GAUSS KERNEL METHOD FOR GENERALIZED SOLUTIONS TO CONSERVATION LAWS IN HETEROGENEOUS MEDIA

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ABSTRACT. Gauss kernel method and Duhamel's integral principle are used to obtain solutions in the sense of Colombeau generalized functions to scalar conservation laws with flux explicitly dependent on space variable, and to the corresponding parabolic approximation. In homogeneous case, the existence and uniqueness of generalized solution are obtained in [15]. This paper presents generalization of results from [15] for wider class of problems.

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1. Introduction

Colombeau generalized functions naturally extend distributions through the convolution transform given by the mollifiers ϕ, $f \mapsto f * \phi$. This fact and the integral transform method are the basic tool in considering the Cauchy problem for a scalar conservation law with a flux-function explicitly dependent on the space variable,

$$\begin{aligned} &\partial_t u(t,x) + \partial_x f(x, u(t,x)) = 0, && x \in \mathbf{R},\ t > 0, \\ &u(0,x) = u_0(x), && x \in \mathbf{R}. \end{aligned} \tag{1}$$

The ideas of integral transform methods are essential ones in our approach to this problem.

The same problem, but in the homogeneous case, i.e. with a flux-function $f = f(u)$ dependent only on the state variable u, is completely solved in [15]. This paper presents generalization of the results from [15] for wider class of problems, i.e. for a more general flux-function f.

In the classical theory, the idea of solving problems as (1) is to consider an approximation of (1) involving small terms added on the right hand side of (1), such as "vanishing viscosity" μu_{xx}, where μ is a small positive parameter tending to zero. In order to obtain a solution to problem (1), one needs to study a limit of a family of solutions $(u^\mu)_\mu$, $\mu \to 0^+$, to the following family of problems

$$\begin{aligned} &\partial_t u(t,x) + \partial_x f(x, u(t,x)) = \mu \partial^2_{xx} u(t,x), && x \in \mathbf{R},\ t > 0, \\ &u(0,x) = u_0(x), && x \in \mathbf{R}. \end{aligned} \tag{2}$$

In the theory of Colombeau generalized functions such procedure can be obtained by viewing μ as a strictly positive generalized constant associated to zero.

If we consider problem (2) with a smooth initial data $u_0 \in C_b^\infty(\mathbf{R})$, whose all derivatives are bounded, then (2) can be reformulated as an integral equation

$$u(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-y^2} u_0(x - 2\sqrt{\mu t} y) dy + \\ + \int_0^t \frac{1}{\sqrt{\pi \mu s}} \int_{-\infty}^{+\infty} y \mathrm{e}^{-y^2} f(x - 2\sqrt{\mu s} y, u(t-s, x - 2\sqrt{\mu s} y)) dy ds, \tag{3}$$

by the use of Duhamel's principle and the heat-kernel $E(t,x) = \frac{1}{2\sqrt{\pi \mu t}} \mathrm{e}^{-\frac{x^2}{4\mu t}}$, where $\mu > 0$. The same formulation, (3), is valid for a representative of a generalized solution, and it is used to prove the existence and uniqueness of a approximated generalized solution to conservation law (1).

1.1. **Algebra $\mathcal{G}_g(\mathbf{R}^2_+)$.** Let us firstly recall the basic definitions and properties we need here. We refer to [4, 9, 12, 14, 15, 16] for more details about generalized functions.

Let $\mathbf{R}^2_+ = (0,\infty) \times \mathbf{R}$ and

$$C^\infty_{\bar{b}}(\mathbf{R}^2_+) := \{u \in C^\infty(\mathbf{R}^2_+) \, : \, \forall \alpha,\, \beta \in \mathbf{N},\, \forall T > 0, \\ \|u\|_{\alpha,\beta;T} := \sup_{(t,x) \in (0,T) \times \mathbf{R}} |\partial_t^\alpha \partial_x^\beta u(t,x)| < \infty\}.$$

If one put $T = \infty$, then the corresponding space is denoted by $C_b^\infty(\mathbf{R}^2_+)$. Excluding $\mathbf{R}_+$ and taking supremum over $x \in \mathbf{R}$, one obtains $C_b^\infty(\mathbf{R})$.

