion-dispersion limit with a weaker balance, $\delta = \mathcal{O}(\varepsilon^2)$, as $\varepsilon \to 0$. It is important to point out that in every of the previously mentioned works the flux which appears in the considered conservation law *does not depend on space and time explicitly.* Authors of [4], dealing with heterogeneous media for discontinuous flux in one-dimensional case, obtained stronger balance between two parameters, $\delta = o(\varepsilon^2)$, as $\varepsilon \to 0$. In order to obtain the results analogous to those from [5] (multidimensional case and optimal diffusion-dispersion ratio), similarly as in [5], we shall use the kinetic formulation of the conservation law under consideration. We will improve the balance result from [4], but with the stronger assumptions on the flux (see (H3) in the next chapter). Further improvements shall be made in [1].

The paper is organized as follows. We give in Section 2 basic notations, assumptions and the statement of the main theorem. In Section 3, we prove a priori inequalities for the family $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$. In Section 4 we prove the main theorem which is based on the Theorem 2.5 from [3].

2 Notations, assumptions and the main result

In the sequel, we put $|g|^2 = \sum_{i=1}^d |g_i|^2$, for a vector valued function $g = (g_1, ..., g_d)$ defined on $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}$. The derivative D_{x_i} at the point (t, x, u), where u depends on (t, x), is defined by $D_{x_i} g(t, x, u) = (\partial_{x_i} g(t, x, \lambda))|_{\lambda=u(t,x)}$. Derivatives ∂_{x_i} and D_{x_i} are connected by the identity $\partial_{x_i} g(t, x, u) = D_{x_i} g(t, x, u) + \partial_u g(t, x, u)\partial_{x_i} u$. For the simplicity, in the sequel we shall write u^ε instead of $u^{\varepsilon,\delta}$ and consider that $\varepsilon \in (0, 1)$. We assume that f, b and u_0 are enough regular, so that solutions u^ε, $\varepsilon \in (0, 1)$ of (2) have enough regularity so that all formal computations below are correct, as well as that $u^\varepsilon, \partial_{x_i} u^\varepsilon$ and $\partial_{x_i x_j} u^\varepsilon$ vanish as $|x| \to \infty$. Now we list additional assumptions. For the initial data we assume that $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $u_0^\varepsilon, \nabla u_0^\varepsilon \in H^1(\mathbb{R}^d)$. Here $u_0^\varepsilon = u_0 \star \delta_\varepsilon$, $\varepsilon \in (0, 1)$ for a suitable $(\delta_\varepsilon)_\varepsilon$ net of smooth compactly supported functions. Notice that from the last assumption $(u_0^\varepsilon)_\varepsilon$ is uniformly bounded in $L^2(\mathbb{R}^d)$. We assume that the diffusion term $b = (b_1, ..., b_d) : \mathbb{R}^d \to \mathbb{R}^d$ fulfils the following hypotheses:

(H1) There exist positive constants C_1, C_2 such that $C_1|\lambda|^2 \leq \lambda \cdot b(\lambda) \leq C_2|\lambda|^2$, for all $\lambda \in \mathbb{R}^d$.

(H2) The gradient matrix $Db(\lambda)$ is a positively definite matrix uniformly in $\lambda \in \mathbb{R}^d$, i.e. there exists positive constant C_3 such that $\xi^T Db(\lambda)\xi \geq C_3|\xi|^2$, for all $\lambda, \xi \in \mathbb{R}^d$.

We assume for flux $f=(f_1,...,f_d):\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^d$ that $f=f(t,x,u)$ and $f_u(t,x,u)$ are continuous and that they have locally integrable derivatives with respect to t and x. These assumptions enable us to make calculations in a priori estimates of the next section. Moreover, we assume that f fulfils the following hypotheses:

(H3) $\partial_u f\in L^\infty(\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R})$, $D_{x_i}f_i\in L^2\cap L^1(\mathbb{R}^+\cap\times\mathbb{R}^d\times\mathbb{R})$ and $|D_{x_i}f_j(t,x,v)|\leq|\zeta_{i,j}(t,x)||v|$, for some $\zeta_{i,j}\in L^\infty(\mathbb{R}^+\times\mathbb{R}^d)$.

