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ON THE UNIQUENESS OF SOLUTION TO GENERALIZED CHAPLYGIN GAS

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Abstract. The main object of the paper is finding a unique solution to Riemann problem for generalized Chaplygin gas model. That is a model of the dark energy in Universe introduced in the last decade. It permits an infinite mass concentration so one has to consider solutions containing the Dirac delta function. Although it was easy to construct solution to any Riemann problem, the usual admissibility conditions, overcompressiveness, do not exclude unwanted delta-type waves when a classical solution exists. We are using Shadow Wave approach in order to solve that uniqueness problem since they are well adopted for using Lax entropy–entropy flux conditions and there is a rich family of convex entropies.

1. Introduction.

A generalized Chaplygin gas appears in a number cosmology theories and it is a model for a compressible fluid with a pressure inversely proportional to a gas energy density, $p = -C/\rho^\alpha$, $C > 0$, $0 < \alpha < 1$, see [2] for the first model, and [9] for some more advanced models. It is used as a model for the dark energy in the Universe. (We will use $C = 1$ in the rest of the paper for simplicity.) The system consists from the mass and momentum conservation laws

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x\Big(\rho u^2 - \frac{1}{\rho^\alpha}\Big) &= 0, \end{aligned}$$

where u denotes a velocity of the gas. In this paper we use the momentum variable $q = \rho u$:

$$\begin{aligned} \partial_t \rho + \partial_x q &= 0 \\ \partial_t q + \partial_x\Big(\frac{q^2}{\rho} - \frac{1}{\rho^\alpha}\Big) &= 0. \end{aligned} \tag{1}$$

The physical region for both systems is $\rho > 0$ and the sound speed of the system tends to zero as $\rho \to \infty$. Note that we do not have vacuum states due to division by zero in the flux. That is, we do not loose a solution by rewriting the original

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system into evolutionary form (1) like in the case of isentropic gas-dynamics model, for example. That property allows a mass concentration in a finite time and one could expect some kind of non-classical solutions containing Dirac delta function. (One of the first definitions up to our knowledge was given in [12].) It was not so difficult to construct solutions of such type for (1). Moreover, using solutions of that kind (let us call them delta shocks) one can always solve arbitrary Riemann problem for that system. Let us note that there is a solution to system (1) with the Riemann data containing a delta shock solution constructed in [21] (and the one for the system with friction in [20]). But in both cases, the authors just claimed and proved that there is an overcompressive delta shock solution in a region without classical ones. They overlooked a fact that there is a region where exists both kinds of solution: a classical and a delta shock ones. That means that the system may not have a unique solution. Our aim is to give some progress in solving that problem. In all systems admitting some kind of the delta function in a solution we have found in the literature, one could use a fact that non-classical solution containing the delta function has to be overcompressive. That condition ruled out unwanted solutions of that kind. In the case of system (1), the overcompressibility condition is not good enough as one can see bellow. There is a region where there are both two-shock and overcompressive delta shock solutions. Using the shadow wave solutions we could try with convex entropies and Lax entropy condition. One can look at [15] about relations between the overcompressibility and Lax entropy condition with (semi-)convex entropies when delta shocks are represented by shadow waves. In most of the cases these conditions are equivalent. But, there is at least one case when overcompressibility is superior to the Lax condition, 2×2 pressureless gas dynamics, as written in [7]. (Interestingly, for the full system, with energy equation added, these two admissibility conditions are equivalent again.)

Concerning system (1), we found few families of convex entropies using standard procedure (see [6]) that can be used for admissibility check. Note that we were not able to find all convex entropies to the system. Using the obtained entropies we got the following results. First, we reduced the area where there are non-uniqueness problems. That clearly means that we have obtained better admissibility conditions than overcompressibility. This is the only such result in the literature up to our knowledge and that is the most interesting result of the paper. We were not able to prove that these entropy functions suffice for uniqueness proof. One can see some examples and some numerical illustrations that give us some signs that it could be possible to prove uniqueness. There are two possible reasons for that failure. One is that convex entropies involves modified Bessel functions of the second kind that are quite tough for proper approximation in both directions to zero or infinity. But maybe we need some wider families of convex entropies in order to prove the uniqueness. We left these questions open.

Let us note that for the well known Chaplygin model $\alpha = 1$ there is a unique solution to a Riemann problem. There is a delta shock solution in some cases (see [3]). It is interesting that in this case a lot of conditions are good enough to obtain a unique solution: overcompressibility, and entropy condition with only one convex entropy functional (mechanical energy, for example). One can look in [16] for that.

Among a numerous number of papers dealing with delta shock waves, we can mention [23], where authors studied a class of non-strictly hyperbolic systems of conservation laws where they manage to find a kind of delta shock waves where both state variables contain the Dirac delta function, unlike most papers where only one state variable contains the Dirac delta function. Let us also mention [17] where one can find a definition for more singular non-classical objects – δ' shock waves, and [10] describing something that looks like $\sqrt{\delta}$ (see also [14] that contains some additional properties of such waves). The papers [5] and [13] contain some examples of delta shock formation from classical shocks. Note that all these objects can be substituted by appropriate shadow waves used in this paper.

The paper is organized as follows. Section 2 contains some basic properties of the given system. Section 3 contains the existence proof to the Riemann problem but without uniqueness. In order to gain uniqueness, in Section 4, we use convex entropy – entropy flux pair. Our intention is to employ entropy conditions in order to weed out inadmissible solutions. Actually, our aim was to prove that a simple shadow wave solution (SDW for short) to our problem is admissible for all convex entropy pairs only at the points that can not be connected by two shock solution. In general, there are two entropy conditions that SDW solution has to satisfy. The first one is connected with δ'-part, and the second one with δ-part. We proved that first entropy condition is true for all convex entropy pairs we have found. But with the second condition we were only partially successful. In Section 5 we gave some partial uniqueness results. We also proved local theorem which gave us the existence of the points that can be connected with the left-hand state by two shock solution where entropy condition is not satisfied while overcompressibility condition is. Thus, we have proved that the entropy conditions are better than well-known overcompressibility condition in that case. That is the first result of that kind up to our knowledge.

2. **Properties of the system.** Let us briefly give the properties of the system. One can use a standard textbooks about conservation law systems, like [4], [6] or [19]. It is strictly hyperbolic system with the eigenvalues $\lambda_1 = \frac{q}{\rho} - \sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}$, $\lambda_2 = \frac{q}{\rho} + \sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}$ and appropriate eigenvectors $r_1 = \left(-1, -\frac{q}{\rho} + \sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}\right)^T$ and $r_2 = \left(1, \frac{q}{\rho} + \sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}\right)^T$. Both fields are genuinely nonlinear.

Using the standard procedures one can find the rarefaction curves:

$$\begin{aligned}
R_1 : q &= \frac{\rho}{\rho_0}q_0 + \frac{2\sqrt{\alpha}}{1+\alpha}\rho\big(\rho^{-\frac{1+\alpha}{2}} - \rho_0^{-\frac{1+\alpha}{2}}\big),\ \rho < \rho_0 \\
R_2 : q &= \frac{\rho}{\rho_0}q_0 - \frac{2\sqrt{\alpha}}{1+\alpha}\rho\big(\rho^{-\frac{1+\alpha}{2}} - \rho_0^{-\frac{1+\alpha}{2}}\big),\ \rho > \rho_0,
\end{aligned}$$

as well as the shock ones:

$$\begin{aligned}
S_1 : q &= \frac{\rho}{\rho_0}q_0 - \sqrt{\frac{\rho}{\rho_0}(\rho-\rho_0)\Big(\frac{1}{\rho_0^\alpha} - \frac{1}{\rho^\alpha}\Big)},\ \rho > \rho_0, \\
S_2 : q &= \frac{\rho}{\rho_0}q_0 - \sqrt{\frac{\rho}{\rho_0}(\rho-\rho_0)\Big(\frac{1}{\rho_0^\alpha} - \frac{1}{\rho^\alpha}\Big)},\ \rho < \rho_0.
\end{aligned} \tag{2}$$

The shock speeds c_i of S_i, $i = 1, 2$ are

$$c_{1,2} = \frac{q_0}{\rho_0} \mp \sqrt{\frac{\rho}{\rho_0}\frac{\rho^\alpha - \rho_0^\alpha}{\rho - \rho_0}\frac{1}{\rho_0^\alpha\rho^\alpha}}.$$

Our aim is to solve Riemann problem, i.e. (1) with the initial data

$$(\rho, q) = \begin{cases} (\rho_0, q_0), & x < 0 \\ (\rho_1, q_1), & x > 0 \end{cases}. \quad (3)$$

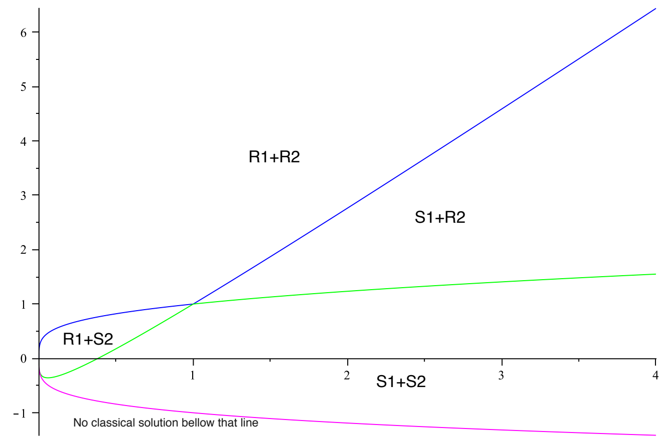

FIGURE 1. Classical waves

A solution is given as a combination of the rarefaction waves for the points (ρ, q) above the curves R_1 and R_2. In areas between the curves R_1 and S_2 (S_1 and R_2, resp.) one can always find a solution in the form $R_1 + S_2$ ($S_1 + R_2$, resp.). Bellow the curves S_1 and S_2 we have a solution consisting from two shocks. But not in the complete area (for more details see [21]): Only if (ρ_1, q_1) is above the curve

$$\Gamma_{ss} = \Gamma_{ss}(\rho_0, q_0): \ q = \Big(\frac{q_0}{\rho_0} - \rho^{-\frac{1+\alpha}{2}} - \rho_0^{-\frac{1+\alpha}{2}}\Big)\rho.$$

That curve is obtained as a boundary of all possible $S_1 + S_2$ combinations and is not included in that area - i.e. there are no classical solutions when $(\rho, q) \in \Gamma_{ss}$. (See Figure 1.)