A net of functions $\{u^\varepsilon\}_{\varepsilon \in (0,1)} \in (C^\infty_{\bar{b}}(\mathbf{R}^2_+))^{(0,1)}$ is called *moderate* if for all $(\alpha, \beta) \in \mathbf{N}^2$ and all $T > 0$ there exists $N \in \mathbf{N}$ such that $\|u\|_{\alpha,\beta;T} = \mathcal{O}(\varepsilon^{-N})$, as $\varepsilon \to 0^+$. Denote by $\mathcal{E}_{M,g}(\mathbf{R}^2_+)$ the set of all moderate nets. A moderate net $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$ is called *negligible* (an element of $\mathcal{N}_g(\mathbf{R}^2_+)$) if for all $(\alpha,\beta) \in \mathbf{N}^2$, $q \in \mathbf{N}$ and all $T > 0$, $\|u\|_{\alpha,\beta;T} = \mathcal{O}(\varepsilon^q)$, as $\varepsilon \to 0^+$. $\mathcal{E}_{M,g}(\mathbf{R}^2_+)$ and $\mathcal{N}_g(\mathbf{R}^2_+)$ are algebras under pointwise multiplication and $\mathcal{N}_g(\mathbf{R}^2_+)$ is an ideal in $\mathcal{E}_{M,g}(\mathbf{R}^2_+)$, so we can define Colombeau algebra $\mathcal{G}_g(\mathbf{R}^2_+) = \mathcal{E}_{M,g}(\mathbf{R}^2_+)/\mathcal{N}_g(\mathbf{R}^2_+)$. In the same way, one can define the algebra $\mathcal{G}_g(\mathbf{R})$.

The equality $C^\infty_{\bar{b}}(\mathbf{R}^2_+) = C^\infty_{\bar{b}}(\overline{\mathbf{R}^2_+})$, $\overline{\mathbf{R}^2_+} = [0,\infty) \times \mathbf{R}$, enables us to define *the restriction* of the generalized function $u \in \mathcal{G}_g(\mathbf{R}^2_+)$ to $\{t = 0\}$, $u|_{t=0} \in \mathcal{G}_g(\mathbf{R})$, as a class of the family $\{u^\varepsilon(0,x)\}_\varepsilon$, where $\{u^\varepsilon(t,x)\}_\varepsilon$ is a representative of the generalized function u (cf. [15] and the references therein).

The composition of a nonlinear function $f : \mathbf{R}^2 \to \mathbf{R}$ and a generalized function $u \in \mathcal{G}_g(\mathbf{R}^2_+)$, $f(x,u)$, such that $f(x,u) \in \mathcal{G}_g(\mathbf{R}^2_+)$, is defined as follows. A smooth function $f : \mathbf{R}^2 \to \mathbf{R}$ is called *slowly increasing* if

$$\forall \alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^2,\ \exists N_\alpha \in \mathbf{N},\ \exists c_\alpha > 0 : \quad |\partial_\xi^{\alpha_1} \partial_\lambda^{\alpha_2} f(\xi,\lambda)| \le c_\alpha (1 + |\lambda|)^{N_\alpha},$$

for all $\xi, \lambda \in \mathbf{R}$. The number $N_{(0,0)}$ is called *the order* of f. If f is slowly increasing and $u \in \mathcal{G}_g(\mathbf{R}^2_+)$, then it is easy to prove that $f(x, u(t,x)) := [\{f(x, u^\varepsilon(t,x))\}_\varepsilon] \in \mathcal{G}_g(\mathbf{R}^2_+)$.

Now we describe *the embedding* of the space of bounded distributions $\mathcal{D}'_{L^\infty}(\mathbf{R})$ into $\mathcal{G}_g(\mathbf{R})$. Let $\rho \in S(\mathbf{R})$ be a rapidly decreasing function such that $\int_{\mathbf{R}} \rho(x) dx = 1$, $\int_{\mathbf{R}} x^n \rho(x) dx = 0,\ n = 1, 2, ...$ and let $\rho_\varepsilon(x) = \frac{1}{\varepsilon} \rho\left(\frac{x}{\varepsilon}\right),\ x \in \mathbf{R},\ \varepsilon > 0$. Let $w \in \mathcal{D}'_{L^\infty}(\mathbf{R})$. Then, the mapping $\iota_\rho : w \mapsto [\{w * \rho_\varepsilon\}_\varepsilon]_{\mathcal{N}_g}$ is aforementioned embedding that commutes with differentiation. Moreover, if $w \in C_b^\infty(\mathbf{R})$, then $\{w_\varepsilon\}_\varepsilon = \{w\}_\varepsilon$ is a representative of the generalized function $\iota_\rho(w)$.

