HYPERBOLIC CONSERVATION LAWS WITH VANISHING NONLINEAR DIFFUSION AND LINEAR DISPERSION IN HETEROGENEOUS MEDIA

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Abstract. We analyze family of solutions to multidimensional scalar conservation law, with flux depending on the time and space explicitly, regularized with vanishing diffusion and dispersion terms. Under a condition on the balance between diffusion and dispersion parameters, we prove that the family of solutions is precompact in $L^1_{\rm loc}$. Our proof is based on the methodology developed in [22], which is in turn based on Panov's extension [18] of Tartar's H-measures [26], or Gerard's micro-local defect measures [5]. This is new approach for the diffusion-dispersion limit problems. Previous results were restricted to scalar conservation laws with flux depending only on the state variable.

1. Introduction

Nonlinear hyperbolic conservation laws model many physical, mechanical and chemical phenomena. Some of well known examples are flow in porous media, sedimentation processes, traffic flow, blood flow. However, these phenomena typically occur in heterogenous media and therefore it is very important to investigate conservation laws involving flux explicitly depending on the position in space (x variable) and time (t variable). Accordingly, the subject of the paper is the following Cauchy problem for multidimensional scalar conservation law

$$\partial_t u(t,x) + {\rm div}_x f(t,x,u) = 0, \quad u(0,x) = u_0(x), \qquad x \in \mathbf{R}^d,\ t \in \mathbf{R}^+. \tag{1}$$

Still, the mentioned conservation law often describes only approximatively appropriate physical situation. More precisely, in order to simplify the model, terms like diffusion or dispersion are neglected since one can often assume that they have no essential influence on the considered process. On the other hand, Cauchy problem (1) admits discontinuous solutions, and it is well known that such solutions are not unique. Therefore, to obtain the information that is physically relevant, it is important to inspect which solution is selected by a specific zero diffusion-dispersion limit.

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Therefore, we consider the following family of problems

$$\partial_t u^{\varepsilon,\delta} + \mathrm{div}_x f(t,x,u^{\varepsilon,\delta}) = \varepsilon \sum_{j=1}^{d} \partial_{x_j} b_j(\nabla_x u^{\varepsilon,\delta}) + \delta \sum_{j=1}^{d} \partial_{x_j x_j x_j} u^{\varepsilon,\delta}, \tag{2}$$

$$u^{\varepsilon,\delta}(0,x) = u_0^{\varepsilon}(x), \quad x \in \mathbf{R}^d, \ \varepsilon,\delta > 0, \tag{3}$$

where the initial data u_0^ε from (3) converge to the initial datum u_0 from (1) strongly in $L^2(\mathbf{R}^d)$, as $\varepsilon \to 0$ (all terms from (1), (2) and (3) are precisely described in the next chapter). In order to obtain a weak entropy solution to (1) as a limit of a subsequence of the solutions to (2)-(3), we study precompactness properties of the family $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$. The goal is to obtain optimal diffusion-dispersion ratio, i.e. the optimal balance of the two parameters ε and δ.

In our investigation we were mainly inspired by the results of S. Hwang, [10], which are followed in the proof of the Theorem 1, and S. A. Sazhenkov, [22], which enable us to consider equation (1) in more general form than the one from [10].

Let us briefly recall already obtained results on the problem of diffusion–dispersion limit. Naturally, first results are obtained for flux independent on space and time. In [23], using compensated compactness argument, the author proves that the family of solutions to KdV-Burgers equation converges to a weak solution to Burgers equation if diffusion parameter ε and dispersion parameter δ are balanced in the sense that $\delta = \mathcal{O}(\varepsilon^2)$, as $\varepsilon \to 0$. Using the same methodology, in [15] the diffusion-dispersion problem is addressed for the case of flux in general form. The balance between two parameters obtained there is analogous to the result from [23].

Multidimensional homogeneous model, completely analogous to (2) (nonlinear diffusion, linear dispersion), was firstly introduced by J. M. Correia and P. G. LeFloch, [1]. Similar problem was considered in [12] but under less restrictive conditions on the relative size between diffusion and dispersion parameters. More precisely, it is proved in [12] that the family of solutions of a scalar conservation law perturbed by diffusion and dispersion converges to a unique entropy admissible weak solution (see [13]) of appropriate conservation law if the diffusion parameter ε predominates the dispersion parameter δ in the sense that $\delta = o(\varepsilon^2)$, as $\varepsilon \to 0$. To accomplish this, authors use the concept of measure valued solutions to conservation laws introduced by DiPerna [3]. Recent step forward with respect to the relative size of diffusion and dispersion parameters is made in [10] (see also [8, 9] for the similar results and methodology). Using the kinetic approach [21] and the averaging lemma [16, 21, 25], the author obtains the diffusion-dispersion limit if the diffusion parameter ε and the dispersion parameter δ are balanced in the sense that $\delta = \mathcal{O}(\varepsilon^2)$, as $\varepsilon \to 0$ (or more precisely $\delta = \mathcal{O}(\varepsilon^{\frac{r+3}{r+1}})$ for appropriate $r \geq 1$; see (H2) below). It is important to stress that in every of the previously mentioned works the flux corresponding to the considered conservation law *does not depend on space and time explicitly.* Thus, unlike the situation we have here, it is tacitly assumed that the authors are dealing with a process in a homogenous media.

Now, we shall analyze more closely the problem we are dealing with. If we assume that the relative size of ε and δ is weaker than in [10], then we can rely on [3] (as in [12]) to state that the family $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ of solutions to (2)-(3) converges to unique entropy admissible weak solution to (1). Also, if we consider one dimensional variant of (2)-(3) and assume that the relative size of ε and δ is the same as in [10], we can use compensated compactness, and even assume that the flux $f = f(t,x,\lambda)$

from (1) is discontinuous in $(t,x) \in \mathbf{R}^+ \times \mathbf{R}$ to obtain the convergence result (see [7]).

In order to obtain the results analogous to those from [10] (multidimensional space, optimal diffusion-dispersion ratio), similarly as there, we shall use kinetic formulation of the conservation law that we are considering (see also [2]).

Note that in [10] the author has used the averaging lemma for an appropriate transport equation [21]. But, only result concerning averaging lemmas for transport equations with a flux explicitly depending on space and time variable is the one from [5]. It is proved in [5] that for the sequence of solutions $(h_n)_n \in L^2(\mathbf{R}^d \times \mathbf{R})$ of transport equations

$$\mathrm{div}_x a(x,\lambda) h_n(x,\lambda) = \sum_{k=0}^{N} \partial_\lambda^k g_n^k(x,\lambda), \quad x \in \mathbf{R}^d,\ \lambda \in \mathbf{R},\ a \in C^1(\mathbf{R}^d \times \mathbf{R}),$$

and every $\rho \in C_0^1(\mathbf{R})$, the sequence of averaged quantities

$$\left(\int_{\mathbf{R}^d} \rho(\lambda) h_n(x,\lambda) d\lambda \right)_n \quad \text{is strongly precompact in } L^1_{\mathrm{loc}}(\mathbf{R}^d),$$

if for every $k = 1,...,N$, the sequences $(g_n^k(x,\lambda))_n$ are strongly precompact in $H^{-1}(\mathbf{R}^d \times \mathbf{R})$.

Furthermore, the result on velocity averaging given in [5] is weaker than the result from e.g. [21], which is used in [10]. Therefore, we need stronger precompactness result than the one from [5]. We will accomplish this by using results of [20] (H-measures), where the results from [5] are improved by the use of additional, more regular assumptions on the sequence $(h_n)_n$ (defined in (10)), whose convergence we want to prove (see also [5, 18, 19, 20, 26] for the H-measure techniques).

The first result involving diffusion-dispersion limits in heterogenous media is given in [7]. It is proved that for the flux $f = f(t,x,\lambda)$, $(t,x,\lambda) \in \mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}$ which is the Caratheodory vector (i.e. measurable in $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$ and continuous in λ) of locally bounded variation, the family of solutions of appropriate scalar conservation law perturbed by diffusion and dispersion parameter converges to a weak solution of appropriate conservation law if the diffusion parameter ε predominates the dispersion parameter δ in the sense that $\delta = o(\varepsilon^2)$, as $\varepsilon \to 0$.

We will improve the balance result from [7], but with the stronger assumptions on the flux (see (H3) in the next chapter).

The paper is organized as follows. In Section 2 we give basic notations, assumptions and the statement of the main theorem. In Section 3 we prove a priori inequalities for the family $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$. In Section 4 we prove necessary precompactness result for a family of solutions of appropriate transport equation. In Section 5 we show how to reduce (2) to transport equation (26). Then, we use results from Section 3 to prove that the family of solutions to (26), as well as (26) itself, satisfy conditions from Section 4 to conclude strong precompactness of the family $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$.