3. **Shadow waves.** In this section we are looking for non-classical (singular) solution below the curve Γ_{ss}. We are using a simple shadow wave type of solution which is defined as robustly as possible in order to improve chances of obtaining some sort of uniqueness. A big advantage of this type of solutions is that it also includes delta and singular shocks as special cases. That was one of the main reasons why we have chosen to look at solution in the form of the simple shadow wave.

Lemma 3.1. *There exists a simple shadow wave (SDW for short) written in the form*

$$(\rho, q) = \begin{cases} (\rho_0, q_0), & x < (c-\varepsilon)t \\ (\rho_{0,\varepsilon}, q_{0,\varepsilon}), & (c-\varepsilon)t < x < ct \\ (\rho_{1,\varepsilon}, q_{1,\varepsilon}), & ct < x < (c+\varepsilon)t \\ (\rho_1, q_1), & x > (c+\varepsilon)t, \end{cases}$$

that solves (1, 3) if and only if

$$(q_0\rho_1 - q_1\rho_0)^2 > (\rho_0 - \rho_1)\Big(\frac{1}{\rho_1^\alpha} - \frac{1}{\rho_0^\alpha}\Big)\rho_0\rho_1.$$

Proof. Using Lemma 1 from [15] one gets the following formulas for its derivatives

$$\partial_t \rho \approx \big(-c[\rho] + (\varepsilon\rho_{0,\varepsilon} + \varepsilon\rho_{1,\varepsilon})\big)\delta - c(\varepsilon\rho_{0,\varepsilon} + \varepsilon\rho_{1,\varepsilon})t\delta'$$

$$\partial_x q \approx [q]\delta + (\varepsilon q_{0,\varepsilon} + \varepsilon q_{1,\varepsilon})t\delta'$$

$$\partial_t q \approx \big(-c[q] + (\varepsilon q_{0,\varepsilon} + \varepsilon q_{1,\varepsilon})\big)\delta - c(\varepsilon q_{0,\varepsilon} + \varepsilon q_{1,\varepsilon})t\delta'$$

$$\partial_x\Big(\frac{q^2}{\rho} - \frac{1}{\rho^\alpha}\Big) \approx \Big[\frac{q^2}{\rho} - \frac{1}{\rho^\alpha}\Big]\delta + \Big(\varepsilon\Big(\frac{q_{0,\varepsilon}^2}{\rho_{0,\varepsilon}} - \frac{1}{\rho_{0,\varepsilon}^\alpha}\Big) + \varepsilon\Big(\frac{q_{1,\varepsilon}^2}{\rho_{1,\varepsilon}} - \frac{1}{\rho_{1,\varepsilon}^\alpha}\Big)\Big)t\delta'.$$

Here and bellow the sign "≈" denotes a limit as $\varepsilon \to 0$ while $[y] := y_1 - y_0$ is the standard designation of jump in a variable y across a shock front. The support of delta function δ and its derivative δ' is the line $x = ct$. One immediately sees that the only possibility to avoid a trivial case (when both $\rho_{i,\varepsilon}$ and $q_{i,\varepsilon}$, $i = 0, 1$, are zero) is $\rho_{i,\varepsilon}, q_{i,\varepsilon} \sim \varepsilon^{-1}$, $i = 0, 1$. So, let us denote

$$\xi_i := \lim_{\varepsilon\to 0} \varepsilon\rho_i,\ \chi_i := \lim_{\varepsilon\to 0} \varepsilon q_i,\ i = 0, 1.$$

Then

$$\varepsilon\Big(\frac{q_{0,\varepsilon}^2}{\rho_{0,\varepsilon}} - \frac{1}{\rho_{0,\varepsilon}^\alpha}\Big) \approx \frac{\chi_i^2}{\xi_i},\ i = 0, 1.$$

and Riemann problem (1, 3) reduces to the system of the following equations

$$\begin{aligned} -c[\rho] + (\xi_0 + \xi_1) + [q] &= 0 \\ c(\xi_0 + \xi_1) &= \chi_0 + \chi_1 \\ -c[q] + (\chi_0 + \chi_1) + \Big[\frac{q^2}{\rho} - \frac{1}{\rho^\alpha}\Big] &= 0 \\ c(\chi_0 + \chi_1) &= \frac{\chi_0^2}{\xi_0} + \frac{\chi_1^2}{\xi_1}. \end{aligned} \tag{4}$$

Denote by $\kappa_1 := c[\rho] - [q]$ and $\kappa_2 = c[q] - \Big[\frac{q^2}{\rho} - \frac{1}{\rho^\alpha}\Big]$ so called Rankine-Hugoniot deficits for the first and second equation of the system, resp. One immediately gets $\kappa_2 = c\kappa_1$ from the second equation. The third and fourth equation determines c

$$c = \frac{[q] \pm \sqrt{[q]^2 - [\rho]\Big[\frac{q^2-\rho^{1-\alpha}}{\rho}\Big]}}{[\rho]}. \tag{5}$$

The only possible relation between unknowns ξ_i, χ_i, $i = 0, 1$, is

$$\xi_0 = \frac{\chi_0}{c} \text{ and } \xi_1 = \frac{\chi_1}{c},$$

and it fixes the fourth equation. The first and the third equation in (4) uniquely determines a strength of SDW

$$\xi := \xi_0 + \xi_1 = \kappa_1,\ \chi := \chi_0 + \chi_1 = \kappa_2 = c\kappa_1.$$

The variable ρ denotes the density so $\kappa_1 > 0$ (the case $\kappa_1 = 0$ corresponds to a shock). From the first equation in (4) we have

$$c = \frac{q_1 - q_0 + \kappa_1}{\rho_1 - \rho_0},$$

and the positivity of κ_1 implies that one has to take plus sign in (5). A simple computation gives

$$\kappa_1 = \sqrt{\frac{(q_0\rho_1 - q_1\rho_0)^2}{\rho_0\rho_1} - (\rho_0 - \rho_1)\Big(\frac{1}{\rho_1^\alpha} - \frac{1}{\rho_0^\alpha}\Big)}.$$

Thus, an SDW solution to (1), (3) exists if and only if

$$(q_0\rho_1 - q_1\rho_0)^2 > (\rho_0 - \rho_1)\Big(\frac{1}{\rho_1^\alpha} - \frac{1}{\rho_0^\alpha}\Big)\rho_0\rho_1 \tag{6}$$

i.e. a point (ρ_1, q_1) has to be below the curve

$$q = \frac{\rho}{\rho_0}q_0 - \sqrt{\frac{\rho}{\rho_0}(\rho_0 - \rho)\Big(\frac{1}{\rho^\alpha} - \frac{1}{\rho_0^\alpha}\Big)} \tag{7}$$

or above the curve

$$q = \frac{\rho}{\rho_0}q_0 + \sqrt{\frac{\rho}{\rho_0}(\rho_0 - \rho)\Big(\frac{1}{\rho^\alpha} - \frac{1}{\rho_0^\alpha}\Big)}. \tag{8}$$

□

Remark 1. Note that in the (simple) SDW given by (3.1) we have only used constant mean-states $(\rho_{0,\varepsilon}, q_{0,\varepsilon})$, $(\rho_{1,\varepsilon}, q_{1,\varepsilon})$ and a constant SDW speed curve $x = ct$. That is the simplest form of a SDW solution, but in the case of our Riemann problem that is enough since the initial data does not contain a delta function and initial states (ρ_0, q_0), (ρ_1, q_1) in the Riemann initial data are constant. Otherwise, one may use a type of SDW called weighted SDW (for more details see [15]).

The curve given by (7) coincides with (2) and is above Γ_{ss}. Therefore, the region of the data (ρ_1, q_1) situated between this curve and Γ_{ss} corresponds exactly to $S_1 + S_2$ solution, meaning that a solution to Riemann problem is not unique: For (ρ_1, q_1) between these curves both S1+S2 and SDW solution exists. Also, both solutions exist above the curve (8). One has to exclude SDW or S1+S2 solution. The overcompressibility condition is often used in order to gain a uniqueness of delta shock – type solutions. It means that $\lambda_i(\rho_0, q_0) \geq c \geq \lambda_i(\rho_1, q_1)$ should be true for $i = 1, 2$.

That relation for system (1) is satisfied if

$$\frac{q_0}{\rho_0} - \sqrt{\alpha}\rho_0^{-\frac{1+\alpha}{2}} \geq \frac{q_1 - q_0 + \kappa_1}{\rho_1 - \rho_0} \geq \frac{q_1}{\rho_1} + \sqrt{\alpha}\rho_1^{-\frac{1+\alpha}{2}}. \tag{9}$$

Let us denote by $x := q_0\rho_1 - q_1\rho_0$ and note that (9) implies $x > 0$.
Take $\rho_1 > \rho_0$ first. Note that condition

$$\frac{q_0}{\rho_0} - \sqrt{\alpha}\rho_0^{-\frac{1+\alpha}{2}} \geq \frac{q_1}{\rho_1} + \sqrt{\alpha}\rho_1^{-\frac{1+\alpha}{2}}$$

implies $x \geq \sqrt{\alpha}\big(\rho_0^{\frac{1-\alpha}{2}}\rho_1 + \rho_0\rho_1^{\frac{1-\alpha}{2}}\big) > \sqrt{\alpha}\rho_0^{\frac{1-\alpha}{2}}(\rho_0 + \rho_1) =: z$. The condition (9) is equivalent to

$$\begin{aligned} f_1(x) &:= x - \sqrt{\alpha}\rho_0^{\frac{1-\alpha}{2}}(\rho_1 - \rho_0) - \rho_0\kappa_1 \\ &= x - \sqrt{\alpha}\rho_0^{\frac{1-\alpha}{2}}(\rho_1 - \rho_0) - \sqrt{\frac{\rho_0}{\rho_1}}\sqrt{x^2 - x_*^2} \geq 0 \end{aligned}$$

$$\begin{aligned} f_2(x) &:= x + \sqrt{\alpha}\rho_1^{\frac{1-\alpha}{2}}(\rho_1 - \rho_0) - \rho_1\kappa_1 \\ &= x + \sqrt{\alpha}\rho_1^{\frac{1-\alpha}{2}}(\rho_1 - \rho_0) - \sqrt{\frac{\rho_1}{\rho_0}}\sqrt{x^2 - x_*^2} \leq 0, \end{aligned} \tag{10}$$

where $x_* := \sqrt{\rho_0\rho_1(\rho_1 - \rho_0)(\rho_0^{-\alpha} - \rho_1^{-\alpha})} > 0$. Note that the condition for SDW existence (6) means that $x > x_*$. So, further on we will look only at x satisfying $x > \max\{x_*, z\} =: x_0$.