Recall, generalized function $u = [\{u_\varepsilon\}_\varepsilon] \in \mathcal{G}_g(\mathbf{R}^2_+)$, is *associated* to a distribution $w \in \mathcal{D}'(\mathbf{R}^2_+)$, $u \approx w$, if $u^\varepsilon \to w$ in $\mathcal{D}'(\mathbf{R}^2_+)$, when $\varepsilon \to 0^+$.

A generalized function $\mu \in \mathcal{G}_g(\mathbf{R})$, is called *generalized constant*, if it has representative $\{\mu_\varepsilon\}_\varepsilon$ such that $\mu_\varepsilon(t,x) = \mu_\varepsilon \in \mathbf{R}$, $\varepsilon \in (0,1)$. It is said that a generalized constant $[\{\mu_\varepsilon\}_\varepsilon]$ is *strictly positive* if

$$\exists N \in \mathbf{N}: \quad \varepsilon^N \leq \mu_\varepsilon \leq \varepsilon^{-N}, \ \ \varepsilon \to 0^+.$$

If μ is a strictly positive generalized constant, then $1/\mu$ is also a strictly positive generalized constant. Strictly positive generalized constant μ is associated to zero, $\mu \approx 0$, if and only if $\mu_\varepsilon \to 0^+$, when $\varepsilon \to 0^+$.

A generalized function $u \in \mathcal{G}_g(\mathbf{R}^2_+)$, is called *of* $\sqrt[r]{\log}$*-type* if it has a representative $\{u^\varepsilon\}_\varepsilon$, such that $\forall T > 0$,

$$\sup_{(t,x)\in(0,T)\times\mathbf{R}} |u^\varepsilon(t,x)| = \mathcal{O}(\sqrt[r]{|\log\varepsilon|}), \ \varepsilon \to 0^+,$$

and it is called *of bounded type* if

$$\sup_{(t,x)\in(0,T)\times\mathbf{R}} |u^\varepsilon(t,x)| = \mathcal{O}(1), \ \varepsilon \to 0^+.$$

2. Generalized solutions

A generalized function $u \in \mathcal{G}_g(\mathbf{R}^2_+)$ that solves problem (1) in $\mathcal{G}_g(\mathbf{R}^2_+)$ is called *generalized solution* to (1). It means that for given $u_0 = [\{u_0^\varepsilon\}] \in \mathcal{G}_g(\mathbf{R})$, $\{u_t^\varepsilon + (f(x,u^\varepsilon))_x\}_\varepsilon \in \mathcal{N}_g(\mathbf{R}^2_+)$ and $\{u^\varepsilon(0,\cdot) - u_0^\varepsilon\}_\varepsilon \in \mathcal{N}_g(\mathbf{R})$. A generalized function $u \in \mathcal{G}_g(\mathbf{R}^2_+)$ that solves problem (1) in $\mathcal{G}_g(\mathbf{R}^2_+)$ with equality replaced by the association, $\approx$, is called *approximated (generalized) solution* to (1). This concept is widely investigated; we refer to some papers concerning generalized solutions [1, 13, 15, 16, 17, 19]. Relations between Colombeau type solutions to (1) and measure valued solutions to (1), in the case when $f(x,u) = f(u)$, are discussed in [1]. We refer to [5, 6, 7, 18] for the measure valued solutions.

Generalized solutions of bounded type to (2) are of special importance because they present approximate generalized solution to conservation law (1). If $u \in \mathcal{G}_g(\mathbf{R}^2_+)$ is of bounded type and μ is generalized constant associated to zero, $\mu \approx 0$, then $\mu u_{xx} \approx 0$, too. That means that u solves conservation law (1) with equality replaced by association, i.e. $u_t + f(x,u)_x \approx 0$.

Another important fact is that if we consider the initial data $u_0 \in L^\infty(\mathbf{R})$, the embedding of u_0, $\iota(u_0) \in \mathcal{G}_g(\mathbf{R})$ is of bounded type. Then, generalized solution to (2) with initial data $\iota(u_0)$ is of bounded type, too, and additionally solves the following problem,

$$u_t + f(x,u)_x \approx 0, \qquad u|_{t=0} \approx u_0.$$

The idea to work with generalized functions of bounded type comes from the classical theory of conservation laws with the flux-function f explicitly dependent on space variable x. If the initial data $u_0 \in L^\infty(\mathbf{R})$ is bounded with some constants $a < b$, $a \leq u_0(x) \leq b$, then the classical solution to (2) stays between the same constants $a < b$, provided that flux-function $f(x,\lambda)$ vanish at $\lambda = a$ and $\lambda = b$, i.e.

$$f(x,a) = f(x,b) = 0, \quad x \in \mathbf{R}, \tag{4}$$

cf. [2, 10, 11].

