ON THE COMPACTNESS FOR TWO DIMENSIONAL SCALAR CONSERVATION LAW WITH DISCONTINUOUS FLUX

J. ALEKSIĆ AND D. MITROVIC

Abstract. We prove that a family of solutions to two dimensional scalar conservation law with discontinuous flux function regularized with the vanishing viscosity and smoothen flux augmented with BV initial data is strongly precompact in $L^1_{\rm loc}$ under a weaker nonlinearity condition then in previous works.

1. Introduction

In the paper we consider the following Cauchy problem for two dimensional scalar conservation law

$$u_t + \operatorname{div} f(x,y,u) = 0, \qquad (1)$$
$$u(x,y,0) = u_0(x,y),$$

where $u = u(x,y,t)$, $x, y \in \mathbf{R}$, $t \in \mathbf{R}^+$ and $f = (f_1, f_2) : \mathbf{R}^3 \to \mathbf{R}^2$ (divergence is taken with respect to x and y). For the initial data u_0 we assume that

$$u_0 \in (BV \cap L^\infty)(\mathbf{R}^2), \quad a \le u_0(x,y) \le b, \quad x, y \in \mathbf{R}. \qquad (2)$$

The flux function $f = (f_1, f_2)$ that we consider here has the following properties:

$$f_i(\cdot,\cdot,\lambda) \in (BV \cap L^\infty)(\mathbf{R}^2), \text{ for all } \lambda \in \mathbf{R}, \qquad (3)$$
$$f_i(x,y,\cdot) \in C(\mathbf{R}), \text{ for all } (x,y) \in \mathbf{R}^2 \qquad (4)$$
$$0 = f_i(\cdot,\cdot,b) = f_i(\cdot,\cdot,a), \ i = 1,2, \text{ for all } (x,y) \in \mathbf{R}^2. \qquad (5)$$

In recent years problems of this kind received lots of attention since they model many physical phenomena. As examples of special importance we emphasize applications in flow in porous media, sedimentation processes, traffic flow, radar shape-from-shading problems, blood flow, and gas flow in a variable duct.

If f_1 and f_2 are smooth functions then existence and uniqueness of an entropy solution is provided by well known method of doubling of variables due to Kružkov [8], or using the measure valued concept by DiPerna [2]. It is well known, cf. [1, 8], that for Lipshitz-continuous flux, the family of solutions to vanishing viscosity regularization of (1) converges to the solution of (1) in the strong topology of $L^1(\mathbf{R}^2 \times \mathbf{R}^+)$. But, if the flux is discontinuous in x, y, we can not apply classical results.

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Existence of solution for the problem of type (1) was settled only recently in [6]. The proof is based on two dimensional variant [5] of celebrated method of compensated compactness [11]. The case when the space is of an arbitrary dimension was completed by Panov [9], using another method of Tartar – H-measures [12] (introduced independently by Gerard [3] who named them microlocal defect measures).

In both papers [6, 9] the following regularization of problem (1) was considered (here and in the sequel Δ stands for the Laplacian, $\Delta u = u_{xx} + u_{yy}$):

$$\partial_t u^{\varepsilon,\delta} + \operatorname{div} f^{\delta}(x,y,u^{\varepsilon,\delta}) = \varepsilon \Delta u^{\varepsilon,\delta}, \tag{6}$$

$$u^{\varepsilon,\delta}|_{t=0} = u_0^{\delta}, \tag{7}$$

where the approximations f_i^{δ} and u_0^{δ} are constructed in the following manner. Let $\omega : \mathbf{R} \to \mathbf{R}$ be arbitrary smooth function such that $\omega(\xi) = 0$ for $|\xi| \geq 1$, and $\int_{\mathbf{R}} \omega(\xi)\, d\xi = 1$. We define

$$f_i^{\delta}(x,y,\lambda) = \frac{1}{\delta^3} \iiint_{\mathbf{R}^3} f_i(\xi,\eta,\zeta) \omega\left(\frac{x-\xi}{\delta}\right) \omega\left(\frac{y-\eta}{\delta}\right) \omega\left(\frac{\lambda-\zeta}{\delta}\right) d\xi d\eta d\zeta$$

and

$$u_0^{\delta}(x,y) = \frac{1}{\delta^2} \iint_{\mathbf{R}^2} u_0(\xi,\eta) \omega\left(\frac{x-\xi}{\delta}\right) \omega\left(\frac{y-\eta}{\delta}\right) d\xi d\eta.$$