Let us first note that $f_1(x) > 0$, when $x > x_0$ if $\rho_1^{-\alpha} < (1-\alpha)\rho_0^{-\alpha}$. Otherwise, $f_1(x) > 0$, for $x > x_1$, where $x_1 > x_*$ is a single root of $f_1(x) = 0$, $x > x_0$. More precisely,

$$x_1 := \sqrt{\alpha}\rho_0^{\frac{1-\alpha}{2}}\rho_1 + \rho_0\rho_1^{\frac{1}{2}}\sqrt{\rho_1^{-\alpha} - (1-\alpha)\rho_0^{-\alpha}}.$$

So, the first overcompressibility condition holds when $x \geq x_1$.

Also, it holds $\rho_0^{-\alpha} \geq (1-\alpha)\rho_1^{-\alpha}$, so $f_2 \leq 0$ if $x \geq x_2$ where $x_2 := \sqrt{\alpha}\rho_0\rho_1^{\frac{1-\alpha}{2}} + \rho_0^{\frac{1}{2}}\rho_1\sqrt{\rho_0^{-\alpha} - (1-\alpha)\rho_1^{-\alpha}}$.

We have that $x_1 < x_2$, since

$$x_1 - x_2 = \rho_0\rho_1^{\frac{1-\alpha}{2}}\Big(\sqrt{1-(1-\alpha)\Big(\frac{\rho_1}{\rho_0}\Big)^{\alpha}} - \sqrt{\alpha}\Big) - \rho_0^{\frac{1-\alpha}{2}}\rho_1\Big(\sqrt{1-(1-\alpha)\Big(\frac{\rho_0}{\rho_1}\Big)^{\alpha}} - \sqrt{\alpha}\Big)$$

equals zero when $\rho_1 = \rho_0$, and its first derivative with respect to ρ_1 is negative for $\rho_1 > \rho_0$.

Therefore, in the case $\rho_1 > \rho_0$, both conditions in (9) hold if $x \geq x_2$.

Let $\rho_1 < \rho_0$, now. Using the same notation and arguments (with ρ_0 and ρ_1 interchanged) as above, one could see that both conditions in (9) are satisfied if $x \geq x_1$.

Therefore, one sees that (ρ_1, q_1) can be connected by an overcompressive SDW with (ρ_0, q_0) if and only if it lies bellow the curve

$$\Gamma_{oc}:\ q = \begin{cases} \dfrac{\rho}{\rho_0}q_0 - \dfrac{1}{\rho_0}\Big(\sqrt{\alpha}\rho_0\rho^{\frac{1-\alpha}{2}} + \rho_0^{\frac{1}{2}}\rho\sqrt{\rho_0^{-\alpha} - (1-\alpha)\rho^{-\alpha}}\Big), & \text{if } \rho_0 \leq \rho, \\ \dfrac{\rho}{\rho_0}q_0 - \dfrac{1}{\rho_0}\Big(\sqrt{\alpha}\rho_0^{\frac{1-\alpha}{2}}\rho + \rho_0\rho^{\frac{1}{2}}\sqrt{\rho^{-\alpha} - (1-\alpha)\rho_0^{-\alpha}}\Big), & \text{if } \rho_0 > \rho. \end{cases}$$

Remark 2. As one could see, the curve (8) lies above Γ_{oc} and SDW solution above (8) is not overcompresive. If (ρ_1, q_1) lies below Γ_{oc} and above Γ_{ss} a solution to (1, 3) is not unique (see Figure 2): One can construct both S1+S2 and the overcompressive SDW solution to that problem. Our aim is to use a possibility of using convex entropy – entropy flux pair for SDWs. That possibility was one of the major reasons of use SDWs to reconstruct non-classical solution to conservation law systems (see [15] for examples).

The solution concepts used in [20] and [21] share that property. Basically, all three concepts give solutions with the same distributional limit. The authors of these papers simply excluded unwanted delta shocks in the above area without an explanation. We will try to use Lax entropy condition. The first task will be to find as broad as possible a family of convex entropies for system (1).

4. **Convex entropies.** Suppose that a conservation laws system posses convex entropy – entropy flux pair (called convex entropy pair bellow) (η, Q). According

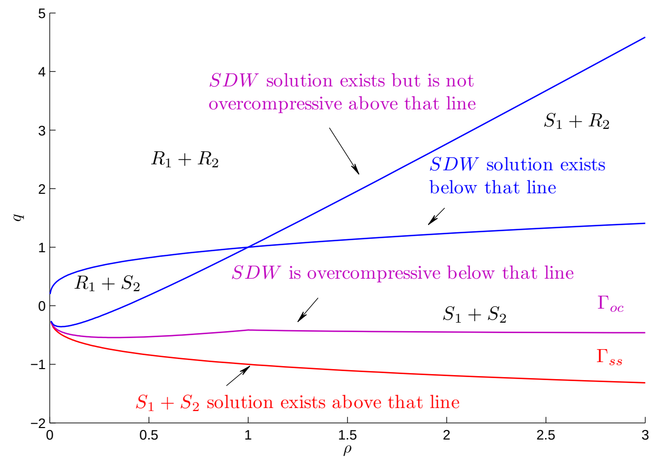

FIGURE 2. Overcompressive SDW vs. S1+S2

to the entropy conditions from [15], a SDW solution (ρ, q) to (1) is admissible if

$$\begin{aligned}\lim_{\varepsilon\to 0} -c(\varepsilon\eta(\rho_{0,\varepsilon}, q_{0,\varepsilon}) + \varepsilon\eta(\rho_{1,\varepsilon}, q_{1,\varepsilon})) + \varepsilon Q(\rho_{0,\varepsilon}, q_{0,\varepsilon}) + \varepsilon Q(\rho_{1,\varepsilon}, q_{1,\varepsilon}) = 0 \\ -c(\eta(\rho_1, q_1) - \eta(\rho_0, q_0)) + Q(\rho_1, q_1) - Q(\rho_0, q_0) \\ + \lim_{\varepsilon\to 0}(\varepsilon\eta(\rho_{0,\varepsilon}, q_{0,\varepsilon}) + \varepsilon\eta(\rho_{1,\varepsilon}, q_{1,\varepsilon})) \le 0.\end{aligned} \tag{11}$$

It is not so hard to find one convex entropy pair. Analogously to the known energy function for other gas dynamic models, we have the following pair of functions

$$\eta = \frac{1}{2}\frac{q^2}{\rho} + \frac{1}{1+\alpha}\rho^{-\alpha}, \quad Q = \frac{1}{2}\frac{q^3}{\rho^2} - \frac{\alpha}{1+\alpha} q\rho^{-(1+\alpha)}.$$

Substitution of these functions in (11) gives a different set of admissible points (ρ_1, q_1) than the overcompressibility condition. But there is still a non-empty intersection of that set with $\{(\rho_1, q_1)$: there exists a S1+S2 solution connecting (ρ_0, q_0) and $(\rho_1, q_1)\}$.

Even more, the overcompressive and entropic sets of admissible states (ρ_1, q_1) are not comparable as one could see on the Figure 3. Note that a situation is different in the case of Chaplygin gas with $\alpha = 1$ (see [16]), where use only of the energy $\eta = \frac{q^2+1}{\rho}$ as a convex entropy is enough to single out a unique solution to Riemann problem, and the overcompressibility condition gives the same one.

Let us now try to find some more convex entropies. Using the standard procedure (see [6] for example) one can find that an entropy function η satisfies

$$\partial_{\rho\rho}\eta + \frac{2q}{\rho}\partial_{\rho q}\eta + \Big(\frac{q^2}{\rho^2} - \frac{\alpha}{\rho^{1+\alpha}}\Big)\partial_{qq}\eta = 0.$$

After a change of variables $v = \frac{q}{\rho} + \frac{2\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}$ and $w = \frac{q}{\rho} - \frac{2\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}$, the equation becomes

$$(v - w)\partial_{vw}\eta = \frac{3+\alpha}{2(1+\alpha)}(\partial_v\eta - \partial_w\eta).$$

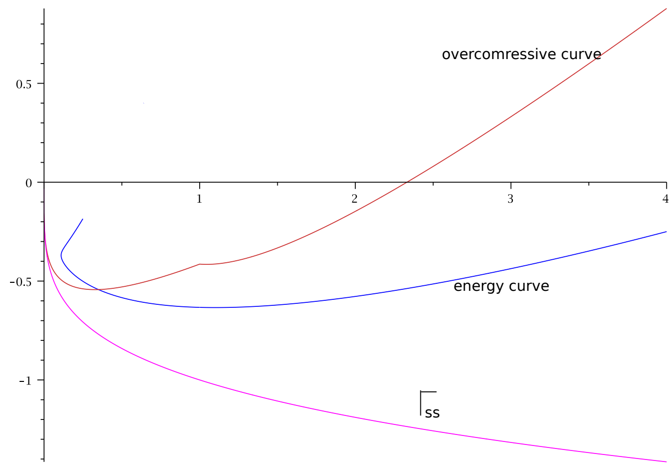

FIGURE 3. Energy entropy condition

If we separate variables by $\eta(v,w) = f(v-w)g(v+w)$, it reduces to

$$\frac{g''(v+w)}{g(v+w)} = \frac{2B}{v-w}\frac{f'(v-w)}{f(v-w)} + \frac{f''(v-w)}{f(v-w)} = l \in \mathbb{R},$$

where $B = \frac{3+\alpha}{2(1+\alpha)}$. For $l \le 0$ a function is not convex and consequently, a function g nor η can not be convex. Fix $l > 0$. Then $g(v+w) = C_1 e^{\sqrt{l}(v+w)} + C_1 e^{-\sqrt{l}(v+w)}$, while f solves $f''(v-w) + \frac{2B}{v-w} f'(v-w) - l f(v-w) = 0$. From [18] one gets

$$f(v-w) = (v-w)^{-\frac{1}{1+\alpha}} \Big(c_1 I_{\frac{1}{1+\alpha}}\big((v-w)\sqrt{l}\big) + c_2 K_{\frac{1}{1+\alpha}}\big((v-w)\sqrt{l}\big)\Big),$$

where $I_\nu(x)$ denote modified Bessel function of the first kind, while $K_\nu(x)$ denote modified Bessel function of the second kind. Using the original variables (ρ, q), we have

$$\eta(\rho,q) = C_1\eta_1(\rho,q) + C_2\eta_2(\rho,q) + C_3\eta_3(\rho,q) + C_4\eta_4(\rho,q),$$

where

$$\eta_1(\rho,q) := e^{\frac{2q}{\rho}\lambda}\rho^{\frac12} K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big),\ \eta_2(\rho,q) := e^{-\frac{2q}{\rho}\lambda}\rho^{\frac12} K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big), \tag{12}$$

$$\eta_3(\rho,q) := e^{\frac{2q}{\rho}\lambda}\rho^{\frac12} I_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big),\ \eta_4(\rho,q) := e^{-\frac{2q}{\rho}\lambda}\rho^{\frac12} I_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big), \tag{13}$$

and $\lambda := \sqrt{l} > 0$.