2. Notations, assumptions and the statement of the main result

In the sequel, for a vector valued function $g = (g_1, ..., g_d)$ defined on $\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}$, we denote

$$|g|^2 = \sum_{i=1}^{d} |g_i|^2.$$

The partial derivative D_{x_i} at the point (t,x,u), where u possibly depends on (t,x), is defined by the formula

$$D_{x_i} g(t,x,\lambda) = (\partial_{x_i} g(t,x,\lambda))|_{\lambda=u(t,x)}.$$

In particular, the full derivative ∂_{x_i} and the partial derivative D_{x_i} are connected by the identity

$$\partial_{x_i} g(t,x,u) = D_{x_i} g(t,x,u) + \partial_u g(t,x,u)\partial_{x_i} u.$$

Now, we list basic assumptions.

1. Assumption on the solutions $u^{\varepsilon,\delta}$ and the initial data:
We assume that $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ has enough regularity so that all formal computations below are correct. Moreover, we assume that $u^{\varepsilon,\delta}, \partial_{x_i} u^{\varepsilon,\delta}$ and $\partial_{x_i x_j} u^{\varepsilon,\delta}$ vanish as $|x| \to \infty$.

For the initial data we assume:

$$\begin{aligned} &u_0 \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d),\\ &u_0^\varepsilon \in H^1(\mathbf{R}^d) \text{ and } \nabla u_0^\varepsilon \in H^1(\mathbf{R}^d;\mathbf{R}^d), \text{ for every fixed } \varepsilon > 0,\\ &u_0^\varepsilon \to u_0 \text{ strongly in } L^2(\mathbf{R}^d) \cap L^1(\mathbf{R}^d), \text{ as } \varepsilon \to 0. \end{aligned}$$

Notice that from the last assumption $(u_0^\varepsilon)_\varepsilon$ is uniformly bounded in $L^2(\mathbf{R}^d)$.

2. Assumptions on the diffusion term $b = (b_1, ..., b_d) : \mathbf{R}^d \to \mathbf{R}^d$:
(H1) There exist $r \geq 1$ and constants C_1, C_2 such that

$$C_1|\lambda|^{1+r} \leq \lambda \cdot b(\lambda) \leq C_2|\lambda|^{1+r}, \text{ for all } \lambda \in \mathbf{R}^d.$$

(H2) The gradient matrix $Db(\lambda)$ is positively definite matrix uniformly in $\lambda \in \mathbf{R}^d$, i.e. there exists a positive constant C_3 such that

$$\xi^T Db(\lambda)\xi \geq C_3|\xi|^2, \text{ for all } \lambda, \xi \in \mathbf{R}^d.$$

3. Assumption on the flux vector $f = (f_1, ..., f_d) : \Pi \times \mathbf{R} \to \mathbf{R}^d$:
(H3) We assume that $f = f(t,x,u)$ and $f_u(t,x,u)$ are continuous and that they have locally integrable derivatives with respect to t and x. These assumptions enable us to make calculations in a priori estimates of the next section. Moreover, we assume:

1) If $r > 1$, appearing in (H1), then $\partial_u f \in L^{\frac{2(r+1)}{r-1}}(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})$; if $r = 1$, then $\partial_u f \in L^\infty(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})$,

2) $|D_{x_i} f_j(t,x,v)| \leq |\zeta_{i,j}(t,x)||v|$, for some $\zeta_{i,j} \in L^\infty(\mathbf{R}^+ \times \mathbf{R}^d)$, $i,j = 1,\dots,d$ and $D_{x_i} f_i \in L^1(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})$.

Remark 1. *The inspection of the proof of Lemma 2 shows that the integrability condition of $D_{x_i} f_i(t,x,v)$ with respect to v reflects the problem of uniform boundedness of the net of solutions u^ε, which would imply the integrability assumption of $D_{x_i} f_i$ with respect to (t,x). Still, it is an open problem to find a weaker condition than (H3).*

4. Genuine nonlinearity condition:
We will assume that vector function f is genuinely nonlinear, i.e. for every $\xi = (\xi_1, ...\xi_d) \in S^{d-1}$, and almost every $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$, the mapping

$$\lambda \mapsto \langle \xi\,,\, f_i(t,x,\lambda)\rangle \quad \text{is not affine in } \lambda \text{ on any nontrivial interval.} \tag{4}$$

The main result of the paper is the following theorem.

Theorem 1. *The family of smooth solutions* $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ *to Cauchy problem* (2)-(3) *is strongly precompact in* $L^1_{\rm loc}(\mathbf{R}^+\times\mathbf{R}^d)$ *if* ε *and* δ *from* (2) *are balanced in the sense that*

$$\delta=\mathcal{O}(\varepsilon^{\frac{r+3}{r+1}}) \text{ as } \varepsilon\to 0.$$

To prove the theorem, we shall combine approaches from [10] and [22]. First, we reduce equation (2) to a family of transport equations (26) and then we use techniques of H-measures from [22] to prove strong precompactness of family of solutions to (26).

Remark 2. *Before we start with the proof, we compare our result with the ones existing in the literature which we generalize (those are in* [1] *and* [10]*).*

Recall once again that model (2) *was introduced in* [1] *in the case of homogeneous flux, i.e.* $f=f(u):\mathbf{R}\to\mathbf{R}^d$. *There, it is proved that if* $\delta=o(\varepsilon^{\frac{r+3}{r+1}})$, $\varepsilon\to 0$, *then the family of solutions* $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ *converges in* $L^s((0,T);L^1_{\rm loc}(\mathbf{R}^d))$, *for all* $s<\infty$ *and* $T>0$, *to a unique entropy admissible solution* $u\in L^\infty((0,T);L^1(\mathbf{R}^d))$ *of* (1).

The same problem with the same assumptions as in [1] *was considered in* [10], *where is proved that if* $\delta=\mathcal{O}(\varepsilon^{\frac{r+3}{r+1}})$, *as* $\varepsilon\to 0$, *then the family of solutions* $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ *converges in* $L^s((0,T);L^1_{\rm loc}(\mathbf{R}^d))$ *for all* $s<\infty$ *and* $T>0$, *to a weak solution* $u\in L^\infty((0,T);L^1(\mathbf{R}^d))$ *of* (1). *Here, we obtain analogical result but for heterogeneous flux and by the use of a new approach.*

3. A priori inequalities

In this section, we shall determine a priori inequalities for the solutions to the problem (2)-(3). For the simplicity, in the sequel we shall write u^ε implying $u^{\varepsilon,\delta}$ and consider that $\varepsilon\in(0,1)$.

Lemma 2. *Under the assumptions (H1) and (H3), the family of solutions* $(u^\varepsilon)_\varepsilon$ *to* (2)-(3) *for every* $t\in[0,T]$ *satisfies the following inequality*

$$\int_{\mathbf{R}^d}|u^\varepsilon(t,x)|^2dx+\varepsilon\int_0^t\int_{\mathbf{R}^d}|\nabla u^\varepsilon(t',x)|^{r+1}dxdt'\leq c, \tag{5}$$

for some $c>0$ *that does not depend on* ε.

Proof: Let $\eta=\eta(u)$, $u\in\mathbf{R}$, be a smooth function. We multiply (2) by $\eta'(u^\varepsilon)$ and define $q=(q_1,...,q_n)$ as

$$q_i(t,x,u)=\int_0^u\eta'(v)\partial_v f_i(t,x,v)dv,\quad i=1,...,d.$$

Thus, (2) becomes

$$
\begin{aligned}
&\partial_t \eta(u^\varepsilon) + \mathrm{div}_x\, q_i(t,x,u^\varepsilon) \\
&\quad - \sum_{i=1}^d \int_0^{u^\varepsilon} D_{x_i}\partial_v f_i(t,x,v)\eta'(v)dv + \sum_{i=1}^d \eta'(u^\varepsilon) D_{x_i} f_i(t,x,u^\varepsilon) \\
&= \varepsilon \sum_{i=1}^d \partial_{x_i}\big(\eta'(u^\varepsilon) b_i(\nabla u^\varepsilon)\big) - \varepsilon \eta''(u^\varepsilon)\sum_{i=1}^d b_i(\nabla u^\varepsilon)\partial_{x_i} u^\varepsilon \\
&\quad + \delta \sum_{i=1}^d \partial_{x_i}\big(\eta'(u^\varepsilon)\partial_{x_i x_i} u^\varepsilon\big) - \frac{\delta}{2}\eta''(u^\varepsilon)\sum_{i=1}^d \partial_{x_i}(\partial_{x_i} u^\varepsilon)^2.
\end{aligned} \tag{6}
$$

