Special $H$-matrices and their Schur and diagonal-Schur complements

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ABSTRACT

It is well known, see [D. Carlson, T. Markham, Schur complements of diagonally dominant matrices, Czech. Math. J. 29 (104) (1979) 246–251 [2]; J. Liu, J. Li, Z. Huang, X. Kong, Some properties of Schur complements and diagonal-Schur complements of diagonally dominant matrices, Linear Alg. Appl. 428 (2008) 1009–1030] [14], that the Schur complement of a strictly diagonally dominant matrix is strictly diagonally dominant, as well as its diagonal-Schur complement. Also, if a matrix is an $H$-matrix, then its Schur complement and diagonal-Schur complement are $H$-matrices, too, see [J. Liu, Y. Huang, Some properties on Schur complements of $H$-matrices and diagonally dominant matrices, Linear Alg. Appl. 389 (2004) 365–380] [13]. Recent research, see [J. Liu, Y. Huang, F. Zhang, The Schur complements of generalized doubly diagonally dominant matrices, Linear Alg. Appl. 378 (2004) 231–244] [12]; J. Liu, J. Li, Z. Huang, X. Kong, Some properties of Schur complements and diagonal-Schur complements of diagonally dominant matrices, Linear Alg. Appl. 428 (2008) 1009–1030] [14], showed that the similar statements hold for some special subclasses of $H$-matrices. The aim of this paper is to give more invariance results of this type, and simplified proofs for some already known results, by using scaling approach.

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1. Introduction

The main idea of the considerations that follow is the fact that a matrix $A$ is an $H$-matrix if and only if there exists a diagonal nonsingular matrix $W$ such that $AW$ is a strictly diagonally dominant (SDD) matrix. In other words, see [16], the class of $H$-matrices is diagonally derived from the class of SDD matrices. Some special subclasses of $H$-matrices could be characterized by the form of the corresponding scaling matrix $W$. These characterizations will be presented in short in the first section, as they have already been proven in [6], and some other subclasses of $H$-matrices will be recalled. In the second section simplified proofs of the statements from [14] will be presented, as well as another result of the same type concerning diagonal-Schur complement and Dashnic–Zusmanovich (DZ) matrices. The third section deals with another subclass of $H$-matrices, called S-Nekrasov matrices, for which we give some closure properties under taking the Schur complement and the diagonal-Schur complement in a similar way, i.e., by using scaling approach.

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Throughout the paper we will use the following notations:

\[ N := \{1, 2, \ldots, n\} \] for the set of indices,

\[ S \] for any nonempty proper subset of \( N \),

\[ \bar{S} := N \setminus S \] for the complement of \( S \),

\[ r_i(A) := \sum_{k \in N, k \neq i} |a_{ik}| \] for \( i \) th row sum, and

\[ r_i^S(A) := \sum_{k \in S, k \neq i} |a_{ik}| \] for part of \( i \) th row sum, which corresponds to the subset \( S \).

Obviously, for arbitrary subset \( S \) and each index \( i \in N \),

\[ r_i(A) = r_i^S(A) + r_i^\bar{S}(A). \]

It is important to emphasize that all the time we are dealing with nonsingular \( H \)-matrices, calling them shortly \( H \)-matrices. To be precise, we recall the definition of \( H \)-matrices, as well as some more preliminaries.

**Definition 1.** A matrix \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \) is called an \( H \)-matrix if its comparison matrix \( \langle A \rangle = [m_{ij}] \) defined by

\[ m_{ii} = |a_{ii}|, \quad m_{ij} = -|a_{ij}|, \quad i, j = 1, 2, \ldots, n, \quad i \neq j \]

is an \( M \)-matrix, i.e., \( \langle A \rangle^{-1} \geq 0 \).

**Definition 2.** A matrix \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \) is called an SDD matrix if, for each \( i \in N \), it holds that

\[ |a_{ii}| > r_i(A). \]

**Theorem 1.** If a matrix \( A \in \mathbb{C}^{n \times n} \) is an SDD matrix, then it is nonsingular, moreover it is an \( H \)-matrix.

The above statement that SDD matrices are nonsingular is an old and recurring result in matrix theory, see [15]. This basic result can be traced back to at least Levy (1881), Desplanques (1887), Minkowski (1900) and Hadamard (1903).

The next theorem was formulated in the present form in [3], but it can be treated as the same result as (M35) of Theorem 2.3 in the chapter 6 of [1].

**Theorem 2.** A matrix \( A \) is an \( H \)-matrix if and only if there exists a diagonal nonsingular matrix \( W \) such that \( AW \) is an SDD matrix. Moreover, we can always assume that \( W \) has only positive diagonal entries.