Lemma 4.1. *Entropy functions η_1 and η_2 defined by (12) are convex, while η_3 and η_4 defined by (13) are non-convex, for each $\lambda > 0$ and $0 < \alpha < 1$.*

Proof. It is known that entropy function is convex if its Hessian matrix is positive definite. So, in order to prove that η_1 and η_2 are convex it is enough to prove that the principal minors of a Hessian matrix of $\eta_{1/2}$ are all positive. We use the following relations in the proof below:

$$K_\nu'(z) = -\frac{1}{2}\Big(K_{\nu-1}(z) + K_{\nu+1}(z)\Big), \quad -\frac{2\nu}{z}K_\nu(z) = K_{\nu-1}(z) - K_{\nu+1}(z),$$
$$K_\nu(z) < K_\mu(z), \quad \text{for } \nu < \mu.$$

Put $x(\rho) = \frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda$ for simplicity. We have

$$\frac{\partial}{\partial\rho}K_{\frac{1}{1+\alpha}}\big(x(\rho)\big) = 2\sqrt{\alpha}\lambda\rho^{-\frac{3+\alpha}{2}}K_{\frac{\alpha}{1+\alpha}}\big(x(\rho)\big) + \frac{1}{2}\rho^{-1}K_{\frac{1}{1+\alpha}}\big(x(\rho)\big),$$
$$\frac{\partial}{\partial\rho}K_{\frac{\alpha}{1+\alpha}}\big(x(\rho)\big) = 2\sqrt{\alpha}\lambda\rho^{-\frac{3+\alpha}{2}}K_{\frac{1}{1+\alpha}}\big(x(\rho)\big) + \frac{1}{2}\alpha\rho^{-1}K_{\frac{\alpha}{1+\alpha}}\big(x(\rho)\big).$$

Then,

$$\frac{\partial}{\partial\rho}\eta_1(\rho,q) = e^{\frac{2q}{\rho}\lambda}\rho^{-\frac{1}{2}}\Big(K_{\frac{1}{1+\alpha}}\big(x(\rho)\big)\big(-2q\rho^{-1}\lambda+1\big) + 2\sqrt{\alpha}\lambda\rho^{-\frac{1+\alpha}{2}}K_{\frac{\alpha}{1+\alpha}}\big(x(\rho)\big)\Big),$$
$$\frac{\partial^2}{\partial\rho^2}\eta_1(\rho,q) = 4\lambda^2 e^{\frac{2q}{\rho}\lambda}\rho^{-\frac{5}{2}}\Big(K_{\frac{1}{1+\alpha}}\big(x(\rho)\big)\big(\frac{q^2}{\rho}+\frac{\alpha}{\rho^\alpha}\big) - 2\sqrt{\alpha}q\rho^{-\frac{1+\alpha}{2}}K_{\frac{\alpha}{1+\alpha}}\big(x(\rho)\big)\Big)$$
$$> 4\lambda^2 e^{\frac{2q}{\rho}\lambda}\rho^{-\frac{5}{2}}K_{\frac{\alpha}{1+\alpha}}\big(x(\rho)\big)\Big(q\rho^{-\frac{1}{2}} - \sqrt{\alpha}\rho^{-\frac{\alpha}{2}}\Big)^2 \geq 0,$$
$$\frac{\partial^2}{\partial q\partial\rho}\eta_1(\rho,q) = 4\lambda^2 e^{\frac{2q}{\rho}\lambda}\rho^{-\frac{5}{2}}\Big(-qK_{\frac{1}{1+\alpha}}\big(x(\rho)\big) + \sqrt{\alpha}\rho^{\frac{1-\alpha}{2}}K_{\frac{\alpha}{1+\alpha}}\big(x(\rho)\big)\Big)$$

and

$$\frac{\partial}{\partial q}\eta_1(\rho,q) = 2\lambda e^{\frac{2q}{\rho}\lambda}\rho^{-\frac{1}{2}}K_{\frac{1}{1+\alpha}}\big(x(\rho)\big), \quad \frac{\partial^2}{\partial q^2}\eta_1(\rho,q) = 4\lambda^2 e^{\frac{2q}{\rho}\lambda}\rho^{-\frac{3}{2}}K_{\frac{1}{1+\alpha}}\big(x(\rho)\big).$$

Determinant of Hessian matrix is given by

$$D_1 := \frac{\partial^2}{\partial\rho^2}\eta_1 \cdot \frac{\partial^2}{\partial q^2}\eta_1 - \Big(\frac{\partial^2}{\partial q\partial\rho}\eta_1\Big)^2$$
$$= 16\alpha\lambda^4 e^{\frac{4q}{\rho}\lambda}\rho^{-4-\alpha}\left(\Big(K_{\frac{1}{1+\alpha}}\big(x(\rho)\big)\Big)^2 - \Big(K_{\frac{\alpha}{1+\alpha}}\big(x(\rho)\big)\Big)^2\right).$$

Since $\frac{1}{1+\alpha} > \frac{\alpha}{1+\alpha}$, for $\alpha \in (0,1)$, it is clear that D_1 is positive.

On the same way as above one gets

$$\frac{\partial^2}{\partial\rho^2}\eta_2(\rho,q) = 4\lambda^2 e^{-\frac{2q}{\rho}\lambda}\rho^{-\frac{5}{2}}\Big(K_{\frac{1}{1+\alpha}}\big(x(\rho)\big)\big(\frac{q^2}{\rho}+\frac{\alpha}{\rho^\alpha}\big) + 2\sqrt{\alpha}q\rho^{-\frac{1+\alpha}{2}}K_{\frac{\alpha}{1+\alpha}}\big(x(\rho)\big)\Big)$$
$$> 4\lambda^2 e^{-\frac{2q}{\rho}\lambda}\rho^{-\frac{5}{2}}K_{\frac{\alpha}{1+\alpha}}\big(x(\rho)\big)\Big(q\rho^{-\frac{1}{2}} + \sqrt{\alpha}\rho^{-\frac{\alpha}{2}}\Big)^2 \geq 0,$$
$$D_2 := 16\alpha\lambda^4 e^{-\frac{4q}{\rho}\lambda}\rho^{-4-\alpha}\left(\Big(K_{\frac{1}{1+\alpha}}\big(x(\rho)\big)\Big)^2 - \Big(K_{\frac{\alpha}{1+\alpha}}\big(x(\rho)\big)\Big)^2\right).$$

Since, $\frac{\partial^2}{\partial\rho^2}\eta_2(\rho,q) > 0$ and $D_2 > 0$, one concludes that η_2 is also a convex function.

In the proof of non-convexity of the functions η_3, η_4 one follows the same arguments and one also uses the following

$$I_\nu'(z) = \frac{1}{2}\Big(I_{\nu-1}(z) - I_{\nu+1}(z)\Big), \quad \frac{2\nu}{z}I_\nu(z) = I_{\nu-1}(z) + I_{\nu+1}(z),$$
$$I_\nu(z) > I_\mu(z), \quad \text{for } \nu < \mu.$$

For example, determinant of the Hessian matrix of the function η_3 is

$$D_3 := 16\alpha\lambda^4 e^{\frac{4q}{\rho}\lambda}\rho^{-4-\alpha}\left(\Big(I_{\frac{1}{1+\alpha}}\big(x(\rho)\big)\Big)^2 - \Big(I_{\frac{\alpha}{1+\alpha}}\big(x(\rho)\big)\Big)^2\right).$$

D_3 is negative which means that η_3 is not a convex function. Same follows for a function η_4. □

Therefore, using the original variables (ρ, q) one can conclude that all convex η obtained by the separation of variables are linear combination of the functions η_1 and η_2 from (12) for every $\lambda > 0$. Appropriate entropy flux functions are given by

$$\begin{aligned} Q_1(\rho,q) :=& \frac{1}{2\lambda}\rho^{-\frac{1}{2}}e^{\frac{2q}{\rho}\lambda}\Bigg((2\lambda q-\rho)K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big) \\ &+ 2\lambda\sqrt{\alpha}\rho^{\frac{1-\alpha}{2}}K_{\frac{2+\alpha}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big)\Bigg), \\ Q_2(\rho,q) :=& \frac{1}{2\lambda}\rho^{-\frac{1}{2}}e^{-\frac{2q}{\rho}\lambda}\Bigg((2\lambda q+\rho)K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big) \\ &- 2\lambda\sqrt{\alpha}\rho^{\frac{1-\alpha}{2}}K_{\frac{2+\alpha}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big)\Bigg). \end{aligned}$$

Remark 3. In order to get more convex entropies one can try to separate variables on the different way. For example, we can separate variables by $\eta(v,w) = f(v-w)g(v)$. As a result we get the following convex entropy function

$$\eta(\rho,q) = e^{\lambda\frac{q}{\rho}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\Big)^{-\frac{1}{1+\alpha}}\rho^{\frac{1}{2}}K_{\frac{1}{1+\alpha}}\Big(\frac{2\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big)$$

and appropriate entropy flux function given by

$$\begin{aligned} Q(\rho,q) = e^{\lambda\frac{q}{\rho}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\Big)^{-\frac{1}{1+\alpha}}\rho^{-\frac{1}{2}}\Bigg(&K_{\frac{1}{1+\alpha}}\Big(\frac{2\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big)q \\ &+ \sqrt{\alpha}\,\rho^{\frac{1-\alpha}{2}}K_{\frac{\alpha}{1+\alpha}}\Big(\frac{2\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big)\Bigg). \end{aligned}$$

We choose to represent results obtained by using convex entropy-entropy flux pairs (η_1, Q_1), (η_2, Q_2). We did not make a significant improvement with the above pair (η, Q) so we omit those results.