Choosing here $\eta(u) = \frac{u^2}{2}$, integrating over $\Pi = [0,t)\times \mathbf{R}^d$, taking into account (H1) and partial integration, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^d} |u^\varepsilon(t,x)|^2 dx + \varepsilon C_1 \int_0^t \int_{\mathbf{R}^d} |\nabla u^\varepsilon(t',x)|^{1+r} dx dt' \\
&\leq^{(\mathrm{H1})} \frac{1}{2}\int_{\mathbf{R}^d} |u^\varepsilon(t,x)|^2 dx + \varepsilon \int_0^t \int_{\mathbf{R}^d} \nabla u^\varepsilon(t',x)\cdot b(\nabla u^\varepsilon(t',x)) dx dt' \\
&= \frac{1}{2}\int_{\mathbf{R}^d} |u_0^\varepsilon(x)|^2 dx + \sum_{j=1}^d \int_0^t \int_{\mathbf{R}^d} \int_0^{u^\varepsilon(t',x)} v\, \partial_v D_{x_j} f_j(t',x,v) dv dx dt' \\
&- \sum_{i=1}^d \int_0^t \int_{\mathbf{R}^d} u^\varepsilon D_{x_i} f_i(t',x,u^\varepsilon) dx dt' \\
&=^{\mathrm{p.i.}} \frac{1}{2}\int_{\mathbf{R}^d} |u_0^\varepsilon(x)|^2 dx - \int_0^t \int_{\mathbf{R}^d} \int_0^{u^\varepsilon(t',x)} \sum_{i=1}^d D_{x_i} f_i(t',x,v) dv dx dt'.
\end{aligned} \tag{7}
$$

Now (H3) immediately implies (5). ❑

Lemma 3. *Under the assumptions (H2) and (H3), for* $|D^2 u|^2 = \sum_{i,k=1}^d |\partial_{x_i x_k} u|^2$, *the family of solutions* $(u^\varepsilon)_\varepsilon$ *to* (2)-(3), *for every* $t \in [0,T]$, *satisfies the following inequality*

$$
\varepsilon^{\frac{r+3}{r+1}} \int_{\mathbf{R}^d} |\nabla u^\varepsilon(t,x)|^2 dx + \varepsilon^{\frac{2(r+2)}{r+1}} \int_0^t \int_{\mathbf{R}^d} |D^2 u^\varepsilon(t',x)|^2 dx dt' \leq c, \tag{8}
$$

for some $c > 0$ *that does not depend on* ε.

Proof: We differentiate (2) in x_k and multiply obtained expression by $\partial_{x_k} u^\varepsilon$. Integrating over $\mathbf{R}^d$, using partial integration and assumption (H2), and then summing in $k = 1, ..., d$ those expressions we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^d} \partial_t |\nabla u^\varepsilon|^2 dx - \sum_{k=1}^d \int_{\mathbf{R}^d} \nabla \partial_{x_k} u^\varepsilon \cdot \left(D_{x_k} f(t,x,u^\varepsilon) + \partial_u f \cdot \partial_{x_k} u^\varepsilon\right) dx \\
&= -\varepsilon \sum_{k=1}^d \int_{\mathbf{R}^d} (\nabla \partial_{x_k} u^\varepsilon)^T Db(\nabla u^\varepsilon) \nabla \partial_{x_k} u^\varepsilon dx \leq^{(\mathrm{H2})} -\varepsilon C_3 \sum_{k=1}^d \int_{\mathbf{R}^d} |\nabla \partial_{x_k} u^\varepsilon|^2 dx.
\end{aligned}
$$

Integrating this over $[0,t]$ and using Cauchy-Schwartz inequality we obtain

$$\begin{aligned}&\frac{1}{2}\int_{\mathbf{R}^d}|\nabla u^\varepsilon(\,\cdot\,,t)|^2dx+\varepsilon C_3\sum_{k=1}^d\int_0^t\int_{\mathbf{R}^d}|\nabla\partial_{x_k}u^\varepsilon|^2dxdt'\\&\leq\frac{1}{2}\int_{\mathbf{R}^d}|\nabla u_0^\varepsilon|^2dx\\&\quad+\sum_{k=1}^d\|\nabla\partial_{x_k}u^\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)}\|D_{x_k}f(\,\cdot\,,\,\cdot\,,u^\varepsilon)+\partial_uf\cdot\partial_{x_k}u^\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)}.\end{aligned}$$

Then, using the Young inequality (C_3 below is the same as above),

$$ab\leq\frac{C_3\varepsilon}{2}a^2+\frac{C_4}{\varepsilon}b^2,\quad a,b\in\mathbf{R},$$

for constants C_3,C_4 independent on ε, it follows

$$\begin{aligned}&\frac{1}{2}\int_{\mathbf{R}^d}|\nabla u^\varepsilon(\,\cdot\,,t)|^2dx+\varepsilon C_3\sum_{k=1}^d\int_0^t\int_{\mathbf{R}^d}|\nabla\partial_{x_k}u^\varepsilon|^2dxdt'\\&\leq\frac{1}{2}\int_{\mathbf{R}^d}|\nabla u_0^\varepsilon|^2dx+C_3\frac{\varepsilon}{2}\sum_{k=1}^d\int_0^t\int_{\mathbf{R}^d}|\nabla\partial_{x_k}u^\varepsilon|^2dxdt'\\&\quad+\frac{C_4}{\varepsilon}\int_0^t\int_{\mathbf{R}^d}\sum_{k=1}^d\Big|D_{x_k}f(t',x,u^\varepsilon(t',x))+\partial_uf\cdot\partial_{x_k}u^\varepsilon\Big|^2dxdt'.\end{aligned}\tag{9}$$

Now, we separate proof for $r>1$ and $r=1$. Using inequality $(a+b)^2\leq 2a^2+2b^2$, Hölder inequality and (H3), for $r>1$ we obtain

$$\begin{aligned}&\frac{1}{2}\int_{\mathbf{R}^d}|\nabla u^\varepsilon(\,\cdot\,,t)|^2dx+\frac{\varepsilon}{2}C_3\sum_{k=1}^d\int_0^t\int_{\mathbf{R}^d}|\nabla\partial_{x_k}u^\varepsilon|^2dxdt'\\&\leq\frac{1}{2}\int_{\mathbf{R}^d}|\nabla u_0^\varepsilon|^2dx+\frac{2C_4}{\varepsilon}\sum_{i,j=1}^d\|\zeta_{i,j}\|^2_{L^\infty(\Pi)}\int_0^t\int_{\mathbf{R}^d}|u^\varepsilon(t',x)|^2dxdt'\\&\quad+\frac{2C_4}{\varepsilon^{\frac{r+3}{r+1}}}\sum_{k=1}^d\|(\partial_uf)^2(t,x,u_\varepsilon(x,t))\|_{L^{\frac{r+1}{r-1}}(\mathbf{R}^+\times\mathbf{R}^d)}\cdot\left(\varepsilon\int_0^t\int_{\mathbf{R}^d}|\partial_{x_k}u^\varepsilon|^{r+1}(x,t')dxdt'\right)^{\frac{2}{r+1}}.\end{aligned}$$

By (5), it follows that $\int_0^t\int_{\mathbf{R}^d}|u^\varepsilon(t',x)|^2dxdt'$, $\varepsilon\int_0^t\int_{\mathbf{R}^d}|\nabla u(s,x)|^{r+1}dxds\leq C$, $t\in[0,T]$. Thus, after multiplying the former expression with $\varepsilon^{\frac{r+3}{r+1}}$, we obtain (8).

By the same arguments, in case when $r=1$, we multiply (9) by ε^2 and obtain

$$\begin{aligned}&\frac{\varepsilon^2}{2}\int_{\mathbf{R}^d}|\nabla u^\varepsilon(t,x)|^2dx+C_3\frac{\varepsilon^3}{2}\int_{\mathbf{R}^d}\int_0^t|D^2u^\varepsilon|^2dxdt\\&\leq\varepsilon^2C_5\int_{\mathbf{R}^d}|\nabla u_0^\varepsilon|^2dxdt'+C_7\|\partial_uf\|^2_{L^\infty(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R})}\\&\quad+\varepsilon C_6\int_0^t\int_{\mathbf{R}^d}\sum_{k=1}^d|D_{x_k}f(t',x,u^\varepsilon(t',x))|^2dxdt'\\&\leq\varepsilon^2C_5\|\nabla u_0^\varepsilon\|^2_{L^2(\mathbf{R}^d)}+C_7\|\partial_uf\|^2_{L^\infty}+\varepsilon C_6\sum_{i,j=1}^d\|\zeta_{i,j}\|^2_{L^\infty}\int_0^t\int_{\mathbf{R}^d}|u^\varepsilon(t',x)|^2dxdt',\end{aligned}$$

with appropriate constants. From (5) and the last inequality we obtain (8). ❑

4. Convergence results

To prove strong precompactness of the family $(u^\varepsilon)_\varepsilon$ of solutions to (2)-(3) we shall use the theory of H-measures [5, 26]. The following theorem is the corner stone of H-measures.