The basic theorem on the existence and uniqueness of the generalized solutions to (2), Theorem 4, is formulated for the initial data of bounded type and real constant μ. The proof of the Theorem 4 follows the procedure given in the proof of the Theorem 3.3 from [15]. Comparing this work with the work done in [15], there are some modifications of the properties of the function f. Also, we use more general Gronwall-type inequality (Lemma 3) and involve some additional estimates on the spatial derivative of the function f, denoted by $D_x f$. The spatial derivative D_x is connected with the full derivative ∂_x in the following sense $\partial_x f(x,u) = D_x f(x,u) + \partial_u f(x,u)\partial_x u$.

We use the following uniform boundedness property for the generalized functions of bounded type.

Lemma 1. *If $v \in \mathcal{G}_g(\boldsymbol{R})$ is of bounded type, then there are constants $a < b$, such that for every representative $\{v^\varepsilon\}_\varepsilon$, there exists $\varepsilon_0 \in (0,1]$*

$$a \le v^\varepsilon(x) \le b, \qquad x \in \boldsymbol{R},\ \varepsilon \in (0,\varepsilon_0). \tag{5}$$

Proof. Let $v = [\{v^\varepsilon\}_\varepsilon] \in \mathcal{G}_g(\mathbf{R})$ be of bounded type, i.e. $\sup_{x\in\mathbf{R}} |v^\varepsilon(x)| = \mathcal{O}(1),\ \varepsilon \to 0$. Thus, for a representative $\{v^\varepsilon\}_\varepsilon$, there exist constants $\tilde{a} < \tilde{b}$, and $\tilde{\varepsilon} \in (0,1]$, such that $\tilde{a} \le v^\varepsilon(x) \le \tilde{b}$, $x \in \mathbf{R}$, $\varepsilon \in (0,\tilde{\varepsilon})$.

Let $\{w^\varepsilon\}_\varepsilon$ be (another) arbitrary representative of v, i.e. $\{v^\varepsilon - w^\varepsilon\}_\varepsilon \in \mathcal{N}_g(\mathbf{R})$. Since $v^\varepsilon - w^\varepsilon \to 0$, as $\varepsilon \to 0$, there are constants $\tilde{\tilde{a}} \in (\tilde{a}-1,\tilde{a})$, $\tilde{\tilde{b}} \in (\tilde{b},\tilde{b}+1)$ and $\tilde{\tilde{\varepsilon}} > 0$, such that $\tilde{\tilde{a}} \le w^\varepsilon(x) \le \tilde{\tilde{b}}$, $x \in \mathbf{R}$, $\varepsilon \in (0,\tilde{\tilde{\varepsilon}})$. Since all representatives of v have bounds in intervals $(\tilde{a}-1,\tilde{a})$ and $(\tilde{b},\tilde{b}+1)$, we take $a := \inf \tilde{\tilde{a}}$ and $b = \sup \tilde{\tilde{b}}$. □

To specify bounds $a < b$ we refer to "a,b-bounded type".

Definition 2. If $v \in \mathcal{G}_g(\mathbf{R})$ is of bounded type, and constants $a < b$ are obtained in the proof of the Lemma 1, then we say that v is *of a,b-bounded type.*

We also use the following Gronwall-type inequality, proved in [3], to (11).

Lemma 3. *Let $w = w(t)$ be nonnegative, continuous function that satisfies*

$$w(t) \le a(t) + \int_0^t b(s)w(s)ds, \quad t \in I,$$

where $a, b \ge 0$, are nonnegative functions. Then

$$w(t) \le a(t) + \int_0^t a(s)b(s)\mathrm{e}^{\int_s^t b(r)dr}ds, \quad t \in I.$$

Theorem 4. *If the initial data $u_0 \in \mathcal{G}_g(\boldsymbol{R})$ is of a,b-bounded type, and the flux-function f is slowly increasing and fulfills* (4)*, then there exists a unique generalized solution $u \in \mathcal{G}_g(\boldsymbol{R}^2_+)$ to* (2)*, and u is of bounded type.*

Proof. Let $\{u_0^\varepsilon\}_\varepsilon$ be a representative of the initial data u_0. For a fixed ε, consider the following problem

$$\begin{aligned} &\partial_t u^\varepsilon(t,x) + \partial_x f(x,u^\varepsilon(t,x)) = \mu\partial^2_{xx}u^\varepsilon, \\ &u^\varepsilon|_{t=0} = u_0^\varepsilon. \end{aligned} \tag{6}$$

Integral equation (3) enables us to prove that for every fixed ε there exists a unique solution $u^\varepsilon \in C^\infty_{\bar{b}}(\mathbf{R}^2_+)$ to problem (6), cf. [15, 13].