The following subclass of \( H \)-matrices has been investigated in [8,9].

**Definition 3.** A matrix \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \) is called a Dashnic–Zusmanovich matrix if there exists an index \( i \in N \) such that

\[ |a_{ii}| \cdot (|a_{ij}| - r_i(A) + |a_{ij}|) > r_i(A) \cdot |a_{ij}|, \quad \text{for all } j \neq i, j \in N. \]

**Theorem 3.** [8]. If a matrix \( A \in \mathbb{C}^{n \times n} \) is a Dashnic–Zusmanovich matrix, then it is nonsingular, moreover it is an \( H \)-matrix.

Class of \( \mathcal{S} \)-SDD matrices was defined in the present form in [4,15]. It is easy to see that this class (which is also the subclass of \( H \)-matrices) is the same one defined in [14] under the name strictly generalized doubly diagonally dominant matrices. Here we will recall one of several equivalent definitions of the \( \mathcal{S} \)-SDD class, for more details see [5].

**Definition 4.** Given any matrix \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \geq 2 \), and given any nonempty proper subset \( S \) of \( N \), then \( A \) is an \( S \)-strictly diagonally dominant (\( S \)-SDD) matrix if

\[ |a_{ii}| > r_i^S(A) \quad \text{for all } i \in S \text{ and}, \]

\[ (|a_{ii}| - r_i^S(A))(|a_{ij}| - r_i^S(A)) > r_i^S(A) r_i^S(A) \quad \text{for all } i \in S, j \in \bar{S}. \]

**Definition 5.** If there exists a nonempty proper subset \( S \) of \( N \), such that \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \geq 2 \) is an \( S \)-SDD matrix, then we will say that \( A \) belongs to class of \( \mathcal{S} \)-SDD matrices.

The following classes have been investigated under different names, see, for example, [11]. In order to be precise, we will recall all definitions we need.

First of all, we define \( h_i(A) \) recursively:

\[ h_1(A) := \sum_{j=1}^{n} |a_{ij}|, \]

\[ h_i(A) := \sum_{j=1}^{i-1} |a_{ij}| h_{i}(A) + \sum_{j=i+1}^{n} |a_{ij}|, \]

\[ r_i^S(A) := \sum_{k \in S, k \neq i} |a_{ik}| \] for part of \( i \) th row sum, which corresponds to the subset \( S \).

Throughout the paper we will use the following notations:
and $h_i^S(A)$:
\[
\begin{gathered}
\forall i \in S, h_i^S(A) := r_i^F(A), \\
\forall i \in S, h_i^S(A) := \sum_{j=1}^{n} \left| a_{ij} \right| \frac{h_j^S(A)}{|a_{ij}|} + \sum_{j=1, j \notin S}^{n} |a_{ij}|.
\end{gathered}
\]

Obviously, for arbitrary subset $S$ and each index $i \in N$,
\[
h_i(A) = h_i^S(A) + h_i^S(A).
\]

**Definition 6.** A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2$ is called Nekrasov matrix if, for each $i \in N$, it holds that $|a_{ii}| > h_i(A)$.

**Definition 7.** Given any matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2$, and given any nonempty proper subset $S$ of $N$, then $A$ is an $S$-Nekrasov matrix if
\[
\begin{aligned}
|a_{ii}| &> h_i^S(A) &\text{for all } i \in S, \\
|a_{i\cdot}| &> h_i^S(A) &\text{for all } j \in S, \quad \text{and,} \\
|a_{i\cdot}| - h_i^S(A) &> h_i^S(A) - h_i^S(A) &\text{for all } i \in S, j \in S.
\end{aligned}
\]

**Definition 8.** If there exists a nonempty proper subset $S$ of $N$, such that $A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2$ is an $S$-Nekrasov matrix, then we will say that $A$ belongs to class of $\mathcal{S}$-Nekrasov matrices.

### 2. Scaling matrices in characterization of some subclasses of $H$-matrices

According to **Theorem 2**, a matrix $A \in \mathbb{C}^{n \times n}$ is an $H$-matrix if and only if there exists a nonsingular diagonal matrix $W$ such that $A W$ is an SDD matrix. But, such a matrix $W$ could be found in a very few special cases. Up to now, we are aware of two such cases: Dashnic–Zusmanovich matrices and $\mathcal{S}$-SDD matrices.