Notice that from (3), for all $\lambda \in \mathbf{R}$,

$$f_i^{\delta}(\cdot,\cdot,\lambda) \in (L^{\infty} \cap BV)(\mathbf{R}^2) \tag{8}$$

and $f_i^{\delta}(\cdot,\cdot,\lambda) \to f_i(\cdot,\cdot,\lambda)$, as $\delta \to 0$, in $L^1_{\rm loc}(\mathbf{R}^2)$.

In [6, 9], the existence of the solution was obtained by proving that a family of solutions to equation (6) (i.e. to (1) regularized with the vanishing viscosity and smoothen flux) is strongly precompact in $L^1_{\rm loc}(\mathbf{R}^2 \times \mathbf{R}^+)$. In order to prove the latter, the following nonlinearity condition was necessary (this is a (weaker) variant used in [9]; for other variants see [6, 7, 10]): Let $S^2 \subset \mathbf{R}^3$ denotes the unit sphere. We say that the flux (f_1, f_2) satisfy a nonlinearity condition if

$$\begin{gathered}\text{for almost every } (x,y) \in \mathbf{R}^2 \text{ and every } \xi \in S^2 \text{ the mapping} \\ \lambda \mapsto \xi_0 \lambda + f_1(x,y,\lambda)\xi_1 + f_2(x,y,\lambda)\xi_2 \\ \text{is not constant in } \lambda \text{ on any nontrivial interval.}\end{gathered} \tag{9}$$

We stress that in one dimensional case one does not need any nonlinearity condition in order to prove existence of a weak solution to a scalar conservation law with a flux discontinuous in space variable. More precisely, if we consider a family of solutions to one dimensional variant of (6), using the compensated compactness argument [4, 7], it is not difficult to prove that a family of entropy admissible solutions [6, 9] to (6) weakly converges along a subsequence to a solution of one dimensional variant of (1). But, we can not state anything about strong $L^1_{\rm loc}$-precompactness. In this paper we shall prove that under relaxed nonlinearity condition (see (10) below), a family of solution to (6) is strongly precompact in $L^1_{\rm loc}(\mathbf{R}^2 \times \mathbf{R}^+)$. As a consequence, in one dimensional situation we are able to prove strong $L^1_{\rm loc}$-precompactness of the family $(u^{\varepsilon,\delta})$ practically merely assuming that the initial data belong to the BV-class.

In order to get the result, we shall use variants of estimates derived in [6], and the following theorem (used also in [4]).

Theorem 1 ([9], Corollary 2)**.** Assume that the vector $\phi(x,u) \in (C(\mathbf{R}_u; BV(\Omega)))^n$, $\Omega \subset \mathbf{R}^n$ is an open set, is genuinely nonlinear, i.e. for a.e. $x \in \Omega$ and for all $\xi \in \mathbf{R}^n$, $\xi \neq 0$, the map $(a,b) \ni u \mapsto (\xi, \phi(x,u)) \neq$ constant on any nontrivial interval.

Then, each bounded sequence $(u_k(x))_k \in L^\infty(\Omega)$, $a \leq u_k(x) \leq b$, satisfying for the Heaviside function H,

$$\operatorname{div}_x \Big[H(u_k(x) - p)(\phi(x, u_k(x)) - \phi(x,p)) \Big] \text{ is precompact in } W^{-1,2}_{\rm loc}(\Omega),$$

contains a subsequence convergent in $L^1_{\rm loc}(\Omega)$.

