Further results on $H$-matrices and their Schur complements

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Abstract

It is well-known [D. Carlson, T. Markham, Schur complements of diagonally dominant matrices, Czech. Math. J. 29 (104) (1979) 246–251, [1]] that the Schur complement of a strictly diagonally dominant matrix is strictly diagonally dominant. Also, if a matrix is an $H$-matrix, then its Schur complement is an $H$-matrix, too [J. Liu, Y. Huang, Some properties on Schur complements of $H$-matrices and diagonally dominant matrices, Linear Algebra Appl. 389 (2004) 365–380, [8]]. Recent research showed that the same type of statement holds for some special subclasses of $H$-matrices, see, for example, Liu et al. [J. Liu, Y. Huang, F. Zhang, The Schur complements of generalized doubly diagonally dominant matrices, Linear Algebra Appl. 378 (2004) 231–244]. The aim of this paper is to show that the proof of these results can be significantly simplified by using “scaling” approach as in Zhang et al. [F. Zhang et al., The Schur Complement and its Applications, Springer, New York, 2005] and to give another invariance result of this type.

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1. Introduction

The core idea of the considerations that follow is the fact that a matrix $A$ is an $H$-matrix if and only if there exists a diagonal nonsingular matrix $W$ such that $AW$ is a strictly diagonally dominant (SDD) matrix. In other words, see [10], the class of $H$-matrices is diagonally derived from the class of SDD matrices. Some special subclasses of $H$-matrices could be characterized by the form of the corresponding scaling matrix $W$. These characterizations will be presented in Section 2, although they have already been proven in [3] as the corresponding Geršgorin-type theorems. In Section 3, a simplified proofs of the main theorems from [7] will be presented. Section 4 deals with one more subclass of $H$-matrices, for which we can prove invariance property of the Schur complement in a similar way, i.e. by using scaling approach.

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Throughout the paper we will use the following notations:

\[ N := \{1, 2, \ldots, n\} \text{ for the set of indices} \]
\[ S \text{ for any nonempty proper subset of } N \]
\[ \overline{S} := N \setminus S \text{ for the complement of } S \]
\[ r_i(A) := \sum_{k \in N, k \neq i} |a_{ik}| \text{ for } i \text{th row sum and} \]
\[ r^S_i(A) := \sum_{k \in S, k \neq i} |a_{ik}| \text{ for part of } i \text{th row sum, which corresponds} \]
\[ \text{to the subset } S. \]

Obviously, for arbitrary subset \( S \) and each index \( i \in N, r_i(A) = r^S_i(A) + r^{\overline{S}}_i(A) \).

It is important to emphasize that all the time we are dealing with nonsingular \( H \)-matrices, calling them shortly \( H \)-matrices. To be precise, we recall the definition of \( H \)-matrices, as well as some more preliminaries.

**Definition 1.** A matrix \( A = [a_{ij}] \in \mathbb{C}^{n,n} \) is called an \( H \)-matrix if its comparison matrix \( \langle A \rangle = [m_{ij}] \) defined by

\[ m_{ii} = |a_{ii}|, \quad m_{ij} = -|a_{ij}|, \quad i, j = 1, 2, \ldots, n, \quad i \neq j \]

is an \( M \)-matrix, i.e. \( \langle A \rangle^{-1} \geq 0 \).

**Theorem 1.** If a matrix \( A \in \mathbb{C}^{n,n} \) is an SDD matrix, then it is nonsingular, moreover it is an \( H \)-matrix.

**Theorem 2.** A matrix \( A \) is an \( H \)-matrix if and only if there exists a diagonal nonsingular matrix \( W \) such that \( AW \) is an SDD matrix. Moreover, we can always assume that \( W \) has only positive diagonal entries.

The following subclass of \( H \)-matrices has been investigated in [5,6].