The final assumption is the following nonlinearity condition **(NLC)**: For almost every $t,x\in\mathbb{R}^+\times\mathbb{R}^d$ and every $\xi\in S^d$ the mapping $\lambda\mapsto\xi_0+\sum\limits_{k=1}^{d}\partial_\lambda f_i(t,x,\lambda)\xi_k$ is not identically equal to zero on any set of positive Lebesgue measure.

Our main result which will be proved in Section 4 is the following theorem.

Theorem 2.1 *Under afore listed assumptions, a family of smooth solutions* $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ *to Cauchy problem* (2)-(3) *is strongly pre-compact in* $L^1_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^d)$ *if* ε *and* δ *from* (2) *are balanced in the sense that* $\delta=\mathcal{O}(\varepsilon^2)$ *as* $\varepsilon\to 0$.

3 A priori inequalities

In this section, we give necessary apriori inequalities.

Lemma 3.1 *Under afore listed assumptions, the family of solutions* $(u^\varepsilon)_\varepsilon$ *to* (2)-(3) *for every* $t\in[0,T]$ *satisfies (for suitable* $c>0$*) the following inequality*

$$\int_{\mathbb{R}^d}|u^\varepsilon(t,x)|^2dx+\varepsilon\int_0^t\int_{\mathbb{R}^d}|\nabla u^\varepsilon(t',x)|^2dxdt'\leq c,\ \textit{for}\ \varepsilon<\varepsilon_0. \tag{4}$$

Proof: Let $\eta=\eta(u)$, $u\in\mathbb{R}$, be a smooth function. We multiply (2) by $\eta'(u^\varepsilon)$ and define $q=(q_1,...,q_n)$ as $q_i(t,x,u)=\int_0^u\eta'(v)\partial_v f_i(t,x,v)dv$, $i=1,...,d$. Thus, (2) becomes

$$\begin{aligned}&\partial_t\eta(u^\varepsilon)+{\rm div}_x\, q_i(t,x,u^\varepsilon)-\sum_{i=1}^d\int_0^{u^\varepsilon}\partial_{x_i v}f_i(t,x,v)\eta'(v)dv+\\&\sum_{i=1}^d\eta'(u^\varepsilon)D_{x_i}f_i(t,x,u^\varepsilon)=\varepsilon\sum_{i=1}^d\partial_{x_i}\big(\eta'(u^\varepsilon)b_i(\nabla u^\varepsilon)\big)-\\&\varepsilon\eta''(u^\varepsilon)\sum_{i=1}^d b_i(\nabla u^\varepsilon)\partial_{x_i}u^\varepsilon+\delta\sum_{i=1}^d\partial_{x_i}\big(\eta'(u^\varepsilon)\partial_{x_ix_i}u^\varepsilon\big)-\\&\frac{\delta}{2}\eta''(u^\varepsilon)\sum_{i=1}^d\partial_{x_i}(\partial_{x_i}u^\varepsilon)^2.\end{aligned}\tag{5}$$

We choose here $\eta(u) = \frac{u^2}{2}$ and integrate over $[0,t) \times \mathbb{R}^d$. Taking into account (H1) and partial integration, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^d} |u^\varepsilon(t,x)|^2 dx + \varepsilon C_1 \int_0^t \int_{\mathbb{R}^d} |\nabla u^\varepsilon(t',x)|^2 dx dt'$$
$$\leq \frac{1}{2}\int_{\mathbb{R}^d} |u_0^\varepsilon(x)|^2 dx - \int_0^t \int_{\mathbb{R}^d} \int_0^{u^\varepsilon(t',x)} \sum_{i=1}^d D_{x_i} f_i(t',x,v) dv dx dt'.$$

This and (H3) imply (4). ❑

Lemma 3.2 *Under afore listed assumptions, for* $|D^2 u|^2 = \sum_{i,k=1}^d |\partial_{x_i x_k} u|^2$, *a family of solutions* $(u^\varepsilon)_\varepsilon$ *to* (2)-(3) *satisfies the following inequality*

$$\varepsilon^2 \int_{\mathbb{R}^d} |\nabla u^\varepsilon(t,x)|^2 dx + \varepsilon^3 \int_0^t \int_{\mathbb{R}^d} |D^2 u^\varepsilon(t',x)|^2 dx dt' \leq c, \tag{6}$$

for suitable $c > 0$, *for every* $t \in [0,T]$ *and* $\varepsilon < \varepsilon_0$.