Theorem 4 ([26]). *If $(u^n)_{n\in\mathbf{N}}$ is a sequence in $L^2_{\rm loc}(\Omega;\mathbf{R}^r)$, $\Omega\subset\mathbf{R}^{d+1}$, such that $u^n\rightharpoonup 0$ in $L^2_{\rm loc}(\Omega)$, then there exists subsequence $(u^{n'})_{n'}\subset(u^n)_n$ and positive complex bounded measure $\mu=\{\mu^{jk}\}_{j,k=1,\dots,r}$ on $\mathbf{R}^{d+1}\times\mathbf{S}^d$ such that for all $\varphi_1,\varphi_2\in C_0(\Omega)$ and $\psi\in C(S^d)$,*

$$\lim_{n'\to\infty}\int_{\mathbf{R}^{d+1}}\mathcal{F}(\varphi_1 u^j_{n'})(\xi)\overline{\mathcal{F}(\varphi_2 u^k_{n'})(\xi)}\psi(\frac{\xi}{|\xi|})d\xi=\langle\mu^{jk},\varphi_1\bar\varphi_2\psi\rangle$$
$$=\int_{\mathbf{R}^{d+1}\times S^d}\varphi_1(x)\overline{\varphi_2(x)}\psi(\xi)d\mu^{jk}(x,\xi).$$

As we can see, the H-measure $\mu=\{\mu^{kj}\}_{k,j\in E}$, where E is a finite set, is defined for sequences $(u^n)_n=(u^n(x,k))_n$, $x\in\Omega$, $k\in E$, weakly converging to zero in $L^2(\Omega)$, for every $\lambda\in E$. The essential fact here is that E is a *finite set.* Using diagonal argument, an H-measure can be defined for a sequence $(u^n)_n=(u^n(\cdot,k))_n$ where $k\in E$ and E is a countable set.

On the other hand, if one assumes that the sequence $(u^n)_n=(u^n(\cdot,k))_n$ is defined e.g. for $k\in\mathbf{R}$, in general it is not possible to find family of measures $\{\mu^{kj}\}_{k,j\in E}$ so that the statement of the Theorem 4 holds, since $\mathbf{R}$ is an uncountable set. But, if we additionally assume that $(u^n)_n$ is uniformly continuous in $\lambda\in E\subset\mathbf{R}$, and that E is a subset of the full measure (meas$(\mathbf{R}\backslash E)=0$), then one can define an H measure $\mu=\{\mu^{kj}\}_{k,j\in E}$ so that Theorem 4 still holds. Indeed, one can choose a countable dense subset of E and define an H measure on that countable subset. Then, using the continuity argument, one extends this H measure for every $k,j\in E$. This fact was noticed and formalized in [18] and we will use it here.

To proceed, recall from the previous section that $(u^\varepsilon)_\varepsilon$ is uniformly bounded in $L^2(\Pi)$, $\Pi=[0,T)\times\mathbf{R}^d$, for every fixed $T\in\mathbf{R}^+$ (Lemma 2). Then there exists a subsequence $(u^k)_k$ and a Young measure $\nu=\{\nu_{t,x}\}$, $\nu_{t,x}\in{\rm Prob}(\mathbf{R})$, such that $\lim_{k\to\infty}f(u^k)=\int_{\mathbf{R}}f(\lambda)\,d\nu_{t,x}(\lambda)$ holds in the sense of distributions for all continuous functions $f(\lambda)=o(|\lambda|^2)$, when $|\lambda|\to\infty$, (cf. [23]). In this section we shall prove that the sequence $(h_k)_{k\in\mathbf{N}}$ of the form

$$h_k(t,x,\lambda)=\begin{cases}1, & \text{for }\ 0<\lambda\le u^k(t,x)\\ -1, & \text{for }\ 0>\lambda\ge u^k(t,x)\\ 0, & \quad\text{otherwise}\end{cases}, \tag{10}$$

satisfying the transport equation

$$\begin{split}&\partial_t h_k(t,x,\lambda)+\sum_{i=1}^d\partial_{x_i}(\partial_\lambda f_i(t,x,\lambda)h_k(t,x,\lambda))\\ &=\partial_\lambda m_k(t,x,\lambda)+\bar m_k(t,x,\lambda)+\sum_{i=1}^d\partial_\lambda\partial_{x_i}g_k(t,x,\lambda)+\sum_{i=1}^d\partial_{x_i}\bar g_k(t,x,\lambda),\end{split}\tag{11}$$

is precompact in $L^p_{\rm loc}(\Pi)$, $p \in [1,+\infty)$. Here $(g_k(t,x,\lambda))_k$ and $(\bar g_k(t,x,\lambda))_k$ are precompact in $L^{1+\frac{1}{r}}(\Pi)$, uniformly in $\lambda \in \mathbf{R}$, i.e. there exist functions $\bar g(\,\cdot\,,\,\cdot\,,\lambda)$ and $g(\,\cdot\,,\,\cdot\,,\lambda)$ in $L^{1+\frac{1}{r}}(\Pi)$ such that, along a subsequence,

$$\begin{aligned}&\sup_{\lambda\in\mathbf{R}} \|g_k(\,\cdot\,,\,\cdot\,,\lambda) - g(\,\cdot\,,\,\cdot\,,\lambda)\|_{L^{1+\frac{1}{r}}(\Pi)} \to 0, \quad \text{as} \quad k\to\infty,\\ &\sup_{\lambda\in\mathbf{R}} \|\bar g_k(\,\cdot\,,\,\cdot\,,\lambda) - \bar g(\,\cdot\,,\,\cdot\,,\lambda)\|_{L^{1+\frac{1}{r}}(\Pi)} \to 0, \quad \text{as} \quad k\to\infty.\end{aligned} \tag{12}$$

Furthermore, we assume that $(m_k)_{k\in\mathbf{N}}$, $(\bar m_k)_{k\in\mathbf{N}}$ are sequences of locally bounded measures on $\Pi\times\mathbf{R}_\lambda$. We use notation $m_k, \bar m_k \in \mathcal{M}_{\rm loc}(\Pi\times\mathbf{R}_\lambda)$, $k\in\mathbf{N}$. From the assumption on local boundedness of $(m_k)_{k\in\mathbf{N}}$, $(\bar m_k)_{k\in\mathbf{N}}$ we conclude that there exist measures $m, \bar m \in \mathcal{M}_{\rm loc}(\Pi\times\mathbf{R}_\lambda)$ such that (up to a subsequence)

$$m_k \rightharpoonup m \quad \text{and} \quad \bar m_k \rightharpoonup \bar m, \quad \text{weakly}-\star \text{ in } \mathcal{M}_{\rm loc}(\Pi\times\mathbf{R}_\lambda), \quad \text{as} \quad k\to\infty, \tag{13}$$

see [4]. Due to the uniform boundedness of the sequence $(h_k)_k$, there exists $h \in L^\infty(\Pi\times\mathbf{R}_\lambda)$, such that (along a subsequence)

$$h_k \rightharpoonup h, \quad \text{weakly}-\star \text{ in } L^\infty(\Pi\times\mathbf{R}_\lambda), \quad \text{as} \quad k\to\infty. \tag{14}$$

On the other hand, if $\phi_2 \in C^\infty(\mathbf{R}_\lambda)$, then

$$\int_{\mathbf{R}_\lambda} \phi_2' h_k\, d\lambda = \phi_2(u^k(t,x)) \rightharpoonup \int_{\mathbf{R}_\lambda} \phi_2\, d\nu_{t,x}(\lambda), \quad \text{in } \mathcal{D}'(\Pi), \quad \text{as} \quad k\to\infty. \tag{15}$$

Using the notion of the Stieltjes parameterized measure, we can write $\nu_{t,x}(\lambda) = \partial_\lambda g(t,x,\lambda)$, where $g(t,x,\lambda) = \int_{\mathbf{R}_\lambda} \chi_{\{s:s\leq\lambda\}} d\nu_{t,x}(s)$, $(t,x)\in\Pi$, is the distribution function of the Young measure ν. Then, we can rewrite the limit in (15) as

$$\int_{\mathbf{R}_\lambda} \phi_2' h_k\, d\lambda \rightharpoonup -\int_{\mathbf{R}_\lambda} \phi_2'(\lambda) g(t,x,\lambda)\, d\lambda, \quad k\to\infty,$$

and conclude that $h(t,x,\lambda) = -g(t,x,\lambda)$, $(t,x)\in\Pi$, $\lambda\in\mathbf{R}$. The limiting function h in (14) is a monotone function in λ, for every $(t,x)\in\Pi$. Therefore, the set of discontinuity points λ of the function h, i.e. the complement of the set

$$\mathcal{E} := \{\lambda_o \in \mathbf{R} \,|\, h(\cdot,\cdot,\lambda) \to h(\cdot,\cdot,\lambda_0), \text{ strongly in } L^1_{\rm loc}(\Pi), \text{ as } \lambda\to\lambda_0\},$$

is countable, at most. So, for any fixed $\lambda\in\mathcal{E}$,

$$h_k(\cdot,\cdot,\lambda) \rightharpoonup h(\cdot,\cdot,\lambda), \quad \text{weakly}-\star \quad \text{in } L^\infty(\Pi).$$

Moreover, the sequence $\big(U_k^\lambda(\cdot,\cdot) \equiv h_k(\cdot,\cdot,\lambda) - h(\cdot,\cdot,\lambda)\big)_{\lambda\in\mathcal{E}}$, $k\in\mathbf{N}$, defines an H-measure μ (cf. [18, Lemma 4]), i.e. there exists an H-measure $\{\mu^{pq}\}_{p,q\in\mathcal{E}}$ on $\Pi\times S^d$ such that for arbitrary $\phi_1,\phi_2 \in C_0(\Pi)$ and $\psi\in C(S^d)$

$$\begin{aligned}&\int_{\Pi\times S^d} \phi_1(t,x)\bar\phi_2(t,x)\psi(y)\, d\mu^{pq}(t,x,y) =\\ &= \lim_{k\to\infty} \int_{\mathbf{R}^{d+1}} \mathcal{F}[\phi_1 U_k^p](\xi)\overline{\mathcal{F}[\phi_2 U_k^q](\xi)}\psi(\frac{\xi}{|\xi|})\, d\xi\end{aligned}$$

holds for every $p,q\in\mathcal{E}$, where $\mathcal{F}$ is the Fourier transform with respect to (t,x) variable.