From $a \le u_0^\varepsilon(x) \le b,\ x \in \mathbf{R}$ and (4) we have that

$$a \le u^\varepsilon(t,x) \le b,\ (t,x) \in (0,T) \times \mathbf{R}. \tag{7}$$

The proof of this assertion is standard, cf. [10, 11]. Let us give the sketch of the proof. Let ε be fixed and v_δ^ε, $\delta > 0$, be the solution to the following problem

$$\begin{aligned} &\partial_t v_\delta^\varepsilon + (f(x, v_\delta^\varepsilon))_x = \mu \partial_{xx}^2 v_\delta^\varepsilon - \delta, \\ &v_\delta^\varepsilon|_{t=0} = u_0^\varepsilon. \end{aligned} \tag{8}$$

Then, $v_\delta^\varepsilon \to u^\varepsilon$, strongly in $L^\infty((0,T)\times\mathbf{R})$, as $\delta \to 0$. Suppose that the set $K = \{(t,x) \in (0,T)\times\mathbf{R} : v_\delta^\varepsilon(t,x) > b\}$ is not empty. Let $K_1 = \{t : (t,x) \in K, \text{ for some } x\}$ and $t_0 = \inf K_1$. Then, $v_\delta^\varepsilon(t_0,x) \le b$, because it is continuous. Also by continuity, there must exists an x_0 such that $v_\delta^\varepsilon(t_0,x_0) = b$ and $v_\delta^\varepsilon(t_0,\cdot)$ has a local maximum at x_0. So,

$$v_\delta^\varepsilon(t_0,x_0) = b, \qquad \partial_x v_\delta^\varepsilon(t_0,x_0) = 0, \qquad \partial_{xx}^2 v_\delta^\varepsilon(t_0,x_0) \le 0.$$

On the other hand $v_\delta^\varepsilon(\cdot,x_0)$ is non-decreasing in some neighborhood of t_0, which means that $\partial_t v_\delta^\varepsilon(t_0,x_0) \ge 0$. Now, consider (8) in (t_0,x_0). We obtain

$$\begin{aligned} 0 &\le \partial_t v_\delta^\varepsilon(t_0,x_0) + D_x f(x, v_\delta^\varepsilon)(t_0,x_0) + \partial_u f(x, v_\delta^\varepsilon)(t_0,x_0)\, \partial_x v_\delta^\varepsilon(t_0,x_0) = \\ &= \mu \partial_{xx}^2 v_\delta^\varepsilon(t_0,x_0) - \delta \le -\delta < 0. \end{aligned} \tag{9}$$

Since $f_b(x) \equiv f(x,b) = 0,\ x \in \mathbf{R}$, it follows that $f_b'(x) = D_x f(x,b) = 0,\ x \in \mathbf{R}$. The contradiction in (9) implies that the set K is empty, i.e. $v_\delta^\varepsilon(t,x) \le b$, $(t,x) \in (0,T)\times\mathbf{R}$, hence also $u^\varepsilon(t,x) \le b$, $(t,x) \in (0,T)\times\mathbf{R}$. Similarly we can prove that $u^\varepsilon(t,x) \ge a$, $(t,x) \in (0,T)\times\mathbf{R}$.