Definition 4.2. An SDW solution to (1) is said to be entropic if (11) holds true for all entropy pairs (η_1, Q_1), (η_2, Q_2), $\lambda > 0$.

We have completed the existence proof in Lemma 3.1 above. Only thing we have to solve is to exclude unwanted SDW solution above the line Γ_{ss} and the solution would be unique.

Theorem 4.3. *The relation*

$$\lim_{\varepsilon\to 0} -c(\varepsilon\eta(\rho_{0,\varepsilon}, q_{0,\varepsilon}) + \varepsilon\eta(\rho_{1,\varepsilon}, q_{1,\varepsilon})) + \varepsilon Q(\rho_{0,\varepsilon}, q_{0,\varepsilon}) + \varepsilon Q(\rho_{1,\varepsilon}, q_{1,\varepsilon}) = 0$$

holds true for an SDW solution, entropy pairs (η_1, Q_1) *and* (η_2, Q_2) *and any* λ.

Proof. Since $\rho_{i,\varepsilon}^{-\frac{1+\alpha}{2}} \to 0$ as $\varepsilon \to 0$, $i = 1, 2$ and since the modified Bessel functions of the second kind satisfy

$$K_\nu(x) \sim \frac{1}{2}\Gamma(\nu)\Big(\frac{2}{x}\Big)^\nu,\ \nu > 0 \text{ as } x \to 0, \tag{14}$$

we have

$$K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\lambda\rho_{i,\varepsilon}^{-\frac{1+\alpha}{2}}\Big)\sim\frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big)\Big(\frac{2(1+\alpha)}{4\sqrt{\alpha}\lambda}\Big)^{\frac{1}{1+\alpha}}\rho_{i,\varepsilon}^{\frac{1}{2}},\ i=1,2$$

and

$$K_{\frac{\alpha}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\lambda\rho_{i,\varepsilon}^{-\frac{1+\alpha}{2}}\Big)\sim\frac{1}{2}\Gamma\Big(\frac{\alpha}{1+\alpha}\Big)\Big(\frac{2(1+\alpha)}{4\sqrt{\alpha}\lambda}\Big)^{\frac{\alpha}{1+\alpha}}\rho_{i,\varepsilon}^{\frac{\alpha}{2}},\ i=1,2.$$

By virtue of the above relation, the first relation in (11) for the entropy pair (η_1,Q_1) becomes

$$\begin{aligned}&-\frac{c}{2}e^{2\lambda c}\Gamma\Big(\frac{1}{1+\alpha}\Big)\Big(\frac{1+\alpha}{2\sqrt{\alpha}\lambda}\Big)^{\frac{1}{1+\alpha}}(\xi_0+\xi_1)\\&+\frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big)e^{2\lambda c}\Big(\frac{1+\alpha}{2\sqrt{\alpha}\lambda}\Big)^{\frac{1}{1+\alpha}}(\chi_1+\chi_2)=0.\end{aligned}$$

It is now clear that that relation is true for every λ if and only if $c(\xi_1+\xi_2)=\chi_0+\chi_1$. But that relation is always true when SDW is a solution to the system as one could see above. Obviously, the same holds for the second entropy pair (η_2,Q_2). □

In order to prove uniqueness of solution one needs to prove that the second relation in (11) is always non-positive for (ρ_1,q_1) lying on $\Gamma_{ss}(\rho_0,q_0)$ and bellow for every $\lambda>0$, while above it is positive at least for some $\lambda>0$. We were not able to complete that process, and we will present partial results about that in the rest of the paper. We left that question open.

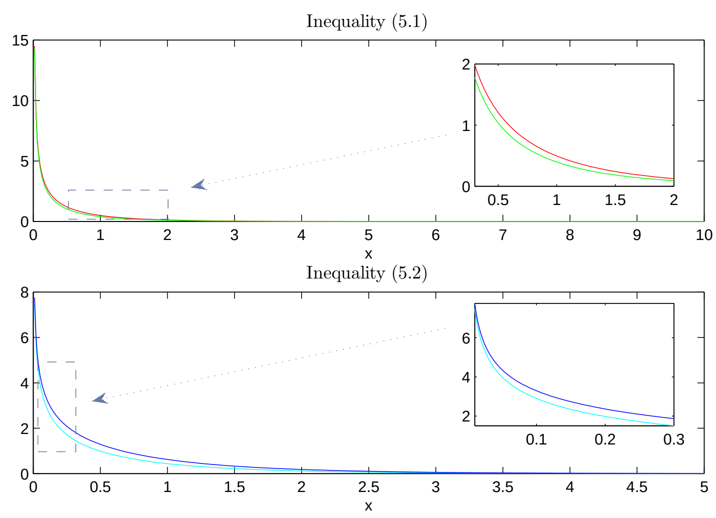

Figure 4. Inequalities (15) and (16)

5. **Partial uniqueness results.** In order to prove that the second relation in (11) is satisfied in some cases we will use the following equality for modified Bessel functions of the second kind

$$K_{\frac{2+\alpha}{1+\alpha}}(x)=\frac{2}{x(1+\alpha)}K_{\frac{1}{1+\alpha}}(x)+K_{\frac{\alpha}{1+\alpha}}(x).$$

Also, the following inequalities hold for every $x > 0$ and $0 < \alpha < 1$ (also see Figure 4)

$$K_{\frac{1}{1+\alpha}}(x) > \frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big)\,\Big(\frac{x}{2}\Big)^{-\frac{1}{1+\alpha}}\,e^{-x}, \tag{15}$$

$$K_{\frac{\alpha}{1+\alpha}}(x) < \frac{1}{2}\Gamma\Big(\frac{\alpha}{1+\alpha}\Big)\,\Big(\frac{x}{2}\Big)^{-\frac{\alpha}{1+\alpha}}\,e^{-x}. \tag{16}$$

Inequality (15) follows from relation

$$x^\nu K_\nu(x)e^x > 2^{\nu-1}\Gamma(\nu),\; x>0,\; \nu > \frac{1}{2}$$

proved in [8], since $\frac{1}{1+\alpha} > \frac{1}{2}$, $0 < \alpha < 1$. Inequality (16) follows from paper [1], where the function $x \mapsto x^\nu e^x K_\nu(x)$ is proved to be monotone decreasing on $(0,\infty)$ for all $\nu < \frac{1}{2}$, while

$$\lim_{x\to 0} x^\nu e^x K_\nu(x) = 2^{\nu-1}\Gamma(\nu).$$

Here, we can use this results since $\frac{\alpha}{1+\alpha} < \frac{1}{2}$, $0 < \alpha < 1$ hold. Put $A = \frac{2\sqrt{\alpha}}{1+\alpha}$ in order to simplify the future notation. Note that $0 < A < 1$ for $0 < \alpha < 1$.

We start our analysis by looking at the second entropy inequality in (11). By a simple substitution and use of the above relations we get the left-hand side of the second relation in (11) to be in the form

$$\begin{aligned}
E^1_\lambda =& \lim_{\varepsilon\to 0}\varepsilon\left(\rho_{0,\varepsilon}^{\frac{1}{2}}K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_{0,\varepsilon}^{-\frac{1+\alpha}{2}}\Big) + \rho_{1,\varepsilon}^{\frac{1}{2}}K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_{1,\varepsilon}^{-\frac{1+\alpha}{2}}\Big)\right)e^{2c\lambda}\\
&-K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)\rho_0^{-\frac{1}{2}}(-c\rho_0+q_0)\,e^{2\lambda\frac{q_0}{\rho_0}} - \sqrt{\alpha}\,K_{\frac{\alpha}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)\rho_0^{-\frac{\alpha}{2}}\,e^{2\lambda\frac{q_0}{\rho_0}}\\
&-K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\Big)\rho_1^{-\frac{1}{2}}(c\rho_1-q_1)\,e^{2\lambda\frac{q_1}{\rho_1}} + \sqrt{\alpha}\,K_{\frac{\alpha}{1+\alpha}}\Big(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\Big)\rho_1^{-\frac{\alpha}{2}}e^{2\lambda\frac{q_1}{\rho_1}}\\
=&\frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big)A^{-\frac{1}{1+\alpha}}\lambda^{-\frac{1}{1+\alpha}}\kappa_1\,e^{2c\lambda} - K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)\rho_0^{-\frac{1}{2}}(-c\rho_0+q_0)\,e^{2\lambda\frac{q_0}{\rho_0}}\\
&-\sqrt{\alpha}\,K_{\frac{\alpha}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)\rho_0^{-\frac{\alpha}{2}}\,e^{2\lambda\frac{q_0}{\rho_0}} - K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\Big)\rho_1^{-\frac{1}{2}}(c\rho_1-q_1)\,e^{2\lambda\frac{q_1}{\rho_1}}\\
&+\sqrt{\alpha}\,K_{\frac{\alpha}{1+\alpha}}\Big(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\Big)\rho_1^{-\frac{\alpha}{2}}\,e^{2\lambda\frac{q_1}{\rho_1}},
\end{aligned}$$

for the first entropy pair (η_1, Q_1). There was used that $\rho_{0,\varepsilon} \sim \rho_{1,\varepsilon} \sim \frac{1}{2\varepsilon}\kappa_1$, as $\varepsilon \to 0$ and $\kappa_1 = c(\rho_1 - \rho_0) - (q_1 - q_0) = c[\rho] - [q]$.