Proof: In the sequel we will use the existence of different constants which indexes will indicate that they are new ones. As well, we will not write $\varepsilon < \varepsilon_0$ for appropriate ε_0 which can be changed. We differentiate (2) with respect to x_k and multiply the obtained expression by $\partial_{x_k} u^\varepsilon$. Then, summing expressions for $k = 1, ..., d$, using partial integration, as well as (H2), we obtain

$$\frac{1}{2}\int_{\mathbb{R}^d} \partial_t |\nabla u^\varepsilon|^2 dx - \sum_{k=1}^d \int_{\mathbb{R}^d} \nabla \partial_{x_k} u^\varepsilon \cdot \left(D_{x_k} f(t,x,u^\varepsilon) + \partial_u f \cdot \partial_{x_k} u^\varepsilon \right) dx$$
$$= -\varepsilon \sum_{k=1}^d \int_{\mathbb{R}^d} (\nabla \partial_{x_k} u^\varepsilon)^T Db(\nabla u^\varepsilon) \nabla \partial_{x_k} u^\varepsilon dx \leq^{(\mathrm{H2})} -\varepsilon C_3 \sum_{k=1}^d \int_{\mathbb{R}^d} |\nabla \partial_{x_k} u^\varepsilon|^2 dx.$$

Now we integrate this over $[0,t]$, use the Cauchy-Schwartz inequality, as well as the Young inequality (C_3 below is the same as above), $ab \leq \frac{C_3 \varepsilon}{2} a^2 + \frac{C_6}{\varepsilon} b^2$, $a, b \in \mathbb{R}$. Then we multiply obtained inequality by ε^2, use inequality $(a+b)^2 \leq 2a^2 + 2b^2$ and the consequence from (4) that $\varepsilon \int_0^t \int_{\mathbb{R}^d} |\nabla u(s,x)|^2 dx ds \leq C$, $t \in [0,T]$ in order to obtain

$$\frac{\varepsilon^2}{2} \int_{\mathbb{R}^d} |\nabla u^\varepsilon(t,x)|^2 dx + C_3 \frac{\varepsilon^3}{2} \int_{\mathbb{R}^d} \int_0^t |D^2 u^\varepsilon|^2 dx dt \leq \varepsilon^2 C_9 \int_{\mathbb{R}^d} |\nabla u_0^\varepsilon|^2 dx dt'$$
$$+ \varepsilon C_{10} \int_0^t \int_{\mathbb{R}^d} \sum_{k=1}^d |D_{x_k} f(t',x,u^\varepsilon(t',x))|^2 dx dt' + C_{11} \|\partial_u f\|^2_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})}$$

with appropriate constants. Taking into account (H3), (6) follows from the last inequality. ❑

4 The proof of the main theorem

To prove strong pre-compactness of the family $(u^\varepsilon)_\varepsilon$ of solutions to (2-3), we use the averaging lemma proved in [3, Theorem 2.5]. We give its variant which is adapted to our purposes.

Theorem 4.1 *Let $(h_n)_n \subset L^2_{\rm loc}(\mathbb{R}^N \times \mathbb{R})$ be a sequence of solutions to the following transport equation*

$$\mathrm{div}_y(F(y,\lambda)h_n(y,\lambda)) = \sum_{i=1}^d \partial_\lambda^{k_i} G_n^i(y,\lambda), \quad y \in \mathbb{R}^N, \quad \lambda \in \mathbb{R},$$

where flux $F = (F_1, ..., F_N) \in C(\mathbb{R}^d \times \mathbb{R})$, and families $(G_n^i)_n$, $i = 1, \dots, N$, are strongly pre-compact in $H^{-1}(\mathbb{R}^N \times \mathbb{R})$. Furthermore, assume that the following non-degeneracy condition is fulfilled: For almost every $y \in \mathbb{R}^N$ and every $\xi \in S^{N-1}$ the mapping

$$\lambda \mapsto \textstyle\sum_{k=1}^N F_k(y,\lambda)\xi_k$$

is not identically equal to zero on any set of positive Lebesgue measure.

Then for every $\rho \in C_0^\infty(\mathbb{R})$ the sequence $(\int_{\mathbb{R}} h_n(x,\lambda)\rho(\lambda)d\lambda)$ is strongly pre-compact in $L^1_{\rm loc}(\mathbb{R}^d \times \mathbb{R})$.