Roughly speaking, the key point of our procedure is the fact that we have $\|u_t(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)}$ bound, for all $t > 0$. Therefore, we can replace u_t by a function $(h(x,y,u))_t$ (u_t will end up on the right hand side) without affecting the precompactness framework. This means that we can replace $\xi_0 \lambda$ from (9) by $\xi_0 h(x,y,\lambda)$ where h is chosen so that (10) is satisfied (this is actually (9) with $\xi_0\lambda$ replaced by $\xi_0 h(x,y,\lambda)$). Then, we can apply Theorem 1 to obtain strong $L^1_{\rm loc}$ precompactness of the family $(u^{\varepsilon,\delta})_{\varepsilon,\delta}$.

The paper is organized as follows. In Section 2 we give a priori estimates. In Section 3, using results from Section 2, we prove the main theorem - Theorem 5. We pay special attention on the one dimensional case. In Section 4 we give an example of scalar conservation law where we apply the new genuine nonlinearity condition, because the usual one, (9), is not fulfilled.

2. A priori lemmas

In order to use Theorem 1, we will need the following a priory estimates (Lemmas 2-4), cf. [5].

Lemma 2. [L^∞-bound] There exists constant $c > 0$ such that for all $t \in (0,T)$,

$$\|u^{\varepsilon,\delta}(\cdot,\cdot,t)\|_{L^\infty(\mathbf{R}^2)} \leq c.$$

Proof: The proof is standard [5, 9] and follows from the assumptions (2) and (5). Actually, $u^{\varepsilon,\delta}$ stays between the same constants a and b from (2). ❑

Lemma 3. [Lipshitz regularity in time] If $\delta = c\varepsilon$ for a constant $c > 0$, then there exists constant c_0, independent of ε, δ such that for all $t > 0$,

$$\iint_{\mathbf{R}^2} |\partial_t u^{\varepsilon,\delta}(\cdot,\cdot,t)|\, dxdy \leq c_0.$$

Proof: Denote $w^{\varepsilon,\delta} = \partial_t u^{\varepsilon,\delta}$. Then, by differentiating (6) in t we see that $w^{\varepsilon,\delta}$ satisfies

$$w^{\varepsilon,\delta}_t + \left(\partial_u f^\delta_1(x,y,u^{\varepsilon,\delta}) \cdot w^{\varepsilon,\delta}\right)_x + \left(\partial_u f^\delta_2(x,y,u^{\varepsilon,\delta}) \cdot w^{\varepsilon,\delta}\right)_y = \varepsilon \Delta w^{\varepsilon,\delta}.$$

Multiplying this by $\operatorname{sign} w^{\varepsilon,\delta}$ we obtain

$$\begin{aligned} |w^{\varepsilon,\delta}|_t &+ \left(\partial_u f^\delta_1(x,y,u^{\varepsilon,\delta}) \cdot |w^{\varepsilon,\delta}|\right)_x + \left(\partial_u f^\delta_2(x,y,u^{\varepsilon,\delta}) \cdot |w^{\varepsilon,\delta}|\right)_y \\ &= \varepsilon \left(\Delta |w^{\varepsilon,\delta}| - \operatorname{sign}' w^{\varepsilon,\delta} \left((w^{\varepsilon,\delta}_x)^2 + (w^{\varepsilon,\delta}_y)^2 \right) \right), \end{aligned}$$

in the sense of distributions. Now we integrate over $\mathbf{R}^2$ and use (8) to obtain