For points $(\rho_1, q_1) \in \Gamma_{ss}(\rho_0, q_0)$ we have $E^1_\lambda = e^{2\lambda\frac{q_0}{\rho_0}}\,\hat{E}^1_\lambda$, where

$$\begin{aligned}
\hat{E}^1_\lambda =& \frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big)A^{-\frac{1}{1+\alpha}}\lambda^{-\frac{1}{1+\alpha}}(\rho_0^{\frac{1-\alpha}{2}} + \rho_1^{\frac{1-\alpha}{2}})e^{-2\lambda\rho_0^{-\frac{1+\alpha}{2}}}\\
&- K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)\,\rho_0^{-\frac{\alpha}{2}} - \sqrt{\alpha}\,K_{\frac{\alpha}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)\,\rho_0^{-\frac{\alpha}{2}}\\
&- K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\Big)\,\rho_1^{-\frac{\alpha}{2}}e^{-2\lambda(\rho_0^{-\frac{1+\alpha}{2}}+\rho_1^{-\frac{1+\alpha}{2}})}\\
&+ \sqrt{\alpha}\,K_{\frac{\alpha}{1+\alpha}}\Big(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\Big)\,\rho_1^{-\frac{\alpha}{2}}e^{-2\lambda(\rho_0^{-\frac{1+\alpha}{2}}+\rho_1^{-\frac{1+\alpha}{2}})}.
\end{aligned} \tag{17}$$

In the same way as above, one can determine that the left-hand side of the second relation in (11) for the second entropy pair (η_2, Q_2) equals

$$E^2_\lambda = \frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big)A^{-\frac{1}{1+\alpha}}\lambda^{-\frac{1}{1+\alpha}}\kappa_1\,e^{-2c\lambda} - K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)\,\rho_0^{-\frac{1}{2}}(-c\rho_0+q_0)\,e^{-2\lambda\frac{q_0}{\rho_0}}$$

$$+\sqrt{\alpha}K_{\frac{\alpha}{1+\alpha}}\left(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\right)\rho_0^{-\frac{\alpha}{2}}\,e^{-2\lambda\frac{q_0}{\rho_0}}-K_{\frac{1}{1+\alpha}}\left(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\right)\rho_1^{-\frac{1}{2}}(c\rho_1-q_1)\,e^{-2\lambda\frac{q_1}{\rho_1}}$$
$$-\sqrt{\alpha}K_{\frac{\alpha}{1+\alpha}}\left(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\right)\rho_1^{-\frac{\alpha}{2}}\,e^{-2\lambda\frac{q_1}{\rho_1}}.$$

For (ρ_1,q_1) lying on $\Gamma_{ss}(\rho_0,q_0)$ we have

$$E_\lambda^2=e^{-2\lambda\left(\frac{q_0}{\rho_0}-\left(\rho_0^{-\frac{1+\alpha}{2}}+\rho_1^{-\frac{1+\alpha}{2}}\right)\right)}\,\hat{E}_\lambda^2,$$

where

$$\begin{aligned}\hat{E}_\lambda^2=&\frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big)A^{-\frac{1}{1+\alpha}}\lambda^{-\frac{1}{1+\alpha}}(\rho_0^{\frac{1-\alpha}{2}}+\rho_1^{\frac{1-\alpha}{2}})e^{-2\lambda\rho_1^{-\frac{1+\alpha}{2}}}\\&-K_{\frac{1}{1+\alpha}}\left(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\right)\rho_1^{-\frac{\alpha}{2}}-\sqrt{\alpha}\,K_{\frac{\alpha}{1+\alpha}}\left(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\right)\rho_1^{-\frac{\alpha}{2}}\\&-K_{\frac{1}{1+\alpha}}\left(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\right)\rho_0^{-\frac{\alpha}{2}}\,e^{-2\lambda\left(\rho_0^{-\frac{1+\alpha}{2}}+\rho_1^{-\frac{1+\alpha}{2}}\right)}\\&+\sqrt{\alpha}\,K_{\frac{\alpha}{1+\alpha}}\left(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\right)\rho_0^{-\frac{\alpha}{2}}\,e^{-2\lambda\left(\rho_0^{-\frac{1+\alpha}{2}}+\rho_1^{-\frac{1+\alpha}{2}}\right)}.\end{aligned}\tag{18}$$

That was the proof of the following technical assertion.

Proposition 1. *If the second relation in (11) holds for (η_1,Q_1) and $(\rho_0,\rho_1)\in\Omega_0\times\Omega_1$, $\Omega_0,\Omega_1\subseteq\mathbb{R}^+$, (ρ_1,q_1) lying on Γ_{ss} then the second entropy condition holds for (η_2,Q_2) and $(\rho_0,\rho_1)\in\Omega_1\times\Omega_0$, (ρ_1,q_1) lying on Γ_{ss}.*

The following theorem is very important. We claim that there exist points above a curve Γ_{ss} that satisfy the overcompressibility condition but not the entropy one. So we may avoid non-uniqueness at least at these points by using the entropy admissibility condition with or without overcompressibility. We still do not know whether there are some points where the overcompressibility is stronger condition than the entropy condition. Let us add that we did not find any numerical example for that until now.

Theorem 5.1. *For every $\alpha\in(0,1)$ and every point (ρ_0,q_0) there exists its neighborhood such that there exist points above the curve Γ_{ss} where overcompressibility condition is satisfied but entropy conditions is not for λ large enough.*

Proof. Define the curve

$$\Gamma_\beta:\ q=\left(\frac{q_0}{\rho_0}-(\beta+(1-\beta)\sqrt{\alpha})\big(\rho_0^{-\frac{1+\alpha}{2}}+\rho^{-\frac{1+\alpha}{2}}\big)\right)\rho,\tag{19}$$

for $0<\beta<1$.

Take $\rho_0=\rho_1$. Then $c=\frac{1}{2}\big(\frac{q_0}{\rho_0}+\frac{q_1}{\rho_1}\big)$, and using q_1 defined by (19) we have

$$c=\frac{q_0}{\rho_0}-(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{-\frac{1+\alpha}{2}}.$$

The speed given by c is continuous with respect to ρ-variable and that is true for ρ_1 in a neighborhood of ρ_0, too.

One can easily check that overcompressibility condition for $\rho_1=\rho_0$ is always satisfied since the inequality $\beta(1-\sqrt{\alpha})>0$ holds for each $\beta\in(0,1)$ and $\alpha\in(0,1)$.

A simple computation gives

$$\begin{aligned}\kappa_1|_{\rho_0=\rho_1}&=2(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{\frac{1-\alpha}{2}},\\-c\rho_0+q_0|_{\rho_0=\rho_1}=c\rho_1-q_1|_{\rho_0=\rho_1}&=(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{\frac{1-\alpha}{2}}.\end{aligned}$$

Due to continuity of all functions used in this analysis overcompressibility condition is satisfied on Γ_β in a neighborhood of ρ_0, too.

Let as now check the second entropy condition for the first entropy pair (η_1, Q_1) using the above data.

We have

$$E^1_{\lambda,\beta}|_{\rho_0=\rho_1} = (\beta + (1-\beta)\sqrt{\alpha})\, \rho_0^{-\frac{\alpha}{2}}\, e^{2\lambda\frac{q_0}{\rho_0}}\, \tilde{E}^1_{\lambda,\beta}|_{\rho_0=\rho_1},$$

where

$$\begin{aligned}\tilde{E}^1_{\lambda,\beta}|_{\rho_0=\rho_1} =&\Gamma\Big(\frac{1}{1+\alpha}\Big)(A\lambda\rho_0^{-\frac{1+\alpha}{2}})^{-\frac{1}{1+\alpha}}e^{-2\lambda(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{-\frac{1+\alpha}{2}}}\\ &- K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)(1+e^{-4\lambda(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{-\frac{1+\alpha}{2}}})\\ &- K_{\frac{\alpha}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)\frac{\sqrt{\alpha}}{\beta+(1-\beta)\sqrt{\alpha}}(1-e^{-4\lambda(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{-\frac{1+\alpha}{2}}}).\end{aligned}$$

Using the relation

$$\frac{K_\nu(x)}{K_{\nu-1}(x)} < \frac{\nu+\sqrt{\nu^2+x^2}}{x},\ \nu\in\mathbb{R}, \tag{20}$$

from [11] and the fact that $K_{-\nu} = K_\nu$, we get

$$\begin{aligned}\tilde{E}^1_{\lambda,\beta}|_{\rho_0=\rho_1} >&\Gamma\Big(\frac{1}{1+\alpha}\Big)(A\lambda\rho_0^{-\frac{1+\alpha}{2}})^{-\frac{1}{1+\alpha}}e^{-2\lambda(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{-\frac{1+\alpha}{2}}}\\ &- K_{\frac{\alpha}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)(1+e^{-4\lambda(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{-\frac{1+\alpha}{2}}})\\ &\cdot\left(\frac{\frac{1}{1+\alpha}+\sqrt{\big(\frac{1}{1+\alpha}\big)^2+\big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\big)^2}}{2A\lambda\rho_0^{-\frac{1+\alpha}{2}}}\right)\\ &- K_{\frac{\alpha}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)\frac{\sqrt{\alpha}}{\beta+(1-\beta)\sqrt{\alpha}}(1-e^{-4\lambda(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{-\frac{1+\alpha}{2}}})\Bigg).\end{aligned}$$

Now, using relation (16) and letting $\lambda\to\infty$, we have that $E^1_{\lambda,\beta}|_{\rho_0=\rho_1} > 0$ if

$$\begin{aligned}&\Gamma\Big(\frac{1}{1+\alpha}\Big)A^{-\frac{1}{1+\alpha}}\rho_0^{\frac{1}{2}}\lambda^{-\frac{1}{1+\alpha}}e^{-2\lambda(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{-\frac{1+\alpha}{2}}}\\ &-\frac{1}{2}\Gamma\Big(\frac{\alpha}{1+\alpha}\Big)A^{-\frac{\alpha}{1+\alpha}}\rho_0^{\frac{\alpha}{2}}\lambda^{-\frac{\alpha}{1+\alpha}}\Big(1+\frac{\sqrt{\alpha}}{\beta+(1-\beta)\sqrt{\alpha}}\Big)e^{-2A\lambda\rho_0^{-\frac{1+\alpha}{2}}} > 0.\end{aligned}$$

Since the exponential function decreases to zero at infinity faster than any power of λ, the above is true if $\beta+(1-\beta)\sqrt{\alpha} < A = \frac{2\sqrt{\alpha}}{1+\alpha}$. The equation $h_\beta(x)=0$, with

$$h_\beta(x) = -\beta + (1+\beta)x - \beta x^2 - (1-\beta)x^3$$

has only one root x_β in the interval $(0,1)$ given by $x_\beta = \frac{1-\sqrt{1+4(1-\beta)\beta}}{2(\beta-1)}$ (note that $x=1$ is one root of the equation $h_\beta(x)=0$ for any $\beta\in(0,1)$). For each $\beta\in(0,1)$ we obtain an interval $(\alpha_\beta,1)$, $\alpha_\beta := x_\beta^2$ such that entropy condition does not hold i.e. the function h_β is positive for all α in the interval $(\alpha_\beta,1)$.