Now, we prove that $\{u^\varepsilon\}_\varepsilon$ is moderate. Since $u_0 \in \mathcal{G}_g(\mathbf{R})$ is of a,b-bounded type, from (5) and (7) we have that $a \le u^\varepsilon(t,x) \le b$, $(t,x) \in (0,T)\times\mathbf{R}$, $\varepsilon \in (0,\varepsilon_0)$. Thus for all $T > 0$,

$$\sup_{(t,x)\in(0,T)\times\mathbf{R}} |u^\varepsilon(t,x)| \le \max\{|a|,|b|\} \cdot \varepsilon^{-1},\ \varepsilon \to 0^+.$$

To estimate the derivative u_x^ε, replace u by u^ε and u_0 by u_0^ε in (3). The differentiation with respect to x implies

$$\begin{aligned} u_x^\varepsilon(t,x) &= \frac{1}{\sqrt{\pi}} \int_{\mathbf{R}} \mathrm{e}^{-y^2} (u_0^\varepsilon)'(x - 2\sqrt{\mu t} y) dy + \\ &+ \int_0^t \frac{1}{\sqrt{\pi\mu s}} \int_{\mathbf{R}} y \mathrm{e}^{-y^2} \Big[D_x f(x - 2\sqrt{\mu s} y, u^\varepsilon(t-s, x - 2\sqrt{\mu s} y)) + \\ &+ \partial_u f\,(x - 2\sqrt{\mu s} y, u^\varepsilon(t-s, x - 2\sqrt{\mu s} y))\, u_x^\varepsilon(t-s, x - 2\sqrt{\mu s} y) \Big] dy ds, \end{aligned} \tag{10}$$

i.e.,

$$\begin{aligned} |u_x^\varepsilon(t,x)| &\le \frac{1}{\sqrt{\pi}} \int_{\mathbf{R}} \mathrm{e}^{-y^2} |(u_0^\varepsilon)'(x - 2\sqrt{\mu t} y)| dy + \\ &+ \int_0^t \frac{1}{\sqrt{\pi\mu s}} \int_{\mathbf{R}} |y| \mathrm{e}^{-y^2} \Big[c_{1,0}(1 + |u^\varepsilon(t-s, x - 2\sqrt{\mu s} y)|)^{N_{1,0}} + \\ &+ |\partial_u f(x - 2\sqrt{\mu s} y, u^\varepsilon(t-s, x - 2\sqrt{\mu s} y))|\, |u_x^\varepsilon(t-s, x - 2\sqrt{\mu s} y)| \Big] dy ds. \end{aligned}$$

Taking supremum with respect to $x \in \mathbf{R}$, for $w(t) = \sup_{x\in\mathbf{R}} |u^\varepsilon_x(t,x)| = \|u^\varepsilon_x(t,\cdot)\|_{L^\infty(\mathbf{R})}$, we obtain

$$w(t) \leq \|(u^\varepsilon_0)'\|_{L^\infty(\mathbf{R})} + c_1(1+\|u^\varepsilon\|_{L^\infty(\mathbf{R}^2_+)})^{N_{1,0}}\sqrt{\frac{t}{\mu}} + \frac{c}{\sqrt{\mu}}\|f_u\|_{L^\infty}\int_0^t \frac{1}{\sqrt{t-s}}w(s)ds. \tag{11}$$

Notice that, since f is slowly increasing, from (7) we have that f_u is bounded.

Now, we apply Lemma 3. Taking for $a(t) = \|(u^\varepsilon_0)'\|_{L^\infty} + c_1(1+\|u^\varepsilon\|_{L^\infty(\mathbf{R}^2_+)})^{N_{1,0}}\sqrt{\frac{t}{\mu}}$, and $b(s) = \frac{c\|f_u\|_{L^\infty}}{\sqrt{\mu}}\frac{1}{\sqrt{t-s}}$, from (11) and previous lemma, we obtain

$$w(t) \leq a(t) + \int_0^t a(s)b(s)\mathrm{e}^{\frac{c\|f_u\|}{\sqrt{\mu}}\sqrt{t-s}}.$$

Taking supremum with respect to $t \in (0,T)$, we obtain

$$\sup_{t\in(0,T)} w(t) \leq \|(u^\varepsilon_0)'\|_{L^\infty} + c_1(1+\|u^\varepsilon\|_{L^\infty(\mathbf{R}^2_+)})^{N_{1,0}}\sqrt{\frac{T}{\mu}} + c_2\|(u^\varepsilon_0)'\|_{L^\infty}\mathrm{e}^{\frac{c\|f_u\|}{\sqrt{\mu}}\sqrt{T}}. \tag{12}$$

Thus, for all $T > 0$, there exists $N_1 \in \mathbf{N}$ such that

$$\sup_{(t,x)\in(0,T)\times\mathbf{R}} |u^\varepsilon_x(t,x)| = \mathcal{O}(\varepsilon^{-N_1}), \quad \varepsilon \to 0^+.$$

In a similar way we can estimate other derivatives of u^ε to conclude that $\{u^\varepsilon\}_\varepsilon \in \mathcal{E}_{M,g}(R^2_+)$. Derivative with respect to t can be estimated in much easier way because the absence od D_x-term. It is important to stress that the derivative we want to estimate in equality like (10) are always multiplied by f_u on the right hand side of (10).