$$\begin{aligned}
&\frac{d}{dt}\iint_{\mathbf{R}^2}|w^{\varepsilon,\delta}|(x,y,\cdot)dxdy=\\
&-\iint_{\mathbf{R}^2}\partial_x\left(\partial_u f_1^\delta(x,y,u^{\varepsilon,\delta})\cdot|w^{\varepsilon,\delta}|\right)+\partial_y\left(\partial_u f_2^\delta(x,y,u^{\varepsilon,\delta})\cdot|w^{\varepsilon,\delta}|\right)\,dxdy\\
&+\varepsilon\iint_{\mathbf{R}^2}|w^{\varepsilon,\delta}|_{xx}+|w^{\varepsilon,\delta}|_{yy}\,dxdy-\varepsilon\iint_{\mathbf{R}^2}\left((w_x^{\varepsilon,\delta})^2+(w_y^{\varepsilon,\delta})^2\right)\operatorname{sign}'(w^{\varepsilon,\delta})dxdy\\
&=-\varepsilon\iint_{\mathbf{R}^2}\left((w_x^{\varepsilon,\delta})^2+(w_y^{\varepsilon,\delta})^2\right)\operatorname{sign}'(w^{\varepsilon,\delta})dxdy\leq 0
\end{aligned}$$

and conclude that $\iint_{\mathbf{R}^2}|w^{\varepsilon,\delta}|(x,y,\cdot)dxdy$ is not increasing in time, i.e. for all $t>0$,

$$\begin{aligned}
&\iint_{\mathbf{R}^2}|w^{\varepsilon,\delta}|(x,y,t)dxdy\leq\iint_{\mathbf{R}^2}|w^{\varepsilon,\delta}|(x,y,0)dxdy\\
&=\iint_{\mathbf{R}^2}\left|-\partial_x f_1^\delta(x,y,u_0^\delta)-\partial_y f_2^\delta(x,y,u_0^\delta)+\varepsilon\left((u_0^\delta)_{xx}+(u_0^\delta)_{yy}\right)\right|dxdy\\
&\leq C+\varepsilon\|(u_0^\delta)_{xx}+(u_0^\delta)_{yy}\|_{L^1(\mathbf{R}^2)}\leq C+\frac{\varepsilon}{\delta}\iint_{\mathbf{R}^2}|(u_0^\delta)_x|+|(u_0^\delta)_y|\,dxdy\\
&\leq C+\frac{\varepsilon}{\delta}\|u_0\|_{BV(\mathbf{R}^2)},
\end{aligned}$$

where the constant C appears due to (8) and (2). Taking into account the assumption that $\delta=c\varepsilon$ and (2) again, we conclude the proof. ❑

Lemma 4. [Entropy dissipation bound] There exists a constant c independent from ε and δ such that

$$\varepsilon\iint_{\mathbf{R}^2}\left(u_x^{\varepsilon,\delta}(\cdot,\cdot,t)\right)^2+\left(u_y^{\varepsilon,\delta}(\cdot,\cdot,t)\right)^2dxdy\leq c,$$

for all $t>0$.

Proof: We multiply (6) by $u^{\varepsilon,\delta}$ and integrate over $\mathbf{R}^2$. This implies

$$\begin{aligned}
&\varepsilon\iint_{\mathbf{R}^2}\left(u_x^{\varepsilon,\delta}(\cdot,\cdot,t)\right)^2+\left(u_y^{\varepsilon,\delta}(\cdot,\cdot,t)\right)^2dxdy\\
&=-\iint_{\mathbf{R}^2}\Bigg[u^{\varepsilon,\delta}u_t^{\varepsilon,\delta}+\left(\int_0^{u^{\varepsilon,\delta}}f_1^\delta(x,y,v)dv\right)_x-\int_0^{u^{\varepsilon,\delta}}\left(f_1^\delta(x,y,v)\right)_x dv\\
&+\left(\int_0^{u^{\varepsilon,\delta}}f_2^\delta(x,y,v)dv\right)_y-\int_0^{u^{\varepsilon,\delta}}\left(f_2^\delta(x,y,v)\right)_y dv\Bigg]dxdy\\
&\leq c\left(\|u_t^{\varepsilon,\delta}\|_{L^\infty(\mathbf{R}^+;L^1(\mathbf{R}^2))}+\iint_{\mathbf{R}^2}\left|\int_0^{u^{\varepsilon,\delta}}\partial_x(f_1^\delta(x,y,v))dv\right|+\left|\int_0^{u^{\varepsilon,\delta}}\partial_y(f_2^\delta(x,y,v))dv\right|\right)\\
&\leq c\left(\|u_t^{\varepsilon,\delta}\|_{L^\infty(\mathbf{R}^+;L^1(\mathbf{R}^2))}+\max_{a\leq v\leq b}\|f_1^\delta(x,y,v)\|_{BV(\mathbf{R}^2)}+\max_{a\leq v\leq b}\|f_2^\delta(x,y,v)\|_{BV(\mathbf{R}^2)}\right).
\end{aligned}$$