Obviously, $\alpha_{\beta_1} = x^2_{\beta_1} < x^2_{\beta_2} = \alpha_{\beta_2}$ for $\beta_1<\beta_2$. With $\beta\to 0$ we have $x_\beta\to 0$, so the function $h_\beta(x)$ is positive for $\alpha\in(0,1)$ and β small enough.

Since the function h is continuous, one can conclude the following: For any $\alpha\in(0,1)$ there exist some $\beta\in(0,1)$ (sufficiently small β if α is sufficiently small) such that entropy condition, when $\lambda \to \infty$, is not satisfied in the neighborhood of point ρ_0, on the curve Γ_β for the first entropy pair. This completes the proof. □

Remark 4. The left-hand side of the entropy condition, for the second entropy pair (η_2, Q_2), q_1 given by Γ_β and $\rho_1 = \rho_0$ equals

$$E^2_{\lambda,\beta}|_{\rho_0=\rho_1} = e^{-2\lambda(\frac{q_0}{\rho_0} - 2(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{-\frac{1+\alpha}{2}})}(\beta + (1-\beta)\sqrt{\alpha})\rho_0^{-\frac{\alpha}{2}}\, \tilde{E}^2_\lambda|_{\rho_0=\rho_1},$$

where

$$\begin{aligned}\tilde{E}^2_{\lambda,\beta}|_{\rho_0=\rho_1} =& \Gamma\Big(\frac{1}{1+\alpha}\Big)\Big(A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)^{-\frac{1}{1+\alpha}} e^{-2\lambda(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{-\frac{1+\alpha}{2}}} \\ &- K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)\Big(1 + e^{-4\lambda(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{-\frac{1+\alpha}{2}}}\Big) \\ &- K_{\frac{\alpha}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)\frac{\sqrt{\alpha}}{\beta+(1-\beta)\sqrt{\alpha}}\Big(1 - e^{-4\lambda(\beta+(1-\beta)\sqrt{\alpha})\rho_0^{-\frac{1+\alpha}{2}}}\Big).\end{aligned}$$

If we compare the entropy conditions for (η_1, Q_1) and (η_2, Q_2), we can see that

$$\tilde{E}^2_{\lambda,\beta}|_{\rho_0=\rho_1} = \tilde{E}^1_{\lambda,\beta}|_{\rho_0=\rho_1}$$

holds. So, we can conclude that the second entropy condition for the second entropy pair, (η_2, Q_2), ρ_1 in the neighborhood of ρ_0 and λ sufficiently large is satisfied if and only if it is satisfied for the first entropy pair.

In order to get a better understanding of the above result one may consider the curve $\Gamma_{0.5}$ (i.e. $\beta = 0.5$) to conclude: For every $\alpha \in (\alpha_0, 1)$, $\alpha_0 = (\sqrt{2}-1)^2 \approx 0.17157$ and every (ρ_0, q_0) there exists $\lambda > 0$ and (ρ_1, q_1) that lies above the curve Γ_{ss} such that overcompressibility condition is satisfied but entropy condition is not.

After proving that there are cases when entropy condition is more restrictive than the overcompressibility one, we will present some results that illustrates usefulness of the entropy condition. We start with the one describing asymptotic behavior of the entropy condition as parameter λ tends to zero or infinity.

Proposition 2. *The relations in (11) are satisfied for all entropy pairs (η_1, Q_1), (η_2, Q_2) and points at Γ_{ss} as $\lambda \to 0$ or $\lambda \to \infty$.*

Proof. Since we have already proved that the first relation in (11) holds true for any $\lambda > 0$, we just need to prove that the second relation in (11) holds true for λ sufficiently small and large. Let as check condition for the first entropy pair (η_1, Q_1) and $\lambda \to 0$. One could easily check that $\lim_{\lambda\to 0} E^i_\lambda = 0$, $i = 1, 2$ follows from (17) and (18). Even more, $\lim_{\lambda\to 0} E^i_\lambda = 0$, $i = 1, 2$ holds for any q, without limiting analysis to the Γ_{ss} curve. We want to show that $\hat{E}^i_\lambda$ are decreasing in $\lambda = 0$. Using the formulas

$$\frac{d}{dx}K_\nu(x) = -K_{\nu-1}(x) - \frac{\nu}{x}K_\nu(x),\ K_{-\nu}(x) = K_\nu(x)$$

one gets

$$\begin{aligned}\frac{\partial}{\partial\lambda}\hat{E}^1_\lambda =& -\Big(\frac{\lambda^{-1}}{1+\alpha} + 2\rho_0^{-\frac{1+\alpha}{2}}\Big)\frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big)(A\lambda)^{-\frac{1}{1+\alpha}}\Big(\rho_0^{\frac{1-\alpha}{2}} + \rho_1^{\frac{1-\alpha}{2}}\Big)e^{-2\lambda\rho_0^{-\frac{1+\alpha}{2}}} \\ &+ \Big(\frac{\lambda^{-1}}{1+\alpha} + 4\rho_0^{-\frac{1+\alpha}{2}}\frac{\alpha}{1+\alpha}\Big)K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big)\rho_0^{-\frac{\alpha}{2}}\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{\alpha}\Big(\frac{\lambda^{-1}\alpha}{1+\alpha}+4\rho_0^{-\frac{1+\alpha}{2}}\frac{1}{1+\alpha}\Big)K_{\frac{\alpha}{1+\alpha}}\left(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\right)\rho_0^{-\frac{\alpha}{2}}\\
&+\Bigg(\Big(\frac{\lambda^{-1}}{1+\alpha}+2\rho_0^{-\frac{1+\alpha}{2}}+2\frac{1-\alpha}{1+\alpha}\rho_1^{-\frac{1+\alpha}{2}}\Big)K_{\frac{1}{1+\alpha}}\left(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\right)\rho_1^{-\frac{\alpha}{2}}\\
&-\sqrt{\alpha}\Big(\frac{\lambda^{-1}\alpha}{1+\alpha}+2\rho_0^{-\frac{1+\alpha}{2}}-2\frac{1-\alpha}{1+\alpha}\rho_1^{-\frac{1+\alpha}{2}}\Big)K_{\frac{\alpha}{1+\alpha}}\left(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\right)\rho_1^{-\frac{\alpha}{2}}\Bigg)\\
&\cdot e^{-2\lambda\left(\rho_0^{-\frac{1+\alpha}{2}}+\rho_1^{-\frac{1+\alpha}{2}}\right)}.
\end{aligned}
$$

In order to prove that $\hat{E}^1_\lambda$ is decreasing in $\lambda=0$ one may use the following equalities taken from [22]

$$
K_\nu(x)=\frac{1}{2}\pi\frac{I_{-\nu}(x)-I_\nu(x)}{\sin(\nu\pi)},\ I_\nu(x)=\left(\frac{x}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2k}}{k!\,\Gamma(\nu+k+1)},
$$

where $I_\nu(x)$ denotes modified Bessel function of the first kind. Using the identity $\Gamma(\nu)\Gamma(1-\nu)=\frac{\pi}{\sin(\pi\nu)}$ one gets

$$
K_\nu(x)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\Gamma(\nu-k)\Big(\frac{x}{2}\Big)^{2k-\nu}+\Gamma(-\nu-k)\Big(\frac{x}{2}\Big)^{2k+\nu}\right). \tag{21}
$$

Replacing the identity (21) into the first derivative with respect to λ given above and arranging the terms in ascending powers of λ one gets the following form of the first derivative

$$
\frac{\partial}{\partial\lambda}\hat{E}^1_\lambda|_{\lambda=0}=\lambda^{-1-\frac{1}{1+\alpha}}\,i_1+\lambda^{-1-\frac{\alpha}{1+\alpha}}\,i_2+\lambda^{-\frac{1}{1+\alpha}}\,i_3+\lambda^{-\frac{\alpha}{1+\alpha}}\,i_4+\lambda^{\frac{\alpha}{1+\alpha}}\,i_5+\mathcal{O}(\lambda^{\frac{1}{1+\alpha}}).
$$

By straightforward calculations we get

$$
\begin{aligned}
&i_1=i_2=i_3=i_4=0,\\
&i_5=\Gamma\Big(\frac{1}{1+\alpha}\Big)A^{-\frac{1}{1+\alpha}}\left(\frac{1+2\alpha}{1+\alpha}\frac{\alpha-1}{1+\alpha}\Big(\rho_0^{-\frac{1+3\alpha}{2}}+\rho_1^{-\frac{1+3\alpha}{2}}\Big)-\rho_0^{-\frac{1+\alpha}{2}}\rho_1^{-\alpha}\frac{2}{1+\alpha}\frac{3+\alpha}{1+\alpha}\right).
\end{aligned}
$$

Since $i_5<0$, for $\alpha\in(0,1)$, one can conclude that $\hat{E}^1_\lambda$ is decreasing in $\lambda=0$.

The same holds for the second entropy pair (η_2,Q_2). So, the relation (11) holds true for $\lambda\to0$ and all entropy pairs (η_1,Q_1) and (η_2,Q_2).

Take now λ to be large enough. Using the notation from (17) and (18) one can easily conclude that $\hat{E}^1_\lambda\le0$ using the following:

- Each term in (17) and (18) are close to zero for λ sufficiently large.
- The terms

$$
K_{\frac{1}{1+\alpha}}\left(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\right)\rho_0^{-\frac{\alpha}{2}}\text{ and }K_{\frac{\alpha}{1+\alpha}}\left(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\right)\rho_0^{-\frac{\alpha}{2}}
$$

  dominate the other three terms in (17) for λ large enough. Same holds for the second entropy pair.