Finally, we conclude that $\{u^\varepsilon\}_\varepsilon$ is a representative of $u \in \mathcal{G}_g(R^2_+)$ that defines a generalized solution to (2).

To prove the uniqueness, assume that u_1 and u_2 are two generalized solutions to (2). Then there exist $N = \{N_\varepsilon\}_\varepsilon \in \mathcal{N}_g(\mathbf{R}^2_+)$ and $n = \{n_\varepsilon\}_\varepsilon \in \mathcal{N}_g(\mathbf{R})$, such that

$$(u^\varepsilon_1 - u^\varepsilon_2)_t + f(x,u^\varepsilon_1)_x - f(x,u^\varepsilon_2)_x = \mu\,(u^\varepsilon_1 - u^\varepsilon_2)_{xx} + N_\varepsilon$$
$$(u^\varepsilon_1 - u^\varepsilon_2)_{t=0} = n_\varepsilon.$$

Using again (3), we obtain

$$\begin{aligned}(u^\varepsilon_1 - u^\varepsilon_2)(t,x) &= \frac{1}{\sqrt{\pi}}\int_{\mathbf{R}} \mathrm{e}^{-y^2} n_\varepsilon(x-2\sqrt{\mu t}y)dy + \\ &+ \frac{1}{\sqrt{\pi}}\int_0^t\int_{\mathbf{R}} \mathrm{e}^{-y^2} N_\varepsilon(t-s, x-2\sqrt{\mu t}y)dyds \\ &+ \int_0^t \frac{1}{\sqrt{\pi\mu s}}\int_{\mathbf{R}} y\mathrm{e}^{-y^2}\Big[f(x-2\sqrt{\mu s}y, u^\varepsilon_1(t-s,x-2\sqrt{\mu s}y)) - \\ &- f(x-2\sqrt{\mu s}y, u^\varepsilon_2(t-s,x-2\sqrt{\mu s}y))\Big]dyds.\end{aligned}$$

This implies

$$\begin{aligned}\|\left(u_1^\varepsilon - u_2^\varepsilon\right)(t,\cdot)\|_{L^\infty} &\le \|n_\varepsilon\|_{L^\infty} + t\|N_\varepsilon\|_{L^\infty((0,T)\times\mathbf{R})} + \\ &+ \frac{c}{\sqrt{\mu}}\|f_u\|_{L^\infty}\int_0^t \frac{1}{\sqrt{t-s}}\|(u_1^\varepsilon - u_2^\varepsilon)(s,\cdot)\|_{L^\infty} ds.\end{aligned}$$

Applying Lemma 3, we obtain

$$\begin{aligned}&\|\left(u_1^\varepsilon - u_2^\varepsilon\right)(t,\cdot)\|_{L^\infty} \le \\ &\le \Big(\|n_\varepsilon\|_{L^\infty} + t\|N_\varepsilon\|_{L^\infty((0,T)\times\mathbf{R})}\Big)\Big(1 + \frac{2c}{\sqrt{\mu}}\|f_u\|_{L^\infty}\sqrt{t}\Big)\mathrm{e}^{\frac{\pi c^2}{\mu}\|f_u\|^2 t}.\end{aligned}$$

From here we conclude that

$$\sup_{(t,x)\in(0,T)\times\mathbf{R}} |\left(u_1^\varepsilon - u_2^\varepsilon\right)(t,x)| = \mathcal{O}(\varepsilon^M), \quad \varepsilon \to 0^+,$$

for all $T > 0$ and $M \in \mathbf{N}$, provided that $N \in \mathcal{N}_g(\mathbf{R}_+^2)$ and $n \in \mathcal{N}_g(\mathbf{R})$. Derivatives of $u_1^\varepsilon - u_2^\varepsilon$ can be estimated in the same way as in the proof of the existence to conclude that $u_1 - u_2 \in \mathcal{N}_g(\mathbf{R}_+^2)$. □

Considering μ as a generalized constant we can obtain similar results.