Proof of the Theorem 2.1: Let $\eta \in C_0^\infty(\mathbb{R})$ and

$$h_\varepsilon(t,x,\lambda) = \begin{cases} 1, & \text{for } \ 0 < \lambda \le u^\varepsilon(t,x), \quad (t,x) \in \Pi, \\ -1, & \text{for } \ 0 > \lambda \ge u^\varepsilon(t,x), \quad (t,x) \in \Pi, \\ 0, & \quad \text{otherwise}, \end{cases}$$

where $\Pi = (0,T) \times \mathbb{R}^d$. Let $\varphi \in C_0^\infty(\Pi)$. We rewrite (5) as

$$\begin{aligned}
(7) \qquad & -\int_{\Pi\times\mathbb{R}} h_\varepsilon(t,x,\lambda)\eta'(\lambda)\varphi_t(t,x)\,d\lambda dx dt \\
& -\sum_{i=1}^d \int_{\Pi\times\mathbb{R}} h_\varepsilon(t,x,\lambda)\partial_\lambda f_i(t,x,\lambda)\eta'(\lambda)\varphi_{x_i}(t,x)\,d\lambda dx dt \\
& +\sum_{i=1}^d \int_{\Pi\times\mathbb{R}} h_\varepsilon(t,x,\lambda)\partial_{x_i\lambda} f_i(t,x,\lambda)\eta''(\lambda)\varphi(t,x) d\lambda dx dt \\
& = -\int_\Pi \sum_{i=1}^d (\varepsilon b_i(\nabla u^\varepsilon) + \delta\partial_{x_i x_i} u^\varepsilon)\,\eta'(u^\varepsilon)\varphi_{x_i}(t,x) dx dt \\
& -\sum_{i=1}^d \int_\Pi \left(\varepsilon b_i(\nabla u^\varepsilon) u^\varepsilon_{x_i} + \delta u^\varepsilon_{x_i}\partial_{x_i x_i} u^\varepsilon\right)\eta''(u^\varepsilon)\varphi(t,x) dx dt.
\end{aligned}$$

As in [5], we represent equation (7) as an equation in $\mathcal{D}'(\Pi\times\mathbb{R})$. With an abuse of notation (see notation in Section 2) we put $H_i^\varepsilon(t,x) = \varepsilon b_i(\nabla u^\varepsilon)$, $\bar{H}_i^\varepsilon(t,x) =$

$\delta\partial_{x_i x_i} u^\varepsilon$, $G_i^\varepsilon(t,x) = \varepsilon b_i(\nabla u^\varepsilon) u^\varepsilon_{x_i}$, $\bar{G}_i^\varepsilon(t,x) = \delta u^\varepsilon_{x_i}\partial_{x_i x_i} u^\varepsilon$, and note that the nets $H_i^\varepsilon, \bar{H}_i^\varepsilon, G_i^\varepsilon(t,x), \bar{G}_i^\varepsilon$ are uniformly bounded in $L^1_{\rm loc}(\Pi\times\mathbb{R})$ (cf. (H1)-(H3) and Lemmas 3.1-3.2). Let $\delta(\lambda-u)$ be a Dirac delta function defined by $\langle\delta(\lambda-u),\eta(\lambda)\rangle = \eta(u)$. Then $m_i^\varepsilon = \delta(\lambda-u^\varepsilon)G_i^\varepsilon$, $k_i^\varepsilon = \delta(\lambda-u^\varepsilon)\bar{G}_i^\varepsilon$, $\pi_i^\varepsilon = \delta(\lambda-u^\varepsilon)H_i^\varepsilon$ and $\bar{\pi}_i^\varepsilon = \delta(\lambda-u^\varepsilon)\bar{H}_i^\varepsilon$, $i=1,...,d$, $\varepsilon<1$, are defined as distributions in $\mathcal{D}'(\Pi\times\mathbb{R})$ via the following tensor products:

$$\begin{aligned}
&\langle m_i^\varepsilon,\varphi\otimes\eta'\rangle = \int_\Pi G_i^\varepsilon(t,x)\varphi(t,x)\eta'(u^\varepsilon(t,x))dxdt,\\
&\langle k_i^\varepsilon,\varphi\otimes\eta'\rangle = \int_\Pi \bar{G}_i^\varepsilon(t,x)\varphi(t,x)\eta'(u^\varepsilon(t,x))dxdt,\\
&\langle \pi_i^\varepsilon,\varphi\otimes\eta'\rangle = \int_\Pi H_i^\varepsilon(t,x)\varphi(t,x)\eta'(u^\varepsilon(t,x))dxdt,\\
&\langle \bar{\pi}_i^\varepsilon,\varphi\otimes\eta'\rangle = \int_\Pi \bar{H}_i^\varepsilon(t,x)\varphi(t,x)\eta'(u^\varepsilon(t,x))dxdt.
\end{aligned}\tag{8}$$