Namely, Dashnic–Zusmanovich class can be characterized as a subclass of $H$-matrices for which the corresponding scaling matrix $W$ belongs to the set $\mathcal{F}$, defined as the set of diagonal matrices, whose diagonal entries are equal to 1, all except one, which is an arbitrary positive number, i.e.,
\[
\mathcal{F} = \{ W = \text{diag}(w_1, w_2, \ldots, w_n) : w_i = \gamma > 0 \text{ for one } i \in N, \text{ and } w_j = 1 \text{ for } j \neq i \}.
\]

From the other hand, the $\mathcal{S}$-SDD class can be characterized as a subclass of $H$-matrices for which the corresponding scaling matrix $W$ belongs to the set $\mathcal{F}$, defined as the set of all diagonal matrices whose diagonal entries are either 1 or $\gamma$, where $\gamma$ is an arbitrary positive number, i.e.,
\[
\mathcal{F} = \bigcup_{S \subseteq N} \mathcal{F}^S,
\]
\[
\mathcal{F}^S = \{ W = \text{diag}(w_1, w_2, \ldots, w_n) : w_i = \gamma > 0 \text{ for } i \in S \text{ and } w_i = 1 \text{ otherwise} \}.
\]

In the next section we will use the following theorems, proved in [6]:

**Theorem 4.** A matrix $A$ is an $\mathcal{S}$-SDD matrix if and only if there exists a matrix $W \in \mathcal{F}$ such that $A W$ is an SDD matrix.

**Theorem 5.** A matrix $A$ is a Dashnic–Zusmanovich matrix if and only if there exists a matrix $W \in \mathcal{F}$ such that $A W$ is an SDD matrix.

Concerning the class of $S$-Nekrasov matrices, at this point we just want to emphasize that it can be characterized as a subclass of $H$-matrices for which the corresponding scaling matrix $W$, which scales it into the class of Nekrasov matrix, belongs to the set $\mathcal{F}$ defined above.

### 3. Diagonal-Schur complement of S-D matrices and DZ matrices

The diagonal-Schur complement of $A$ with respect to a proper subset of $N, \alpha$, is denoted by $A(\alpha, \alpha)$ and defined to be
\[
A(\alpha, \alpha) - \{ A(\alpha, \beta)(A(\beta))^{-1}A(\alpha, \beta) \} \circ I
\]
\[
A(\alpha, \beta) \text{ stands for the submatrix of } A \in \mathbb{C}^{n \times n} \text{ lying in the rows indexed by } \alpha \text{ and the columns indexed by } \beta, \text{ while } A(\alpha, \alpha) \text{ is abbreviated to } A(\alpha). \text{ For } A = (a_{ij}) \in \mathbb{C}^{n \times n} \text{ and } B = (b_{ij}) \in \mathbb{C}^{n \times n}, \text{ the Hadamard product of } A \text{ and } B \text{ is the matrix } (a_{ij}b_{ij}), \text{ which we denote by } A \odot B. \text{ Throughout the paper we assume that } A(\alpha) \text{ is a nonsingular matrix.}
In [14] the following theorem has been proven.

**Theorem 6.** Let $A \in SGDD_{n}^{N_{1},N_{2}}, \alpha \subset N$. If $N_{1} \subseteq \alpha$ or $N_{2} \subseteq \alpha$, then

$$A/_{/\alpha} \in SDn_{\alpha}[.].$$

If $N_{1} \subseteq \alpha$ and $N_{2} \not\subseteq \alpha$, then

$$A/_{/\alpha} \in SDGDD_{n}^{N_{1},N_{2} - \alpha}.$$  

First, let us explain the above notation. A matrix $A$ from $C^{n,n}$ is called a strictly generalized doubly diagonally dominant matrix in $C^{n,n}$ if there exist proper subsets $N_{1},N_{2}$ of $N$ such that $N_{1} \cap N_{2} = \emptyset, N_{1} \cup N_{2} = N$ and

$$(|a_{ii}| - \alpha_{i})(|a_{ij}| - \beta_{j}) > \beta_{i}\alpha_{j}$$

for all $i \in N_{1}$ and $j \in N_{2}$, where, with $s = i$ or $j$,

$$\alpha_{s} = \sum_{t \in N_{1} \cup s} |a_{st}|,$$

$$\beta_{s} = \sum_{t \in N_{2} \cup s} |a_{st}|.$$

For this choice of $N_{1},N_{2}$, we write $A \in SGDD_{n}^{N_{1},N_{2}}$. But, obviously, $SGDD_{n}^{N_{1},N_{2}}$ is the same set as the one that we call $N_{1}$-SDD matrices, while the set $SGDD_{n}$ of all strictly generalized doubly diagonally dominant matrices in $C^{n,n}$ is, in fact, our set $\mathcal{S}$-SDD. The set $SDn$ is actually the set of all strictly diagonally dominant (SDD) matrices in $C^{n,n}$.