To employ techniques of H-measures, we shall also need some facts concerning multipliers, and the special ones - the Riesz potentials, [24]. The Riesz potential $\mathcal{J}_\alpha$, $0<\alpha<d$, is defined by the formula

$$\mathcal{F}[\mathcal{J}_\alpha[\varphi]](\xi) = (2\pi|\xi|)^{-\alpha}\mathcal{F}[\varphi](\xi), \quad \varphi\in C_0^\infty(\mathbf{R}^{d+1}).$$

The Riesz potential $\mathcal{J}_1$ is characterized by the following lemma.

Lemma 5. *If $p > d$ the Riesz potential $\mathcal{J}_1$ is a compact operator from $L^p(\mathbf{R}^d)$ to $C(\mathbf{R}^d)$. If $1 < p \leq d$ the Riesz potential $\mathcal{J}_1$ is a compact operator from $L^p(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$ for an arbitrary $q \in [1, pd(d-p)^{-1}]$.*

A multiplier $\mathcal{A}$ with a symbol $\psi \in C(S^d)$ is defined by the formula

$$\mathcal{F}[\mathcal{A}[\varphi]](\xi) = \psi(\xi/|\xi|)\mathcal{F}[\varphi](\xi).$$

The multiplier $\mathcal{R}_j$, $j = 0, ..., d$, with symbol $i\xi_j/|\xi|$ is called the Riesz transform. Recall [24],

$$(\mathcal{J}_\alpha \circ \mathcal{J}_\beta)[\varphi] = \mathcal{J}_{\alpha+\beta}[\varphi]$$
$$\mathcal{J}_1[\partial_{x_j}\varphi] = \mathcal{R}_j[\varphi], \quad j = 0, ..., d, \ x_0 := t.$$

We provide basic properties of multipliers, [24]. As a consequence of the Hörmander-Mikhlin theorem, [17], the following proposition, given in [24, Sect. 3.2, Example 2], is important.

Proposition 6. *For every $p \in (1, \infty)$,*

$$\|\mathcal{A}[\varphi]\|_{L^p(\mathbf{R}^d)} \leq c_p\|\varphi\|_{L^p(\mathbf{R}^d)} \quad \forall\varphi \in L^p(\mathbf{R}^d), \tag{16}$$

where $\mathcal{A}$ is a multiplier with a symbol $\psi \in C^\kappa(S^{d-1})$ and $\mathbf{N} \ni \kappa > \frac{d}{2}$.

Now we give the main property of the H-measure μ.

Theorem 7. *The H-measure μ satisfies the following integral identity:*

$$\int_{\mathbf{R}_\lambda}\Big(\int_{\Pi\times S^d}\Big(y_0 + \sum_{i=1}^d \partial_\lambda f_i(t,x,\lambda)y_i\Big)\beta(t,x,\lambda,y)d\mu^{\lambda\lambda}(t,x,y)\Big)d\lambda = 0, \tag{17}$$

for every $\beta \in C_0(\Pi \times \mathbf{R}_\lambda; C(S^d_y))$.

Proof: We consider equation (11) in $\mathcal{D}'(\Pi \times \mathbf{R}_\lambda)$. So for every $\theta \in C^2_0(\Pi \times \mathbf{R}_\lambda)$,

$$\begin{aligned}
&\int_{\Pi\times\mathbf{R}_\lambda}\Big(\theta_t + \sum_{i=1}^d\Big(\theta_{x_i}\partial_\lambda f_i(t,x,\lambda)\Big)\Big)h_k(t,x,\lambda)\,dxdtd\lambda\\
&-\int_{\Pi\times\mathbf{R}_\lambda}\theta_\lambda dm_k(t,x,\lambda) + \int_{\Pi\times\mathbf{R}_\lambda}\theta\, d\bar{m}_k(t,x,\lambda)+\\
&+\int_{\Pi\times\mathbf{R}_\lambda} g_k(t,x,\lambda)\sum_{i=1}^d\theta_{\lambda x_i}\,dxdtd\lambda - \int_{\Pi\times\mathbf{R}_\lambda}\bar{g}_k(t,x,\lambda)\sum_{i=1}^d\theta_{x_i}\,dxdtd\lambda = 0,
\end{aligned}\tag{18}$$

Using (12), (13) and (14), from (18) it follows

$$\begin{aligned}
&\int_{\Pi\times\mathbf{R}_\lambda} U_k^\lambda(t,x)\Big(\theta_t + \sum_{i=1}^d\theta_{x_i}\partial_\lambda f_i(t,x,\lambda)\Big)\,dxdtd\lambda\\
&\qquad -\int_{\Pi\times\mathbf{R}_\lambda}\theta_\lambda dM_k(t,x,\lambda) + \int_{\Pi\times\mathbf{R}_\lambda} G_k(t,x,\lambda)\sum_{i=1}^d\theta_{\lambda\, x_i}\,dxdtd\lambda\\
&\qquad +\int_{\Pi\times\mathbf{R}_\lambda}\theta\, d\bar{M}_k(t,x,\lambda) - \int_{\Pi\times\mathbf{R}_\lambda}\bar{G}_k(t,x,\lambda)\sum_{i=1}^d\theta_{x_i}\,dxdtd\lambda = 0,
\end{aligned}\tag{19}$$

where

$$U_k^\lambda(t,x) := h_k(t,x,\lambda) - h(t,x,\lambda),$$
$$M^k(t,x,\lambda) := m^k(t,x,\lambda) - m(t,x,\lambda), \quad \bar{M}^k(t,x,\lambda) := \bar{m}^k(t,x,\lambda) - \bar{m}(t,x,\lambda),$$
$$G_k(t,x,\lambda) = g_k(t,x,\lambda) - g(t,x,\lambda), \quad \bar{G}_k(t,x,\lambda) = \bar{g}_k(t,x,\lambda) - \bar{g}(t,x,\lambda).$$

Now multiplying (19) with $\int_{\mathbf{R}_p} \phi_0(p)\,dp$, where $\phi_0 \in C_0^2(\mathbf{R})$, and knowing that the set $\{\theta(t,x,\lambda)\cdot\phi_0(p) : \theta \in C_0^2(\Pi\times\mathbf{R}_\lambda), \phi_0 \in C_0^2(\mathbf{R}_p)\}$ is dense in $C_0^2(\Pi\times\mathbf{R}_\lambda\times\mathbf{R}_p)$, we obtain that

$$\begin{aligned}
&\int_{\Pi\times\mathbf{R}^2_{\lambda,p}} U_k^\lambda(t,x)\Big(\phi_t + \sum_{i=1}^d \phi_{x_i}\partial_\lambda f_i(t,x,\lambda)\Big)\,dxdtd\lambda dp - \\
&- \int_{\mathbf{R}_p}\int_{\Pi\times\mathbf{R}_\lambda} \phi_\lambda\, dM_k(t,x,\lambda)\,dp + \int_{\Pi\times\mathbf{R}^2_{\lambda,p}} G_k(t,x,\lambda)\sum_{i=1}^d \phi_{\lambda\, x_i}\,dxdtd\lambda dp + \\
&+ \int_{\mathbf{R}_p}\int_{\Pi\times\mathbf{R}_\lambda} \phi\, d\bar{M}_k(t,x,\lambda)dp - \int_{\Pi\times\mathbf{R}^2_{\lambda,p}} \bar{G}_k(t,x,\lambda)\sum_{i=1}^d \phi_{x_i}\,dxdtd\lambda dp = 0,
\end{aligned} \tag{20}$$

holds for every $\phi \in C_0^2(\Pi\times\mathbf{R}^2_{\lambda,p})$.

Following [22], in the rest of the proof we will substitute ϕ in (20) by suitable test functions. So let $\phi \in C_0^1(\Pi\times\mathbf{R}^2_{\lambda,p})$ be of the form

$$\phi(t,x,\lambda,p) := \phi_1(t,x,\lambda,p)\cdot(\mathcal{J}_1\circ\mathcal{A})[\phi_2\cdot U_k^p](t,x), \tag{21}$$

where $\phi_1 \in C_0^2(\Pi\times\mathbf{R}^2_{\lambda,p})$, $\phi_2 \in C_0^2(\Pi)$, $\mathcal{J}_1$ is the Riesz's potential and $\mathcal{A}$ is a multiplier on $\mathbf{R}^{d+1}$ with a symbol $\psi \in C^\kappa(S^d)$, $\mathbf{N}\ni\kappa > d/2$.