**Definition 3.** A matrix \( A = [a_{ij}] \in \mathbb{C}^{n,n} \) is called a Dashnic–Zusmanovich matrix if there exists an index \( i \in N \) such that

\[ |a_{ii}| \cdot (|a_{jj}| - r_j(A) + |a_{ji}|) > r_i(A) \cdot |a_{ji}|, \quad \text{for all } j \neq i, \quad j \in N. \]

**Theorem 3.** If a matrix \( A \in \mathbb{C}^{n,n} \) is a Dashnic–Zusmanovich matrix, then it is nonsingular, moreover it is an \( H \)-matrix.

Class of \( \mathcal{S} \)-SDD matrices was defined in the present form in [2,9]. It is easy to see that this class (which is also the subclass of \( H \)-matrices) is the same one defined in [7] under the name strictly generalized doubly diagonally dominant matrices. Here we will recall one of several equivalent definitions of the \( \mathcal{S} \)-SDD class, for more details see [4].

**Definition 4.** Given any matrix \( A = [a_{ij}] \in \mathbb{C}^{n,n}, n \geq 2 \), and given any nonempty proper subset \( S \) of \( N \), then \( A \) is an \( S \)-strictly diagonally dominant (\( S \)-SDD) matrix if

\[ |a_{ii}| > r^S_i(A) \quad \text{for all } i \in S \quad \text{and} \quad \left( |a_{ii}| - r^S_i(A) \right) \left( |a_{jj}| - r^S_j(A) \right) > r^S_j(A) r^S_i(A) \quad \text{for all } i, j \in S, \quad j \in \overline{S}. \]

**Definition 5.** If there exists a nonempty proper subset \( S \) of \( N \), such that \( A = [a_{ij}] \in \mathbb{C}^{n,n}, n \geq 2 \) is an \( S \)-SDD matrix, then we will say that \( A \) belongs to class of \( \mathcal{S} \)-SDD matrices.

2. Scaling matrices in characterization of some subclasses of \( H \)-matrices

According to Theorem 2, a matrix \( A \in \mathbb{C}^{n,n} \) is an \( H \)-matrix if and only if there exists a nonsingular diagonal matrix \( W \) such that \( AW \) is an SDD matrix. But, such a matrix \( W \) could be found in a very few special cases. Up to now, we are aware of two such cases: Dashnic Zusmanovich matrices and \( \mathcal{S} \)-SDD matrices.
Namely, Dashnic Zusmanovich class can be characterized as a subclass of $H$-matrices for which the corresponding scaling matrix $W$ belongs to the set $\mathcal{F}$, defined as the set of diagonal matrices, whose diagonal entries are equal to 1, all except one, which is an arbitrary positive number, i.e

$$\mathcal{F} = \{ W = \text{diag}(w_1, w_2, \ldots, w_n) : w_i = \gamma > 0 \text{ for one } i \in N, \text{ and } w_j = 1 \text{ for } j \neq i \}. \quad (1)$$

From the other hand, the $\mathcal{S}$-SDD class can be characterized as a subclass of $H$-matrices for which the corresponding scaling matrix $W$ belongs to the set $\mathcal{W}$, defined as the set of all diagonal matrices whose diagonal entries are either 1 or $\gamma$, where $\gamma$ is an arbitrary positive number, i.e

$$\mathcal{W} = \bigcup_{S \subseteq N} \mathcal{W}_S,$$

$$\mathcal{W}_S = \{ W = \text{diag}(w_1, w_2, \ldots, w_n) : w_i = \gamma > 0 \text{ for } i \in S \text{ and } w_i = 1 \text{ otherwise} \}. \quad (2)$$

Although the above two facts are already known (see, for example, the related statements in the eigenvalue localization field in [3]), we find appropriate to present them here as Theorems, along with their elegant proofs.

**Theorem 4.** A matrix $A$ is an $\mathcal{S}$-SDD matrix if and only if there exists a matrix $W \in \mathcal{W}$ such that $AW$ is an SDD matrix.

**Proof.** Let $A$ be an $\mathcal{S}$-SDD matrix, i.e. there exists a nonempty proper subset $S \subseteq N$ such that $A$ is an $S$-SDD matrix. Define diagonal matrix $W(S) \in \mathcal{W}$ in the following way:

$$W(S) = \text{diag}(w_1, w_2, \ldots, w_n),$$

where

$$w_i = \begin{cases} \gamma, & i \in S, \\ 1, & i \in \bar{S}. \end{cases}$$

We will show that for a given $S$, $A$ is an $S$-SDD matrix if and only if $AW(S)$ is an SDD matrix.