Applying Lemma 3 and (8), we conclude the proof. ❑

3. New genuine nonlinearity condition and the main result

On the beginning of the section we introduce a generalization of nonlinearity condition (9) that will be used in the proof of Theorem 5. We assume that

$$\begin{gathered}\text{exists } h(x,y,\lambda)\in C^1(\mathbf{R}_\lambda;L^\infty(\mathbf{R}_x\times\mathbf{R}_y)) \text{ such that for all } \xi\in S^2 \text{ the mapping}\\ \lambda\mapsto \xi_0\cdot h(x,y,\lambda)+\xi_1\cdot f_1(x,y,\lambda)+\xi_2\cdot f_2(x,y,\lambda)\\ \text{is not constant in } \lambda \text{ on any nontrivial interval.}\end{gathered}\tag{10}$$

In order to use (10), we rewrite (6) as

$$\begin{aligned}&h(x,y,u^{\varepsilon,\delta})_t+f_1^\delta(x,y,u^{\varepsilon,\delta})_x+f_2^\delta(x,y,u^{\varepsilon,\delta})_y\\ &\qquad=h(x,y,u^{\varepsilon,\delta})_t-u_t^{\varepsilon,\delta}+\varepsilon(u^{\varepsilon,\delta}_{xx}+u^{\varepsilon,\delta}_{yy}).\end{aligned}\tag{11}$$

Denote $\eta'(\lambda)=H(\lambda-k)$, for some constant k (here H stands for the Heaviside step function) and define corresponding entropy fluxes:

$$\begin{aligned}q_0(x,y,\lambda)&=H(\lambda-k)(h(x,y,\lambda)-h(x,y,k)),\\ q_i(x,u,\lambda)&=H(\lambda-k)(f_i(x,y,\lambda)-f_i(x,y,k)),\quad i=1,2,\\ q_i^\delta(x,y,\lambda)&=H(\lambda-k)(f_i^\delta(x,y,\lambda)-f_i^\delta(x,y,k)),\quad i=1,2.\end{aligned}$$

We multiply (11) by $\eta'(u^{\varepsilon,\delta})$ and add $\partial_x q_1(x,y,u^{\varepsilon,\delta})$ and $\partial_y q_2(x,y,u^{\varepsilon,\delta})$ on both sides of equality (11) to obtain

$$\begin{aligned}&\partial_t q_0(x,y,u^{\varepsilon,\delta})+\partial_x q_1(x,y,u^{\varepsilon,\delta})+\partial_y q_2(x,y,u^{\varepsilon,\delta})\\ &\quad=H(u^{\varepsilon,\delta}-k)\Big(\partial_t h(x,y,u^{\varepsilon,\delta})-D_x f_1^\delta(x,y,k)-D_y f_2^\delta(x,y,k)-u_t^{\varepsilon,\delta}\Big)\\ &\quad+\varepsilon(\partial_x(u_x^{\varepsilon,\delta}\eta'(u^{\varepsilon,\delta}))-(u_x^{\varepsilon,\delta})^2\eta''(u^{\varepsilon,\delta})+\partial_y(u_y^{\varepsilon,\delta}\eta'(u^{\varepsilon,\delta}))-(u_y^{\varepsilon,\delta})^2\eta''(u^{\varepsilon,\delta}))\\ &\quad+\partial_x(q_1-q_1^\delta)(x,y,u^{\varepsilon,\delta})+\partial_y(q_2-q_2^\delta)(x,y,u^{\varepsilon,\delta})\\ &\quad\leq H(u^{\varepsilon,\delta}-k)\Big(\partial_t h(x,y,u^{\varepsilon,\delta})-D_x f_1^\delta(x,y,k)-D_y f_2^\delta(x,y,k)-u_t^{\varepsilon,\delta}\Big)\\ &\quad+\varepsilon(\partial_x(u_x^{\varepsilon,\delta}\eta'(u^{\varepsilon,\delta}))+\partial_y(u_y^{\varepsilon,\delta}\eta'(u^{\varepsilon,\delta})))\\ &\quad+\partial_x(q_1-q_1^\delta)(x,y,u^{\varepsilon,\delta})+\partial_y(q_2-q_2^\delta)(x,y,u^{\varepsilon,\delta})\end{aligned}\tag{12}$$