So, replace ϕ in (20) with the one from (21). Since the multipliers on $C^\kappa(S^d)$ commute mutually and with partial derivatives, and $\mathcal{J}_1[\partial_{x_j}\phi] = \mathcal{R}_j[\phi]$ holds for $j = 0,...,d$, $x_0 \equiv t$, from (20) we obtain,

$$\begin{aligned}
0 = &\int_{\Pi\times\mathbf{R}^2}\Big(\phi_{1_t}(\mathcal{J}_1\circ\mathcal{A})[\phi_2 U_k^p] + \phi_1(\mathcal{A}\circ\mathcal{R}_0)[\phi_2 U_k^p]\Big)U_k^\lambda\,dxdtd\lambda dp \\
&+ \int_{\Pi\times\mathbf{R}^2}\sum_{i=1}^d\Big(\phi_{1_{x_i}}(\mathcal{J}_1\circ\mathcal{A}) + \phi_1(\mathcal{A}\circ\mathcal{R}_i)\Big)[\phi_2 U_k^p]U_k^\lambda f_{i_\lambda}\,dxdtd\lambda dp \\
&- \int_{\mathbf{R}_p}\int_{\Pi\times\mathbf{R}_\lambda} \phi_{1_\lambda}(\mathcal{J}_1\circ\mathcal{A})[\phi_2 U_k^p]\,dM_k(t,x,\lambda)\,dp \\
&+ \int_{\mathbf{R}_p}\int_{\Pi\times\mathbf{R}_\lambda} \phi_1(\mathcal{J}_1\circ\mathcal{A})[\phi_2 U_k^p]\,d\bar{M}_k(t,x,\lambda)\,dp \\
&- \int_{\Pi\times\mathbf{R}^2} \bar{G}_k(t,x,\lambda)\sum_{i=1}^d\Big(\phi_{1_{x_i}}(\mathcal{J}_1\circ\mathcal{A}) + \phi_1(\mathcal{A}\circ\mathcal{R}_i)\Big)[\phi_2 U_k^p]\,dxdtd\lambda dp \\
&+ \int_{\Pi\times\mathbf{R}^2} G_k\sum_{i=1}^d\Big(\phi_{1_{\lambda\, x_i}}(\mathcal{J}_1\circ\mathcal{A})[\phi_2 U_k^p] + \phi_{1_\lambda}(\mathcal{A}\circ\mathcal{R}_i)[\phi_2 U_k^p]\Big)\,dxdtd\lambda dp.
\end{aligned} \tag{22}$$

We are ready now to pass to the limit as $k\to\infty$. Using the compactness of the Riesz potential as a mapping from $L^p(\mathbf{R}^{d+1})$ to $C(\mathbf{R}^{d+1})$ for $p > d+1$ (cf. Lemma

5) we obtain that

$$\lim_{k\to\infty}\Big[\int_{\Pi\times\mathbf{R}^2}(\phi_1 U_k^\lambda(\mathcal{A}\circ\mathcal{R}_0)[\phi_2 U_k^p])\,dxdtd\lambda dp + \int_{\Pi\times\mathbf{R}^2}\sum_{i=1}^d(\phi_1 U_k^\lambda f_{i_\lambda}(\mathcal{A}\circ\mathcal{R}_i)[\phi_2 U_k^p])\,dxdtd\lambda dp\Big] = 0. \tag{23}$$

Indeed, every term from (22) containing $\mathcal{J}_1$ converges to zero, while for the last two terms from (22) we have

$$\begin{aligned}&\Big|\int_{\Pi\times\mathbf{R}^2} G_k\sum_{i=1}^d \phi_{1_\lambda}(\mathcal{A}\circ\mathcal{R}_i)[\phi_2 U_k^p]\,dxdtd\lambda dp\Big|\\ &\leq \sum_{i=1}^d \|G_k\|_{L^{1+\frac{1}{r}}(\mathbf{R}^{d+1})}\|\phi_{1_\lambda}(\mathcal{A}\circ\mathcal{R}_i)[\phi_2U_k^p]\|_{L^\beta(\mathbf{R}^{d+1})}\\ &\leq^{(16)} C\|\phi_{1_\lambda}\|_{L^\infty(\Pi\times\mathbf{R}^2)}\|G_k\|_{L^{1+\frac{1}{r}}(\Pi)}\mathrm{meas}(\mathrm{supp}\phi_1)\|\phi_2U_k^p\|_{L^\beta(\Pi)}\to^{(12)} 0,\end{aligned}$$

where $\frac{1}{1+\frac{1}{r}}+\frac{1}{\beta}=1$. Similar estimate holds for the term

$$\int_{\Pi\times\mathbf{R}^2}\bar{G}_k\sum_{i=1}^d\phi_1(\mathcal{A}\circ\mathcal{R}_i)[\phi_2U_k^p]\,dxdtd\lambda dp.$$

We use now the following representation of H-measures via multipliers

$$\int_{\Pi\times S^d}\phi_1\phi_2\psi\,d\mu^{pq}(t,x,y) = \lim_{k\to\infty}\int_\Pi(\phi_1U_k^p)\mathcal{A}[\phi_2U_k^q]\,dxdt, \tag{24}$$

where $\mathcal{A}$ is the multiplier with the symbol $\psi\in C^\kappa(S^d)$, $\mathbf{N}\ni\kappa>d/2$, (cf. [5, 22]). Applying (24) in (23) we obtain that

$$\int_{\mathbf{R}^2}\int_{\Pi\times S^d}\phi_1\phi_2\psi\Big(y_0+\sum_{i=1}^d f_{i_\lambda}y_i\Big)\,d\mu^{\lambda p}(t,x,y)d\lambda dp=0,$$

where $y=\xi/|\xi|\in S^d$. We replace $\phi_1(t,x,\lambda,p)\phi_2(t,x)\psi(y)\in C_0^2(\Pi\times\mathbf{R}^2;C(S_y^d))$ by a test function

$$\frac{1}{\varepsilon}\phi_5(t,x,y)\phi_6\Big(\frac{\lambda-p}{\varepsilon}\Big)\phi_7\Big(\frac{\lambda+p}{2}\Big),$$

(see also [13]) where ϕ_6 is even with the unit mean value, ϕ_6, $\phi_7\in C_0^2(\mathbf{R})$ and $\phi_5\in C_0^2(\Pi;C(S_y^d))$. Then, changing variable $p=\varepsilon\kappa+\lambda$, and passing to limit $\varepsilon\to0$, we obtain that

$$\int_{\mathbf{R}_\lambda}\int_{\Pi\times S^d}\phi_5(t,x,y)\phi_7(\lambda)\Big(y_0+\sum_{i=1}^d f_{i_\lambda}y_i\Big)\,d\mu^{\lambda\lambda}(t,x,y)d\lambda=0,$$

provided that the mapping $(p,q)\mapsto\mu^{pq}$ is continuous from $\mathcal{E}\times\mathcal{E}\to\mathcal{M}(\Pi\times S^d)$, [18, Theorem3, 2)] . The proof is finished by the fact that test functions of the form $\phi_5(t,x,y)\phi_7(\lambda)$ are dense in $C_0^2(\Pi\times\mathbf{R}_\lambda;C(S_y^d))$. ❑

Corollary 8 (the localization principle)**.** *The support of the H-measure $\mu^{\lambda\lambda}$ belongs to the set*

$$\Big\{(t,x,y)\in\Pi\times S^d\,:\,y_0+\sum_{i=1}^d\partial_\lambda f_i(t,x,\lambda)y_i=0\Big\},$$

for almost every $\lambda \in \mathbf{R}$.

Proof: We will use an idea from [22] to provide the integrand in (17) to be nonnegative. Therefor we put

$$\beta(t,x,\lambda,y)=\Big(y_0+\sum_{i,j=1}^{d}\partial_\lambda f_i(t,x,\lambda)y_i\Big)\beta_1^2(t,x,\lambda),$$

where $\beta_1 \in C_0(\Pi\times\mathbf{R}_\lambda)$ is arbitrary chosen. Inserting that test function into (17) and using arbitrariness of β_1 we complete the proof. ❑

Corollary 9. *Assume that the flux vector is genuinely nonlinear (see* (4)*). Then, the sequence* $(h_k(t,x,\lambda))_k$ *is strongly precompact in* $L^p_{\rm loc}(\Pi\times\mathbf{R})$*,* $p\in[1,+\infty)$*.*

Proof: From Corollary 8 and the genuine nonlinearity condition (4), we have that meas(supp$\mu^{\lambda\lambda}$) $=0$, i.e. $\mu^{\lambda\lambda}=0$ for a.e. $\lambda\in\mathbf{R}$. According to the theory of the H-measures, [5, 26], we have that $\mu^{\lambda\lambda}=0$ for a.e. $\lambda\in\mathbf{R}$ if and only if $h^k(\cdot,\cdot,\lambda)\to h(\cdot,\cdot,\lambda)$, $k\to\infty$, strongly in $L^2_{\rm loc}(\Pi)$, for $\lambda\in\mathcal{E}$. Then, using Lebegue dominated convergence theorem we conclude that $h^k(\cdot,\cdot,\lambda)\to h(\cdot,\cdot,\lambda)$, $k\to\infty$, strongly in $L^p_{\rm loc}(\Pi)$ for every $p\in[1,+\infty)$, too, due to boundedness of the sequence $(h_k)_k$. ❑

5. Main theorem

In this section we shall prove Theorem 1. We will follow the procedure from [10]. At the crucial point in the proof (the *Step 3* below), we will apply Corollary 9 that we have proved in the previous section using techniques developed in [22].