The mapping $\eta(\lambda)\mapsto\langle\delta(\lambda-u^\varepsilon)G_i^\varepsilon(t,x),\eta(\lambda)\rangle = \langle\delta(\lambda-u^\varepsilon)\varepsilon b_i(\nabla u^\varepsilon)u^\varepsilon_{x_i},\eta(\lambda)\rangle = \varepsilon u^\varepsilon b_i(\nabla u^\varepsilon)\partial_x\eta(u^\varepsilon(t,x))$, with values in $\mathcal{D}'(\Pi)$, is continuous, so (8) holds and we can prove other identities in the same way. Thus, (7) can be rewritten as equation in $\mathcal{D}'(\Pi\times\mathbb{R})$ as follows

$$\begin{aligned}
&\partial_t h_\varepsilon(t,x,\lambda) + \sum_{i=1}^d \partial_{x_i}(h_\varepsilon(t,x,\lambda)\partial_\lambda f_i(t,x,\lambda)) = \\
&\sum_{i=1}^d \partial_\lambda(h_\varepsilon(t,x,\lambda)\partial_{x_i} f_i(t,x,\lambda)) + \sum_{i=1}^d \left(\partial_{x_i}(\pi_i^\varepsilon+\bar{\pi}_i^\varepsilon) + \partial_\lambda(m_i^\varepsilon+k_i^\varepsilon)\right).
\end{aligned}\tag{9}$$

Now we estimate terms on the right-hand side of (9). By Lemmas 3.1-3.2 and (H1) (as in [5]), we obtain the following results ((10) and (11)):

$$\pi_i^\varepsilon = \bar{g}_i^\varepsilon + \partial_\lambda g_i^\varepsilon,\quad \bar{\pi}_i^\varepsilon = \bar{p}_i^\varepsilon + \partial_\lambda p_i^\varepsilon,\quad i=1,...,d,\tag{10}$$

with $\bar{g}_i^\varepsilon, g_i^\varepsilon\to 0$ and $\bar{p}_i^\varepsilon, p_i^\varepsilon\to 0$ as $\varepsilon\to 0$ in $L^2(\Pi\times\mathbb{R})$;

$$(m_i^\varepsilon)_\varepsilon, (k_i^\varepsilon)_\varepsilon \ \text{ lie in a bounded set of } \mathcal{M}(\Pi\times\mathbb{R}),\quad \text{for every } i=1,...,d,\tag{11}$$

where $\mathcal{M}(\Pi\times\mathbb{R})$ stands for the space of bounded measures. The proof is technical(cf. [5]) and will be omitted. Consider now the remaining term on the right hand side of (9). Denote by $\Pi_i^\varepsilon(t,x,\lambda) = \partial_\lambda(h_\varepsilon(t,x,\lambda)\partial_{x_i} f_i(t,x,\lambda))$, $(t,x,\lambda)\in\Pi\times\mathbb{R}$, $i=1,...,d$. Let $\theta(t,x,\lambda)\in C_0^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R})$ and $i=1,...,d$. Then,

$$\begin{aligned}
\langle\Pi_i^\varepsilon,\theta\rangle &= \Big|\int_{\Pi\times\mathbb{R}} h_\varepsilon(t,x,\lambda)\partial_{x_i} f_i(t,x,\lambda)\theta_\lambda(t,x,\lambda)dtdxd\lambda\Big|\\
&\le \|\theta_\lambda\|_{C^0(\Pi\times\mathbb{R})}\int_{{\rm supp}\theta}|\partial_\lambda f_i(t,x,\lambda)|dtdxd\lambda \le C\|\theta_\lambda\|_{C^0(\Pi\times\mathbb{R})},
\end{aligned}$$