**Theorem 6** has been proven in [14] using various algebraic inequalities. We will show here the simplified proof for both statements in this theorem.

**Theorem 7** (The same as **Theorem 6**). Let $A = [a_{ij}] \in C^{n,n}$ be an S-SDD matrix. Then for any nonempty proper subset $\alpha$ of $N$:

- such that $S \subseteq \alpha$ or $\bar{S} \subseteq \alpha$, $A/_{/\alpha}$ is an SDD matrix;
- $A/_{/\alpha}$ is also an $\mathcal{S}$-SDD matrix. More precisely, if $A$ is an S-SDD matrix, then $A/_{/\alpha}$ is an $(S \setminus \alpha)$-SDD matrix.

**Proof.** Let $A$ be an S-SDD matrix. Then, from **Theorem 4**, there exists a matrix $W \in \mathcal{W}$ (defined by (2)), such that $AW$ is an SDD matrix. As the diagonal-Schur complement of a strictly diagonally dominant matrix is strictly diagonally dominant, too, we conclude that $AW/_{/\alpha}$ is strictly diagonally dominant matrix. As in [6], it is easy to see that

$$(AW)/_{/\alpha} = (A/_{/\alpha}) \cdot W(\alpha).$$

- Since $W(\alpha)$ is either the identity matrix, $I$ (if $S \subseteq \alpha$), or $\gamma \cdot I$ (if $\bar{S} \subseteq \alpha$), it will not affect the strict diagonal dominance. Therefore, $A/_{/\alpha}$ is a strictly diagonally dominant matrix.
- Since $W(\alpha) \in \mathcal{W}$, i.e., the class $\mathcal{W}$ is closed under taking principal submatrices, from **Theorem 4** we obtain that $A/_{/\alpha}$ is an $\mathcal{S}$-SDD matrix. To complete the proof it is enough to see that the matrix $W(\alpha)$ is of the form

$$W(\alpha) = diag(w_{1},w_{2},\ldots,w_{\ell})$$

with

$$w_{i} = \gamma > 0 \text{ for } i \in S \setminus \alpha \text{ and } w_{i} = 1 \text{ otherwise}.$$  

Obviously, using diagonal scaling, the proofs can be significantly shortened, but this technique allows us to get invariance theorems for some other subclasses of $H$-matrices, as we can see from the following theorem.

**Theorem 8.** Let $A = [a_{ij}] \in C^{n,n}$ be a Dashnic–Zusmanovich matrix. Then for any nonempty proper subset $\alpha$ of $N$, $A/_{/\alpha}$ is also a Dashnic–Zusmanovich matrix.

**Proof.** Let $A = [a_{ij}] \in C^{n,n}$ be a Dashnic–Zusmanovich matrix. Then, from **Theorem 5**, there exists a matrix $W \in \mathcal{F}$ (defined by (1)), such that $AW$ is an SDD matrix. As the diagonal-Schur complement of a strictly diagonally dominant matrix is strictly diagonally dominant, $AW/_{/\alpha}$ is strictly diagonally dominant, too. Since

$$(AW)/_{/\alpha} = (A/_{/\alpha}) \cdot W(\alpha)$$

with $W(\alpha) \in \mathcal{F}$, **Theorem 5** provides that $A/_{/\alpha}$ is a Dashnic–Zusmanovich matrix.  

Moreover, if for the given matrix $A$ there exists a scaling matrix $W \in \mathcal{F}$ with $w_{i} = \gamma > 0$ where $\{i\} \subseteq \alpha$ or $N \setminus \{i\} = \alpha$, then $A/_{/\alpha}$ is a strictly diagonally dominant matrix. This can be derived from **Theorem 8** with $S = \{i\}$. 

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Notes:
- $C^{n,n}$ refers to the set of all $n \times n$ matrices.
- $\mathcal{S}$-SDD refers to strictly diagonally dominant matrices.
- $\mathcal{W}$ refers to a class of matrices.
- $diag(w_{1},w_{2},\ldots,w_{\ell})$ denotes a diagonal matrix with diagonal entries $w_{1},w_{2},\ldots,w_{\ell}$.
- $w_{i} = \gamma > 0$ indicates scaling by a factor $\gamma$.
- $\mathcal{F}$ refers to another class of matrices.
- The proofs rely on algebraic inequalities and scaling techniques.
- The theorems provide insights into the structure and properties of these matrices.
4. Schur and diagonal-Schur complements of Nekrasov and S-Nekrasov matrices

As we have already mentioned before, S-Nekrasov matrices can be characterized by the form of scaling matrices which transform them to Nekrasov matrices. Here, we present this fact as a theorem, for its proof see [7].