First, assume that $A$ is an $S$-SDD matrix. We choose $\gamma$ from the interval $(\gamma_1(A), \gamma_2(A))$, with:

$$0 \leq \gamma_1(A) := \max_{i \in S} \frac{r_i^S(A)}{a_{ii}}, \quad \gamma_2(A) := \min_{j \in S} \frac{|a_{ji}| - r_j^S(A)}{r_j^S(A)},$$

where if $r_j^S(A) = 0$ for some $j \in \bar{S}$, the final fraction is defined to be $+\infty$. Note that, according to the definition of $S$-SDD matrices, the interval $(\gamma_1(A), \gamma_2(A))$ is not empty. Now, it is easy to check that $AW(S)$ is an SDD matrix.

Second, if we suppose that $AW(S)$ is an SDD matrix, then $\gamma$ has to be chosen from the interval $(\gamma_1(A), \gamma_2(A))$, which means that this interval is not empty. But, this implies that matrix $A$ is an $S$-SDD matrix. \(\square\)

Similarly, we can prove the following theorem:

**Theorem 5.** A matrix $A$ is a Dashnic–Zusmanovich matrix if and only if there exists a matrix $W \in \mathcal{F}$ such that $AW$ is an SDD matrix.

### 3. Schur complement of $S$-SDD matrices

In [7], the following theorems have been proven:

**Theorem 6.** Let $A \in \text{SGDD}_n^{n_1, n_2}$. If $n_1 \subseteq m \subseteq n$ or $n_2 \subseteq m \subseteq n$, then

$$A/m \in \text{SD}_n^{-|m|}.$$

**Theorem 7.** Let $A \in \text{SGDD}_n^{n_1, n_2}$. Then for any proper subset $m$ of $n$,

$$A/m \in \text{SGDD}_n^{-|m|^{n_1-m}n_2-m}.$$
First, let us explain the above notation. A matrix $A$ from $\mathbb{C}^{n \times n}$ is called a strictly generalized doubly diagonally dominant matrix in $\mathbb{C}^{n \times n}$ if there exist proper subsets $n_1, n_2$ of $n$ such that $n_1 \cap n_2 = \emptyset$, $n_1 \cup n_2 = n$ and

$$\left(|a_{ii}| - \sum_{j \in n_1 \cap n_j} |a_{ij}| \right) - \beta_j \alpha_j > 0$$

for all $i \in n_1$ and $j \in n_2$, where with $s = i$ or $j$,

$$\alpha_s = \sum_{j \in n_1 \cap n_s} |a_{ij}|, \quad \beta_s = \sum_{j \in n_2 \cap n_s} |a_{ij}|.$$ 

For this choice of $n_1, n_2$, we write $A \in \text{SGDD}_{n,n}$. But, obviously, $\text{SGDD}_{n,n}$ is the same set as the one that we call $n_1$-DD matrices, while the set $\text{SGDD}_n$ of all strictly generalized doubly diagonally dominant matrices in $\mathbb{C}^{n \times n}$ is, in fact, our set $\mathcal{F}$-SDD. The set $\text{SD}_n$ is actually the set of all strictly diagonally dominant (SDD) matrices in $\mathbb{C}^{n \times n}$.

The Schur complement of $A$ with respect to a proper subset of $n$, $\alpha$, is denoted by $A/\alpha$ and defined to be:

$$A(\bar{\alpha}) = A(\bar{\alpha}, \alpha)(A(\alpha))^{-1}A(\alpha, \bar{\alpha}),$$

where $A(\alpha, \beta)$ stands for the submatrix of $A \in \mathbb{C}^{n \times n}$ lying in the rows indexed by $\alpha$ and the columns indexed by $\beta$, while $A(\alpha, \bar{\alpha})$ is abbreviated to $A(\alpha)$. Throughout the paper we assume that $A(\alpha)$ is a nonsingular matrix.