in $\mathcal{D}'((0,T)\times\mathbf{R}^2)$.

Theorem 5. Assume that the functions f_1,f_2 from (1) satisfy (3)-(5) and (10). If $\varepsilon=c\delta$, then the family of solutions $(u^\varepsilon)_\varepsilon\equiv(u^{\varepsilon,\delta})_{\varepsilon,\delta}$ to (6) is strongly precompact in $L^1((0,T)\times\mathbf{R}^2)$.

Proof: In the sequel, for $\Omega=(0,T)\times\mathbf{R}^2$, $W^{-1,2}_{\rm c,loc}(\Omega)$ stands for families of functions that are precompact in $W^{-1,2}_{\rm loc}(\Omega)$, while $\mathcal{M}_{\rm b,loc}(\Omega)$ stands for families of functions which are locally bounded in the space of Radon measures $\mathcal{M}(\Omega)$.

In order to use Theorem 1 we have to show that

$$\operatorname{div}_{(t,x,y)}\left[\big(q_0,q_1,q_2\big)(x,y,u^\varepsilon)\right]\in W^{-1,2}_{\rm c,loc}(\Omega).\tag{13}$$

From (12) and the Schwartz lemma on nonnegative distributions it follows that there exists bounded measure $\mu_k \in \mathcal{M}(\Omega)$ such that

$$\begin{aligned}&\partial_t q_0(x,y,u^\varepsilon) + \partial_x q_1(x,y,u^\varepsilon) + \partial_y q_2(x,y,u^\varepsilon)\\ &\quad = H(u^\varepsilon - k)\left(\partial_t h(x,y,u^{\varepsilon,\delta}) - D_x f_1^\delta(x,y,k) - D_y f_2^\delta(x,y,k) - u_t^\varepsilon\right)\\ &\quad + \partial_x(q_1 - q_1^\delta)(x,y,u^\varepsilon) + \partial_y(q_2 - q_2^\delta)(x,y,u^\varepsilon)\\ &\quad + \varepsilon(\partial_x(u_x^\varepsilon \eta'(u^\varepsilon)) + \partial_y(u_y^\varepsilon \eta'(u^\varepsilon))) + \mu_k(t,x,y),\end{aligned}$$

i.e,

$$\begin{aligned}&\partial_t H(u^\varepsilon - k)(h(x,y,u^\varepsilon) - h(x,y,k)) + \partial_x H(u^\varepsilon - k)(f_1(x,y,u^\varepsilon) - f_1(x,y,k))\\ &\quad + \partial_y H(u^\varepsilon - k)(f_2(x,y,u^\varepsilon) - f_2(x,y,k))\\ &\quad = \partial_x(q_1(x,y,u^\varepsilon) - q_1^\delta(x,y,u^\varepsilon)) + \partial_y(q_2(x,y,u^\varepsilon) - q_2^\delta(x,y,u^\varepsilon))\\ &\quad + H(u^\varepsilon - k)\left(\partial_u h(x,y,u^\varepsilon)\partial_t u^\varepsilon - D_x f_1^\delta(x,y,k) - D_y f_2^\delta(x,y,k)\right)\\ &\quad + \varepsilon(\partial_x(u_x^\varepsilon \eta'(u^\varepsilon)) + \partial_y(u_y^\varepsilon \eta'(u^\varepsilon))) + \mu_k(t,x,y).\end{aligned}$$