Proof of the Theorem 1: We divide the proof into three steps.

Step 1: Let $\eta\in C_0^\infty(\mathbf{R})$. For the functions

$$h_\varepsilon(t,x,\lambda)=\begin{cases}1, & \text{for}\ \ 0<\lambda\le u^\varepsilon(t,x),\\ -1, & \text{for}\ \ 0>\lambda\ge u^\varepsilon(t,x),\\ 0, & \quad \text{otherwise},\end{cases}$$

we can rewrite (6) as

$$\begin{aligned}
&\partial_t\int_{\mathbf{R}_\lambda}h_\varepsilon(t,x,\lambda)\eta'(\lambda)d\lambda+\sum_{i=1}^d\partial_{x_i}\int_{\mathbf{R}_\lambda}h_\varepsilon(t,x,\lambda)\partial_\lambda f_i(t,x,\lambda)\eta'(\lambda)d\lambda\\
&\quad+\sum_{i=1}^d\int_{\mathbf{R}_\lambda}h_\varepsilon(t,x,\lambda)D_{x_i\lambda}f_i(t,x,\lambda)\eta''(\lambda)d\lambda\\
&=\varepsilon\sum_{i=1}^d\partial_{x_i}(\eta'(u^\varepsilon)b_i(\nabla u^\varepsilon))-\varepsilon\eta''(u^\varepsilon)\sum_{i=1}^d b_i(\nabla u^\varepsilon)u^\varepsilon_{x_i}\\
&\quad+\delta\sum_{i=1}^d\partial_{x_i}(\eta'(u^\varepsilon)\partial_{x_ix_i}u^\varepsilon)-\delta\eta''(u^\varepsilon)\sum_{i=1}^d\partial_{x_i}\partial_{x_ix_i}u^\varepsilon u^\varepsilon_{x_i}.
\end{aligned}$$

Testing the last relation on an arbitrary $\varphi \in C_0^\infty(\Pi)$ we obtain

$$
\begin{aligned}
&-\int_{\Pi\times\mathbf{R}} h_\varepsilon(t,x,\lambda)\eta'(\lambda)\varphi_t(t,x)\,d\lambda dxdt\\
&-\sum_{i=1}^d\int_{\Pi\times\mathbf{R}} h_\varepsilon(t,x,\lambda)\partial_\lambda f_i(t,x,\lambda)\eta'(\lambda)\varphi_{x_i}(t,x)\,d\lambda dxdt\\
&+\sum_{i=1}^d\int_{\Pi\times\mathbf{R}} h_\varepsilon(t,x,\lambda)D_{x_i\lambda} f_i(t,x,\lambda)\eta''(\lambda)\varphi(t,x)d\lambda dxdt\\
&=-\int_\Pi\sum_{i=1}^d\left(\varepsilon b_i(\nabla u^\varepsilon)+\delta\partial_{x_ix_i}u^\varepsilon\right)\eta'(u^\varepsilon)\varphi_{x_i}(t,x)dxdt\\
&-\sum_{i=1}^d\int_\Pi\left(\varepsilon b_i(\nabla u^\varepsilon)u^\varepsilon_{x_i}+\delta u^\varepsilon_{x_i}\partial_{x_ix_i}u^\varepsilon\right)\eta''(u^\varepsilon)\varphi(t,x)dxdt,
\end{aligned}
\tag{25}
$$

where $\int_{\Pi\times\mathbf{R}}$ and $\int_\Pi$ denote integrals over $\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R}$ and $\mathbf{R}^+\times\mathbf{R}^d$, respectively.

As in [10], we represent equation (25) as an equation in the sense of distributions $\mathcal{D}'(\Pi\times\mathbf{R})$. Put

$$
\begin{aligned}
H_i^\varepsilon(t,x)&=\varepsilon b_i(\nabla u^\varepsilon),\quad \bar H_i^\varepsilon(t,x)=\delta\partial_{x_ix_i}u^\varepsilon,\\
G_i^\varepsilon(t,x)&=\varepsilon b_i(\nabla u^\varepsilon)u^\varepsilon_{x_i},\quad \bar G_i^\varepsilon(t,x)=\delta u^\varepsilon_{x_i}\partial_{x_ix_i}u^\varepsilon,
\end{aligned}
$$

and note that nets $(H_i^\varepsilon)_\varepsilon,(\bar H_i^\varepsilon)_\varepsilon,(G_i^\varepsilon)_\varepsilon,(\bar G_i^\varepsilon)_\varepsilon$ are uniformly bounded in $L^1_{\rm loc}(\Pi\times\mathbf{R})$ (cf. (H1)-(H3) and Lemmas 2-3).

Let $\delta(\lambda-u)$ be a Dirac delta function defined by $\langle\delta(\lambda-u),\eta(\lambda)\rangle=\eta(u)$. Then, the functionals

$$
\begin{aligned}
m_i^\varepsilon&=\delta(\lambda-u^\varepsilon)G_i^\varepsilon,\quad k_i^\varepsilon=\delta(\lambda-u^\varepsilon)\bar G_i^\varepsilon,\\
\pi_i^\varepsilon&=\delta(\lambda-u^\varepsilon)H_i^\varepsilon,\quad \bar\pi_i^\varepsilon=\delta(\lambda-u^\varepsilon)\bar H_i^\varepsilon,\quad i=1,...,d,
\end{aligned}
$$

are defined as distributions in $\mathcal{D}'(\Pi\times\mathbf{R})$ via the following tensor products:

$$
\begin{aligned}
\langle m_i^\varepsilon,\varphi\otimes\eta'\rangle&=\int_\Pi G_i^\varepsilon(t,x)\varphi(t,x)\eta'(u^\varepsilon(t,x))dxdt,\\
\langle k_i^\varepsilon,\varphi\otimes\eta'\rangle&=\int_\Pi \bar G_i^\varepsilon(t,x)\varphi(t,x)\eta'(u^\varepsilon(t,x))dxdt,\\
\langle \pi_i^\varepsilon,\varphi\otimes\eta'\rangle&=\int_\Pi H_i^\varepsilon(t,x)\varphi(t,x)\eta'(u^\varepsilon(t,x))dxdt,\\
\langle \bar\pi_i^\varepsilon,\varphi\otimes\eta'\rangle&=\int_\Pi \bar H_i^\varepsilon(t,x)\varphi(t,x)\eta'(u^\varepsilon(t,x))dxdt.
\end{aligned}
$$

Indeed, since the mapping $\eta(\lambda)\mapsto G_i^\varepsilon(t,x)\eta'(u^\varepsilon(t,x))$ is continuous, this presentation follows from the Schwartz kernel theorem. Thus, (25) can be rewritten as equation in $\mathcal{D}'(\Pi\times\mathbf{R})$ as follows

$$
\begin{aligned}
&\partial_t h_\varepsilon(t,x,\lambda)+\sum_{i=1}^d\partial_{x_i}(h_\varepsilon(t,x,\lambda)\partial_\lambda f_i(t,x,\lambda))=\\
&\sum_{i=1}^d\partial_\lambda(h_\varepsilon(t,x,\lambda)D_{x_i}f_i(t,x,\lambda))+\sum_{i=1}^d\left(\partial_{x_i}(\pi_i^\varepsilon+\bar\pi_i^\varepsilon)+\partial_\lambda(m_i^\varepsilon+k_i^\varepsilon)\right).
\end{aligned}
\tag{26}
$$

Step 2 In this step we estimate terms on the right-hand side of (26). Relying upon Lemmas 2-3 (as in [10]), we obtain the following results:

$$\pi_i^\varepsilon = \bar{g}_i^\varepsilon + \partial_\lambda g_i^\varepsilon, \quad \bar{\pi}_i^\varepsilon = \bar{p}_i^\varepsilon + \partial_\lambda p_i^\varepsilon, \quad i = 1, ..., d, \tag{27}$$

with

$$\bar{g}_i^\varepsilon, g_i^\varepsilon \to 0 \ \text{ as } \ \varepsilon \to 0 \ \text{ in } \ L^{\frac{r+1}{r}}(\Pi \times \mathbf{R}),$$

and

$$\bar{p}_i^\varepsilon, p_i^\varepsilon \to 0 \ \text{ in } \ L^2(\Pi \times \mathbf{R}).$$