The previous two theorems have been proven in [7] using various algebraic inequalities. We will show here the simplified proof for both of them.

**Theorem 8** (Same as Theorem 6). Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be an S-SDD matrix. Then for any nonempty proper subset $\alpha$ of $n$ such that $S \subseteq \alpha$ or $\bar{S} \subseteq \alpha$, $A/\alpha$ is an SDD matrix.

**Proof.** Let $A$ be an S-SDD matrix. Then, from Theorem 4, there exists a matrix $W \in \mathcal{W}$ (defined by (2)), such that $AW$ is an SDD matrix. As the Schur complement of a strictly diagonally dominant matrix is strictly diagonally dominant, too, we conclude that $AW/\alpha$ is strictly diagonally dominant matrix. As in [10], it is easy to see that

$$(AW)/\alpha = (A/\alpha) \cdot (W/\alpha),$$

where $W/\alpha$ is actually $W(\bar{\alpha})$. Since $W/\alpha$ is either the identity matrix, $I$ (if $S \subseteq \alpha$), or $\gamma \cdot I$ (if $\bar{S} \subseteq \alpha$), it will not affect the strict diagonal dominance. Therefore, $A/\alpha$ is a strictly diagonally dominant matrix. \(\square\)

**Theorem 9** (Same as Theorem 7). Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be an $\mathcal{F}$-SDD matrix. Then for any nonempty proper subset $\alpha$ of $n$, $A/\alpha$ is also an $\mathcal{F}$-SDD matrix. More precisely, if $A$ is an S-SDD matrix, then $A/\alpha$ is an $(S \setminus \alpha)$-SDD matrix.

**Proof.** Let $A$ be an $\mathcal{F}$-SDD matrix. Then, from Theorem 4, there exists a matrix $W \in \mathcal{W}$ (defined by (2)), such that $AW$ is an SDD matrix. As the Schur complement of a strictly diagonally dominant matrix is strictly diagonally dominant, too, we conclude that $AW/\alpha$ is strictly diagonally dominant matrix. Again, we have

$$(AW)/\alpha = (A/\alpha) \cdot (W/\alpha).$$

Since $W/\alpha \in \mathcal{W}$, i.e. the class $\mathcal{W}$ is closed under taking principal submatrices, from Theorem 4, we obtain that $A/\alpha$ is an $\mathcal{F}$-SDD matrix. To complete the proof it is enough to see that the matrix $W/\alpha$ is of the form

$$W/\alpha = \text{diag}(w_{i_1}, w_{i_2}, \ldots, w_{i_l})$$

with

$$w_{i_j} = \gamma > 0 \quad \text{for} \quad i_j \in S \setminus \alpha \quad \text{and} \quad w_{i_j} = 1 \quad \text{otherwise}. \quad \square$$

### 4. New invariance result for Schur complement

Let us now consider another subclass of $H$-matrices and corresponding invariance theorem.
Theorem 10. Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a Dashnic–Zusmanovich matrix. Then for any nonempty proper subset $\alpha$ of $N$, $A/\alpha$ is also a Dashnic–Zusmanovich matrix.

Proof. Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a Dashnic–Zusmanovich matrix. Then, from Theorem 5, there exists a matrix $W \in \mathcal{F}$ (defined by (1)), such that $AW$ is an SDD matrix. As the Schur complement of a strictly diagonally dominant matrix is strictly diagonally dominant, $AW/\alpha$ is strictly diagonally dominant, too. Since

$$(AW)/\alpha = (A/\alpha) \cdot (W/\alpha)$$

with $W/\alpha \in \mathcal{F}$, Theorem 5 provides that $A/\alpha$ is a Dashnic–Zusmanovich matrix. $\square$

Moreover, if for the given matrix $A$, there exists a scaling matrix $W \in \mathcal{F}$ with $w_i = \gamma > 0$, where $\{i\} \subseteq \alpha$ or $N \setminus \{i\} = \alpha$, then $A/\alpha$ is a strictly diagonally dominant matrix. This can be derived from Theorem 8 with $S = \{i\}$.

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