In order to prove (13) we shall need Murat's lemma stating that

$$(\operatorname{div} Q_\varepsilon)_\varepsilon \in W^{-1,2}_{\mathrm{c,loc}} \quad \text{if} \quad \operatorname{div} Q_\varepsilon = p_\varepsilon + q_\varepsilon,$$

with $(q_\varepsilon)_\varepsilon \in W^{-1,2}_{\mathrm{c,loc}}(\Omega)$ and $(p_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathrm{b,loc}}(\Omega)$. Indeed, from Lemma 3

$$H(u^\varepsilon - k)\left(\partial_\lambda h(x,y,u^\varepsilon)\partial_t u^\varepsilon - \partial_t u^\varepsilon\right) \in \mathcal{M}_{\mathrm{b,loc}}(\Omega). \tag{14}$$

Lemma 4 implies

$$\partial_x(\varepsilon \partial_x u^\varepsilon H(u^\varepsilon - k)) + \partial_y(\varepsilon \partial_y u^\varepsilon H(u^\varepsilon - k)) \in W^{-1,2}_{\mathrm{c,loc}}(\Omega), \tag{15}$$

provided that

$$\varepsilon \partial_x u^\varepsilon H(u^\varepsilon - k) \to 0 \text{ in } L^2_{\mathrm{loc}}(\Omega)$$

and

$$\int_\Omega |\varepsilon \partial_x u^\varepsilon H(u^\varepsilon - k)|^2\, dxdydt \le \varepsilon^2 \int_\Omega |\partial_x u^\varepsilon|^2 dxdydt \le Tc\varepsilon \to 0, \quad \varepsilon \to 0.$$

Furthermore,

$$\left(D_x f_1^\delta(x,y,k) + D_y f_2^\delta(x,y,k)\right) H(u^\varepsilon - k)) \in \mathcal{M}_{\mathrm{b,loc}}(\Omega), \tag{16}$$

since $f_i^\delta \in BV(\Omega)$. Finally,

$$\partial_x(q_1 - q_1^\delta), \partial_y(q_2 - q_2^\delta) \in W^{-1}_{\mathrm{c,loc}}(\Omega), \tag{17}$$

since

$$\begin{aligned}|q_i - q_i^\delta| &\le |f_i^\delta(x,y,u^\varepsilon) - f_i(x,y,u^\varepsilon)| + |f_i^\delta(x,y,k) - f_i(x,y,k)|\\ &\le 2 \max_{a \le p \le b} |f_i^\delta(x,y,p) - f_i(x,y,p)| \to 0 \text{ in } L^2_{\mathrm{loc}}(\mathbf{R}^2).\end{aligned}$$

Collecting (14-17), from Murat's lemma we obtain (13). Applying Theorem 1 we conclude the proof. ❑

Now, we shall apply the previous theorem on one dimensional case of the considered problem

$$\begin{aligned}u_t + (f(x,u))_x &= 0,\\ u|_{t=0} = u_0(x) &\in (BV \cap L^\infty)(\mathbf{R}) \quad a \le u_0 \le b,\end{aligned}$$

where (3)-(5) (with $\mathbf{R}^2$ replaced by $\mathbf{R}$) are satisfied. We need to assume that for almost every $x \in \mathbf{R}$ the mapping

$$[a,b] \ni \lambda \mapsto f(x,\lambda), \tag{18}$$

is different from a constant on any nontrivial interval.

Corollary 6. A sequence of solutions $(u^\varepsilon)_\varepsilon$ of the problem

$$\begin{aligned} u^\varepsilon_t + (f^\varepsilon(x,u^\varepsilon))_x =& \varepsilon u^\varepsilon_{xx}, \\ u^\varepsilon|_{t=0} =& u^\varepsilon_0(x,y), \end{aligned}$$

where the notation is taken from (6)-(7), is strongly precompact in $L^1_{\rm loc}(\mathbf{R}^+ \times \mathbf{R})$.