Theorem 5. *Let μ be a strictly positive generalized constant such that $1/\mu$ is of* log*-type, i.e.* $1/\mu_\varepsilon = \mathcal{O}(|\log\varepsilon|)$, $\varepsilon \to 0^+$. *Let u_0 be of a,b-bounded type, and let f be slowly increasing and satisfies* (4). *Than there exists unique $u \in \mathcal{G}_g(\boldsymbol{R}_+^2)$, of bounded type, that solves problem* (2).

Proof. The proof follows in the same manner as in the proof of Theorem 4, using that $1/\mu_\varepsilon = \mathcal{O}(\log(1/\varepsilon))$, $\varepsilon \to 0^+$, in inequalities like (12). □

Remark 6. Notice that if $\mu \approx 0$, then the solution obtained in Theorem 5 is approximated (generalized) solution to (1).

Remark 7. Our assumption on $\frac{1}{\mu}$ in Theorem 5 means that $\frac{1}{\mu}$ is a slow-scale generalized constant. We refer to [8] for this notation which is useful but not used in this paper.

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References

[1] J. Aleksić et al., *Approximate generalized solutions and measure-valued solutions to conservation laws.* Integral Transforms Spec. Funct., **20 (3-4)** (2009), 163–170.

[2] J. Aleksić, D. Mitrovic, *On the compactness for two dimensional scalar conservation law with discontinuos flux.* Commun. Math. Sci., **7 (4)** (2009), 963–971.

[3] R. Bellman, *The stability of solutions of linear differential equations.* Duke Math. J. **10 (4)** (1943), 643-647.

[4] J.F. Colombeau, *Elementary Introduction to New Generalized Functions.* North-Holland Mathematics Studies (Amsterdam: Elsevier Science Publishers B.V.), 1985.

[5] R.J. DiPerna, *Generalized solutions to conservation laws.* In: J.M. Ball (Ed), Systems of nonlinear partial differential equations (NATO ASI Series, Dordrecht: C. Reidel) (1983), 305–309.

[6] ———, *Measure-valued solutions to conservation laws.* Arch. Rat. Mech. Anal. **88 (3)** (1985), 223–270.

[7] R.J. DiPerna, A.J. Majda, *Oscillations and concentrations in weak solutions of the incompressible fluid equations.* Commun. Math. Phys., **108 (4)** (1987), 667–689.

[8] C. Garetto, G. Hörmann, *Microlocal analysis of generalized functions: pseudodifferential techniques and propagation of singularities.* Proc. Edinb. Math. Soc. **48 (3)** (2005), 603–629.

[9] M. Grosser et al., *Geometric Theory of Generalized Functions with Applications to General Relativity.* (Dordrecht: Kluwer Academic Publishers), 2001.

[10] K. H. Karlsen, M. Rascle, E. Tadmor, *On the existence and compactness of a two-dimensional resonant system of conservation laws.* Commun. Math. Sci. **5 (2)** (2007), 253–265.

[11] K. H. Karlsen, N. H. Risebro, J. D. Towers, *On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient.* Electron. J. Differential Equations **93** (2002), 23 pp.

[12] M. Nedeljkov, S. Pilipović, D. Scarpalézos, *The linear theory of Colombeau generalised functions.* Pitman Research Notes in Mathematics Series (Harlow: Longman Scientific & Technical), 1998.

[13] M. Nedeljkov, S. Pilipović, D. Rajter-Ćirić, *Heat equation with singular potential and singular data.* Proc. Roy. Soc. Edinburgh Sect. A **135 (4)** (2005), 863–886.

[14] M. Oberguggenberger, *Multiplication of distributions and applications to partial differential equations.* Pitman Research Notes in Mathematics Series, 259. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1992.

[15] M. Oberguggenberger, Y.-G. Wang, *Generalised solutions to conservation laws.* J. Anal. Appl., **13 (1)** (1994), 7–18.

[16] M. Oberguggenberger, *Generalized functions in nonlinear models—a survey.* Proceedings of the Third World Congress of Nonlinear Analysts, Part 8 (Catania, 2000). Nonlinear Anal. **47 (8)** (2001), 5029–5040.

[17] D. Pražák, *A remark on characterization of entropy solutions.* Comput. Math. Appl. **53 (3-4)** (2007), 453–460.

[18] M. E.Schonbek, *Convergence of solutions to nonlinear dispersive equations.* Comm. Partial Differential Equations **7 (8)** (1982), 959–1000.

[19] Y. G. Wang, M. Oberguggenberger, *Nonlinear parabolic equations with regularized derivatives.* J. Math. Anal. Appl. **233 (2)** (1999) 644–658.