Indeed, for arbitrary $\theta(t,x,\lambda) \in C_0^\infty(\Pi \times \mathbf{R})$, using (H1), Lemma 2 and Hölder inequality, we estimate

$$\begin{aligned}
|\langle \pi_i^\varepsilon, \theta\rangle| &\le C_3 \varepsilon \int_\Pi |\nabla u^\varepsilon|^r |\theta| \, dxdt \\
&\le C_3 \varepsilon^{1-\frac{r}{r+1}} \left(\varepsilon \int_\Pi |\nabla u^\varepsilon|^{r+1} dxdt \right)^{\frac{r}{r+1}} \|\theta(t,x,u^\varepsilon(t,x))\|_{L^{r+1}(\Pi)} \\
&\le c\varepsilon^{1-\frac{r}{r+1}} \|\theta\|_{L^{r+1}(\Pi; W^{1,r+1}(\mathbf{R}))}.
\end{aligned}$$

From here we conclude that $\pi_i^\varepsilon \to 0$ in $L^{\frac{r+1}{r}}(\Pi; W^{-1,r+1}(\mathbf{R}))$, so π_i^ε can be represented via (27). Now, using Schwartz inequality and Lemma 3, we estimate $\bar{\pi}_i^\varepsilon$,

$$|\langle \bar{\pi}_i^\varepsilon, \theta\rangle| \le C \frac{\delta}{\varepsilon^{\frac{r+2}{r+1}}} \|\theta\|_{L^2(\Pi; H^1(\mathbf{R}))}.$$

From here we conclude that if $\delta = \mathcal{O}(\varepsilon^{\frac{r+3}{r+1}})$, then $\bar{\pi}_i^\varepsilon \to 0$ in $L^2(\Pi; H^{-1}(\mathbf{R}))$, so that it can be represented via (27).

Also (as in [10]) we obtain that for every $i = 1, ..., d$, the nets

$$(m_i^\varepsilon)_\varepsilon, (k_i^\varepsilon)_\varepsilon \ \text{ lie in a bounded set of } \mathcal{M}(\Pi \times \mathbf{R}), \tag{28}$$

where $\mathcal{M}(\Pi \times \mathbf{R})$ stands for the space of bounded measures. Indeed, from (H1) and Lemma 2 we estimate

$$|\langle m_i^\varepsilon, \theta\rangle| \le \varepsilon \int_\Pi |\nabla u^\varepsilon| |b_i(u^\varepsilon)| |\theta(t,x,u^\varepsilon)| dxdt \le C \sup_{\Pi \times \mathbf{R}} |\theta(t,x,\lambda)|.$$

Finally, from Lemmas 2-3 and the inequality $ab\theta \le \varepsilon a^2\theta + \varepsilon^{-1} b^2 \theta$, we use the estimate

$$|\langle k_i^\varepsilon, \theta\rangle| \le C \frac{\delta}{\varepsilon^{\frac{r+3}{r+1}}} \sup_{\Pi \times \mathbf{R}} |\theta(t,x,\lambda)|$$

and conclude (28).

Then consider the remaining term on the right hand side of (26). Denote by

$$\Pi_i^\varepsilon = \partial_\lambda (h_\varepsilon(t,x,\lambda) D_{x_i} f_i(t,x,\lambda)), \quad i = 1, ..., d.$$

Let $\theta(t,x,\lambda) \in C_0^\infty(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})$ and $i = 1, ..., d$. Then,

$$\begin{aligned}
\langle \Pi_i^\varepsilon, \theta\rangle &= |\int_{\Pi \times \mathbf{R}} h_\varepsilon(t,x,\lambda) D_{x_i} f_i(t,x,\lambda) \theta_\lambda(t,x,\lambda) dtdxd\lambda| \\
&\le \|\theta_\lambda\|_{C^0(\Pi \times \mathbf{R})} \int_{\text{supp}\theta} |D_\lambda f_i(t,x,\lambda)| dtdxd\lambda \le C \|\theta_\lambda\|_{C^0(\Pi \times \mathbf{R})},
\end{aligned}$$

where C is a constant depending only on the support of a test function θ. Thus, for every $i = 1, ..., d$ the family $(\Pi_i^\varepsilon)_\varepsilon$ lies in a locally bounded subset of the space of bounded measures $\mathcal{M}(\Pi \times \mathbf{R})$.

Step 3 From *Step 2* we conclude that (26) can be rewritten as

$$\begin{aligned}&\partial_t h_\varepsilon(t,x,\lambda)+\sum_{i=1}^d \partial_{x_i}(h_\varepsilon(t,x,\lambda)\partial_\lambda f_i(t,x,\lambda))\\ &=\sum_{i=1}^d \partial_{x_i}\left(\partial_\lambda Q_i^\varepsilon(t,x,\lambda)+\bar{Q}_i^\varepsilon(t,x,\lambda)\right)+\sum_{i=1}^d\left(\partial_\lambda P_i^\varepsilon+\bar{P}_i^\varepsilon\right).\end{aligned}$$

Since $2\geq \frac{r+1}{r}$, we have that $(Q_i^\varepsilon)_\varepsilon$ and $(\bar{Q}_i^\varepsilon)_\varepsilon$, $i=1,...,d$, are precompact in $L_{\rm loc}^{\frac{r+1}{r}}(\Pi\times\mathbf{R})$, while $(P_i^\varepsilon)_\varepsilon$ and $(\bar{P}_i^\varepsilon)_\varepsilon$, $i=1,...,d$, are locally bounded in the space of bounded measures $\mathcal{M}(\Pi\times\mathbf{R})$.

Therefore, we can apply Corollary 9 for the net $(h_\varepsilon)_\varepsilon$ and conclude that a subsequence $(h_k)_k\subset(h_\varepsilon)_\varepsilon$ satisfies

$$\left(\int_{-R}^R h_k(t,x,\lambda)d\lambda\right)_{k\in\mathbf{N}} \quad \text{is convergent in} \quad L^1_{\rm loc}(\mathbf{R}^+\times\mathbf{R}^d), \tag{29}$$

for every $R\in\mathbf{N}$. Furthermore,

$$\begin{aligned}\Big|u^\varepsilon-\int_{-R}^R h_\varepsilon(t,x,\lambda)d\lambda\Big|&=\Big|\int_\lambda h_\varepsilon(t,x,\lambda)d\lambda-\int_{-R}^R h_\varepsilon(t,x,\lambda)d\lambda\Big|\\ &=\Big|\int_R^\infty h_\varepsilon(t,x,\lambda)d\lambda+\int_{-\infty}^{-R}h_\varepsilon(t,x,\lambda)d\lambda\Big|\\ &=H(u^\varepsilon-R)(u^\varepsilon-R)+H(-u^\varepsilon-R)(-u^\varepsilon-R).\end{aligned}\tag{30}$$

Furthermore, from Lemma 2, we have that there exists $K_1>0$ that does not depend on ε, so that

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}}\big[H(u^\varepsilon-R)(u^\varepsilon-R)+H(-u^\varepsilon-R)(-u^\varepsilon-R)\big]dxdt\\ &\leq\int_{|u^\varepsilon|>R}|u^\varepsilon|dxdt\leq\frac{1}{R}\int_0^t\int_x|u^\varepsilon|^2dxdt\leq\frac{K_1}{R},\end{aligned}\tag{31}$$

since $\int_{|u^\varepsilon|>R}R|u^\varepsilon|dxdt\leq\int_{|u^\varepsilon|>R}|u^\varepsilon|^2dxdt<\tilde{K_1}$. Therefore, from (30) and (31) it follows

$$\int_0^t\int_{\mathbf{R}}\left|u^\varepsilon-\int_{-R}^R h_\varepsilon(t,x,\lambda)d\lambda\right|dtdx\leq\frac{K_1}{R}. \tag{32}$$

From here, it is easy to prove that $(u^k)_k$, where the indexing is taken from (29), is Cauchy sequence in $L^1_{\rm loc}(\Pi)$. Indeed, for every compact set $K\subset\subset\Pi$, we have

$$\begin{aligned}&\int_K|u^{k_1}-u^{k_2}|dxdt\\ &\leq\int_K|u^{k_1}-\int_{-R}^R h_{k_1}(t,x,\lambda)d\lambda|dxdt+\int_K|u^{k_2}-\int_{-R}^R h_{k_2}(t,x,\lambda)d\lambda|dxdt\\ &+\int_K|\int_{-R}^R h_{k_1}(t,x,\lambda)d\lambda-\int_{-R}^R h_{k_2}(t,x,\lambda)d\lambda|dxdt\leq\frac{2K_1}{R}+\gamma(k_1,k_2),\end{aligned}$$

where $\frac{2K_1}{R}$ appears due to (32), and γ is a function tending to zero as $k_i\to\infty$, $i=1,2$ and it is here since $(h_k)_k$ is convergent in $L^1_{\rm loc}(\Pi\times\mathbf{R})$.

Thus, we see that the subsequence $(u^k)_k \subset (u^\varepsilon)_\varepsilon$ is the Cauchy sequence in $L^1_{loc}(\Pi)$ implying $L^1_{loc}(\Pi)$-precompactness of the family $(u^\varepsilon)_\varepsilon$. ❑

Remark 3. *Notice that if $\delta = o(\varepsilon^{\frac{r+3}{r+1}})$, $\varepsilon \to 0$, then $(u^k)_k$ tends to a unique entropy solution to (1). The proof is analogous to the one from* [1].

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