Proof: According to the previous theorem, it is enough to find a function $h(x,\lambda)$ such that the mapping

$$\lambda \mapsto h(x,\lambda)\xi_0 + f(x,\lambda)\xi_1 \tag{19}$$

is different from a constant on any nontrivial interval. Taking

$$h(x,\lambda) = f^2(x,\lambda)$$

we conclude that (19) will not be satisfied only if there exists a nonzero set $\Omega \subset \mathbf{R}$ such that for $x \in \Omega$ there exists $(\xi_0,\xi_1) \in \mathbf{R}^2 \backslash \{0\}$ satisfying

$$f(x,\lambda) = \frac{-\xi_1 \pm \sqrt{\xi_1^2 + 4\xi_0 c}}{2\xi_0},$$

for a constant c, contradicting (18). ❑

4. Example

Consider the following Cauchy problem

$$\begin{aligned} &u_t + (k(x)g(u))_x + (l(y)f(u))_y = 0 \\ &u|_{t=0} = u_0(x,y) \in BV(\mathbf{R}^2) \end{aligned}$$

with

$$-1 \le u_0(x,y) \le 1$$

$$g(u) = \begin{cases} 0, & \text{for } |u| \ge 1 \\ u+1, & \text{for } -1 < u < 0 \\ 1-u^2, & \text{for } 0 < u < 1 \end{cases}$$

$$k(x) = \begin{cases} 3, & \text{for } x \ge 0 \\ 1, & \text{for } x < 0, \end{cases}$$

and

$$f(u) = \begin{cases} 0, & \text{for } |u| \ge 1 \\ 1-u^2, & \text{for } -1 < u < 0 \\ 1-u, & \text{for } 0 < u < 1 \end{cases}$$

$$l(y) = \begin{cases} 4, & \text{for } y \ge 0 \\ 2, & \text{for } y < 0, \end{cases}$$

Clearly, the flux vector $(k(x)g(u), l(x)f(u))$ does not satisfy classical genuine nonlinearity condition (9). Therefore, hitherto it has not been possible to state that the family $(u^\varepsilon)_\varepsilon$ of solutions to the equation

$$u_t^\varepsilon + (k_\varepsilon(x)g(u^\varepsilon))_x + (l(y)f(u_\varepsilon))_y = \varepsilon(u_{xx}^\varepsilon + u_{yy}^\varepsilon)$$

where

$$k_\varepsilon(x) = \begin{cases} 3, & \text{for } \ x \geq \varepsilon \\ \frac{x}{\varepsilon} + 2, & \text{for } \ -\varepsilon < x < \varepsilon \\ 1, & \text{for } \ x \leq -\varepsilon, \end{cases}$$

and

$$l_\varepsilon(y) = \begin{cases} 4, & \text{for } \ x \geq \varepsilon \\ \frac{x}{\varepsilon} + 3, & \text{for } \ -\varepsilon < x < \varepsilon \\ 2, & \text{for } \ x \leq -\varepsilon, \end{cases}$$

is strongly precompact in $L^1_{\rm loc}(\mathbf{R}^+ \times \mathbf{R}^2)$. Still, it is true according to Theorem 5. Indeed, take h from (10) to be $h(x,u) = u^3$. In that case, the vector field $(h(x,u), k(x)g(u), l(y)f(u))$ satisfies the conditions from Theorem 5, since $k \in BV$. Therefore, Theorem 5 provides strong $L^1_{\rm loc}$-precompactness of the family $(u^\varepsilon)_\varepsilon$.

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Jelena Aleksić, Department of Mathematics and informatics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia
E-mail address: jelena.aleksic@im.ns.ac.yu

Darko Mitrovic, Faculty of Mathematics and Natural Sciences, University of Montenegro, Cetinjski put bb, 81000 Podgorica, Montenegro
E-mail address: matematika@t-com.me