**Theorem 9.** A matrix $A$ is an $\mathcal{S}$-Nekrasov matrix if and only if there exists a matrix $W \in \mathcal{W}$ such that $AW$ is a Nekrasov matrix.

To prove some properties of the Schur and diagonal-Schur complements, we need some additional notation and preliminaries.

**Definition 9.** The Schur complement of $A$ with respect to a proper subset of $\mathcal{N}$, $\mathcal{A}$, is denoted by $A(\mathcal{A})$ and defined to be

$A(\mathcal{A}) = A(\mathcal{N}, \mathcal{A})(A(\mathcal{A}))^{-1}A(\mathcal{A}, \mathcal{N})$.

**Definition 10.** For a given proper subset of the index set, $\mathcal{A}$, we say that a matrix class $\mathcal{C}$ is $\mathcal{A}$-SC-closed if for any $A \in \mathcal{C}$, $A(\mathcal{A}) \in \mathcal{C}$.

**Definition 11.** A matrix class $\mathcal{C}$ is SC-closed if $\mathcal{C}$ is $\mathcal{A}$-SC-closed for all $\mathcal{A}$.

**Definition 12.** We say that a matrix class $\mathcal{C}$ is $\mathcal{A}$-diagonal-SC-closed if for any $A \in \mathcal{C}$, $A(\mathcal{A})^{(1)} \in \mathcal{C}$.

**Definition 13.** A matrix class $\mathcal{C}$ is diagonal-SC-closed if $\mathcal{C}$ is $\mathcal{A}$-diagonal-SC-closed for all $\mathcal{A}$.

**Theorem 10** (Sequential property of Schur complement [16]). Let $A \in \mathbb{C}^{n \times n}$ be principally nonsingular and suppose that $\mathcal{A}$ is a proper subset of $\mathcal{N}$ and that $\mathcal{B}$ is a proper subset of $\mathcal{A}$. Then,

$(A(\mathcal{A}))^{(\mathcal{B})} = A(\mathcal{A} \cup \mathcal{B})$.

It is important to note that the same property does not hold for the diagonal-Schur complement.

In [10], it has been proven that Nekrasov property is hereditary for Gaussian elimination, which implies the following:

**Corollary.** The Nekrasov class is $\{1\}$-SC-closed.

Using the scaling characterization, from the above fact we obtain:

**Theorem 11.** If $A$ is S-Nekrasov matrix, then $A(\{1\})$ is $S(\{1\})$-Nekrasov matrix.

**Proof.** Let $A$ be an S-Nekrasov matrix. Then, from Theorem 4, there exists a matrix $W \in \mathcal{W}$ (defined by (2)), such that $AW$ is an Nekrasov matrix. As the $\{1\}$-Schur complement of a Nekrasov matrix is an Nekrasov matrix, too, we conclude that $AW(\{1\})$ is a Nekrasov matrix. We have also

$(AW)(\{1\}) = (A(\{1\})) \cdot W(\{\mathcal{T}\}))$.  

Since $W(\{\mathcal{T}\}) \in \mathcal{W}$ is of the form

$W(\{\mathcal{T}\}) = \text{diag}(w_1, w_2, \ldots, w_\ell)$

with

$w_i = \gamma > 0$ \quad for $i \in S \setminus \{1\}$ and $w_i = 1$ otherwise,

from Theorem 4 we obtain that $A(\{1\})$ is an $S(\{1\})$-Nekrasov matrix. □

Or, in other words:

**Theorem 12.** The $\mathcal{S}$-Nekrasov class is $\{1\}$-SC-closed. Moreover, from the sequential property of Schur complement, it is $\mathcal{A}$-SC-closed for all $\mathcal{A} = \{1, 2, \ldots, m\}$.

The similar closure properties hold for diagonal-Schur complement:

**Theorem 13.** The Nekrasov class is $\{1\}$-diagonal-SC-closed.

**Theorem 14.** If $A$ is S-Nekrasov matrix, then $A(\{1\})$ is $S(\{1\})$-Nekrasov matrix.

**Theorem 15.** The $\mathcal{S}$-Nekrasov class is $\{1\}$-diagonal-SC-closed.

We are concluding this section with the following remark: As the sequential property doesn’t hold for diagonal-Schur complement, we don’t have immediately the $\mathcal{A}$-diagonal-SC-closure for $\mathcal{A} = \{1, 2, \ldots, m\}$ as before.
References