where C is a constant depending only on the support of a test function θ. Thus, for every $i=1,...,d$ the family $(\Pi_i^\varepsilon)_\varepsilon$ lies in a locally bounded subset of the space of bounded measures $\mathcal{M}(\Pi\times\mathbb{R})$. Knowing that every sequence of measures bounded in $\mathcal{M}(\Pi\times\mathbb{R})$ is pre-compact in $H^{-1}(\Pi\times\mathbb{R})$(cf. [2] Theorem 5), we can apply Theorem 4.1 for the net $(h_\varepsilon)_\varepsilon$ and conclude that a subsequence $(h_k)_k$ of $(h_\varepsilon)_\varepsilon$ satisfies

$$\text{(12)}\qquad \Big(\int_{-R}^{R} h_k(t,x,\lambda)d\lambda\Big)_{k\in\mathbf{N}} \quad \text{is convergent in} \quad L^1_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^d),$$

for every $R\in\mathbf{N}$. Furthermore,

$$\text{(13)}\qquad \begin{aligned} \Big|u^\varepsilon-\int_{-R}^{R} h_\varepsilon(t,x,\lambda)d\lambda\Big| &= \Big|\int_\lambda h_\varepsilon(t,x,\lambda)d\lambda-\int_{-R}^{R} h_\varepsilon(t,x,\lambda)d\lambda\Big| \\ &= \Big|\int_R^\infty h_\varepsilon(t,x,\lambda)d\lambda+\int_{-\infty}^{-R} h_\varepsilon(t,x,\lambda)d\lambda\Big| \\ &= H(u^\varepsilon-R)(u^\varepsilon-R)+H(-u^\varepsilon-R)(-u^\varepsilon-R). \end{aligned}$$

Thus by Lemma 3.1, we have that there exists constant $K_1>0$, so that

$$\text{(14)}\qquad \begin{aligned} &\int_0^t\int_{\mathbb{R}}[H(u^\varepsilon-R)(u^\varepsilon-R)+H(-u^\varepsilon-R)(-u^\varepsilon-R)]dxdt \\ &\leq \int_{|u^\varepsilon|>R}|u^\varepsilon|dxdt \leq \frac{1}{R}\int_0^t\int_x |u^\varepsilon|^2dxdt \leq \frac{K_1}{R}, \end{aligned}$$

since $\int_{|u^\varepsilon|>R} R|u^\varepsilon|dxdt \leq \int_{|u^\varepsilon|>R}|u^\varepsilon|^2dxdt < \tilde{K_1}$. Therefore, from (13) and (14) it follows

$$\text{(15)}\qquad \int_0^t\int_{\mathbb{R}}\Big|u^\varepsilon-\int_{-R}^{R} h_\varepsilon(t,x,\lambda)d\lambda\Big|dtdx \leq \frac{K_1}{R}$$

Now by (15) it is easy to prove that $(u^k)_k$ (where the indexing is taken from (12)) is a Cauchy sequence in $L^1_{\rm loc}(\Pi)$. Indeed, for every compact set $K\subset\subset\Pi$, we have

$$\begin{aligned} &\int_K |u^{k_1}-u^{k_2}|dxdt \\ &\leq \int_K\Big|u^{k_1}-\int_{-R}^{R} h_{k_1}(t,x,\lambda)d\lambda\Big|dxdt+\int_K\Big|u^{k_2}-\int_{-R}^{R} h_{k_2}(t,x,\lambda)d\lambda\Big|dxdt \\ &+\int_K\Big|\int_{-R}^{R} h_{k_1}(t,x,\lambda)d\lambda-\int_{-R}^{R} h_{k_2}(t,x,\lambda)d\lambda\Big|dxdt \leq \frac{2K_1}{R}+\gamma(k_1,k_2), \end{aligned}$$

where $\frac{2K_1}{R}$ appears due to (15), and γ is a function tending to zero as $k_i\to\infty$, $i=1,2$, because $(h_k)_k$ is convergent in $L^1_{\rm loc}(\Pi\times\mathbb{R})$. Thus, we see that the subsequence $(u^k)_k$ of $(u^\varepsilon)_\varepsilon$ is the Cauchy sequence in $L^1_{\rm loc}(\Pi)$. This implies that the family $(u^\varepsilon)_\varepsilon$ is pre-compact in $L^1_{\rm loc}(\Pi)$. ❑

Remark 4.1 *Notice that if $\delta = o(\varepsilon^2)$, $\varepsilon \to 0$, then $(u^k)_k$ tends to a unique entropy solution to (1). The proof is analogous to the one from [6].*

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