□

Remark 5. We have performed a lot of tests in order to check the validity of the above proposition for each $\lambda>0$. It seems that Γ_{ss} is always in the entropic region, but we did not succeed to prove it. One can look at Figures 6 and 7. It should be noted that lack of precise enough approximations for Bessel function of

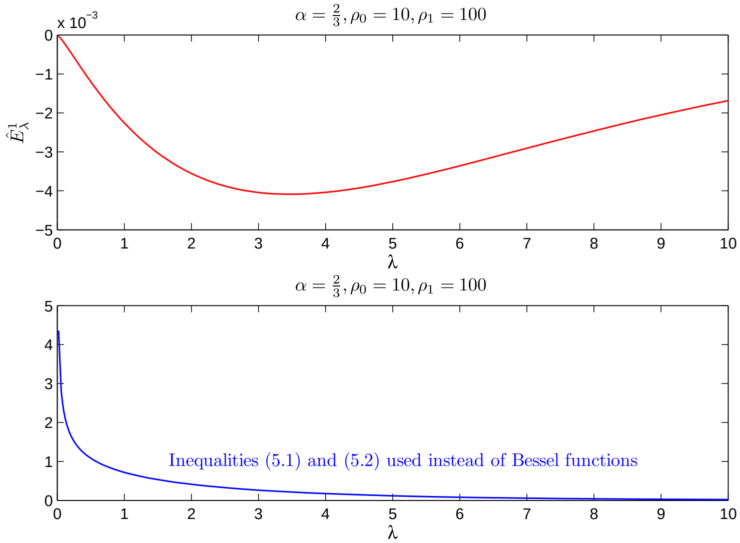

FIGURE 5. Inequalities (15) and (16) are not enough to prove non-positivity of $\hat{E}^1_\lambda$

the second kind represents a serious setback in this analysis. Even though it looks like that inequalities (15) and (16) can be quite helpful, they are not enough to prove (global) non-positivity of E^i_λ, $i = 1, 2$ (for example see Figure 5). As one can see below, in some special cases those inequalities were very helpful, but only locally. Note that inequalities (15) and (16) give us only one (lower or upper) bound for Bessel functions. In order to get the other bound one can use inequality (20). Using (20) one gets the lower bound for $K_{\frac{\alpha}{1+\alpha}}(x)$

$$K_{\frac{\alpha}{1+\alpha}}(x) > \frac{x\,K_{\frac{1}{1+\alpha}}(x)}{\frac{1}{1+\alpha} + \sqrt{(\frac{1}{1+\alpha})^2 + x^2}},\ x > 0.$$

But, one can easily check that not even the inequality given above is enough to prove non-positivity of E^i_λ, $i = 1, 2$. Numerical experiments we have done[1] confirmed our assertion. So, one has to look for better approximations of the Bessel functions or to find an alternative way to prove non-positivity of the entropy functions.

If we use the entropy condition as the admissible one that means that we have to prove that Γ_{ss} is entropic in order to have a solution. If the curve Γ_{ss} is optimal, that would imply prove uniqueness. That would be the second open question we left open.

In the rest of this section we shall present some special cases when Γ_{ss} is entropic.

Example 1. The relations in (11) are satisfied if (ρ_1, q_1) is lying on Γ_{ss} and one of the following conditions is satisfied

(i) $\rho_0^{-\frac{1+\alpha}{2}} + \rho_1^{-\frac{1+\alpha}{2}}$ sufficiently small.
(ii) α is close enough to 1.

[1]All necessary numerical illustrations and calculations were performed by *Matlab*.

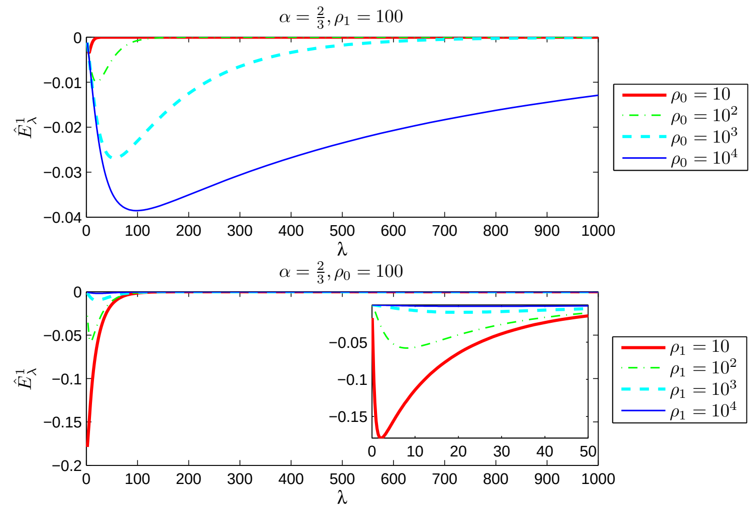

FIGURE 6. Entropies at Γ_{ss} curve – the first entropy pair

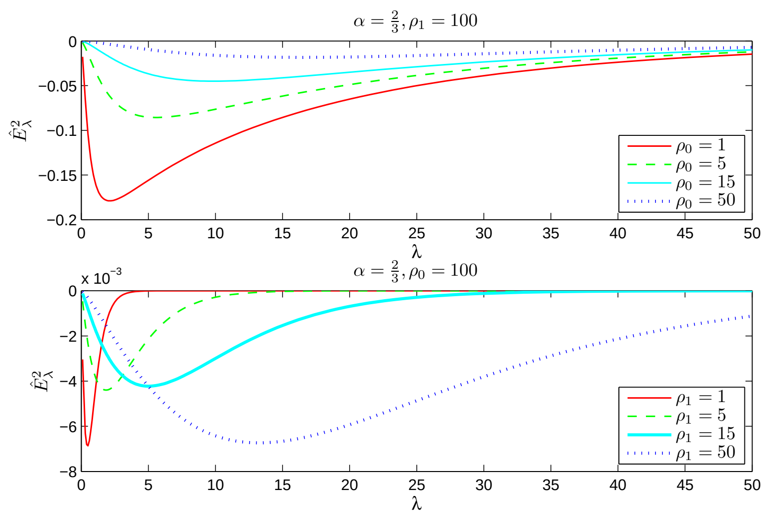

FIGURE 7. Entropies at Γ_{ss} curve – the second entropy pair

Proof. Again, it is enough to prove that the second relation in (11) holds true.
(i) Note that $\rho_0^{-\frac{1+\alpha}{2}} + \rho_1^{-\frac{1+\alpha}{2}} \to 0$ if and only if $\rho_0^{-\frac{1+\alpha}{2}} \to 0$ and $\rho_1^{-\frac{1+\alpha}{2}} \to 0$.

For the first entropy pair (η_1, Q_1), (ρ_1, q_1) lying on Γ_{ss} and $\rho_0^{-\frac{1+\alpha}{2}} + \rho_1^{-\frac{1+\alpha}{2}}$ close to zero we have

$$E^1_\lambda \sim e^{2\lambda \frac{q_0}{\rho_0}} \Big(\frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big) A^{-\frac{1}{1+\alpha}} \lambda^{-\frac{1}{1+\alpha}} (\rho_0^{\frac{1-\alpha}{2}} + \rho_1^{\frac{1-\alpha}{2}})\, e^{-2\lambda \rho_0^{-\frac{1+\alpha}{2}}}$$
$$- \frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big) A^{-\frac{1}{1+\alpha}} \lambda^{-\frac{1}{1+\alpha}} (\rho_0^{\frac{1-\alpha}{2}} + \rho_1^{\frac{1-\alpha}{2}} e^{-2\lambda(\rho_0^{-\frac{1+\alpha}{2}} + \rho_1^{-\frac{1+\alpha}{2}})})$$

$$- \sqrt{\alpha}\frac{1}{2}\Gamma\Big(\frac{\alpha}{1+\alpha}\Big)A^{-\frac{\alpha}{1+\alpha}}\lambda^{-\frac{\alpha}{1+\alpha}}(1-e^{-2\lambda(\rho_0^{-\frac{1+\alpha}{2}}+\rho_1^{-\frac{1+\alpha}{2}})})\Big).$$

Since

$$e^{-2\lambda\rho_0^{-\frac{1+\alpha}{2}}}, e^{-2\lambda(\rho_0^{-\frac{1+\alpha}{2}}+\rho_1^{-\frac{1+\alpha}{2}})} \to 1 \text{ as } \rho_0^{-\frac{1+\alpha}{2}}+\rho_1^{-\frac{1+\alpha}{2}} \to 0,$$

it is clear that $E^1_\lambda \leq 0$ holds. So, the second entropy condition holds for (η_1, Q_1). The same holds for the second entropy pair (η_2, Q_2).

(ii) Let as take α close enough to 1. Then A will be close to 1 and

$$K_{\frac{1}{1+\alpha}}(x) \approx K_{\frac{\alpha}{1+\alpha}}(x) \approx K_{\frac{1}{2}}(x),\ \Gamma\Big(\frac{1}{1+\alpha}\Big) \approx \Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}.$$

So,

$$\begin{aligned}\hat{E}^1_\lambda \sim & \frac{\sqrt{\pi}}{2}\lambda^{-\frac{1}{2}}(1+1)e^{-2\lambda\rho_0^{-1}} - K_{\frac{1}{2}}\left(2\lambda\rho_0^{-1}\right)\rho_0^{-\frac{1}{2}}(1+\sqrt{1}) \\ & - K_{\frac{1}{2}}\left(2\lambda\rho_1^{-1}\right)\rho_1^{-\frac{1}{2}}(1-\sqrt{1})e^{-2\lambda(\rho_0^{-1}+\rho_1^{-1})} \\ \leq & 2\,\frac{\sqrt{\pi}}{2}\lambda^{-\frac{1}{2}}e^{-2\lambda\rho_0^{-1}} - 2\,\frac{\sqrt{\pi}}{2}\lambda^{-\frac{1}{2}}\rho_0^{-\frac{1}{2}}\rho_0^{\frac{1}{2}}e^{-2\lambda\rho_0^{-1}} = 0,\end{aligned}$$

where the relation

$$K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big) \geq \frac{\sqrt{\pi}}{2}(A\lambda\rho_0^{-\frac{1+\alpha}{2}})^{-\frac{1}{2}}e^{-2A\lambda\rho_0^{-\frac{1+\alpha}{2}}},\ \alpha\in(0,1)$$

was used.

So, $\hat{E}^1_\lambda \leq 0$, for α close enough to 1 and for all $\rho_0, \rho_1 \geq 0$ and so is $\hat{E}^2_\lambda \leq 0$. □

Example 2. The relations in (11) for the first entropy pair (η_1, Q_1) holds true if one of the following conditions is true

(i) ρ_0 is sufficiently small and (ρ_1, q_1) is lying on Γ_{ss},
(ii) ρ_1 is sufficiently small, (ρ_1, q_1) is lying on Γ_{ss},
(iii) $\rho_0^{-\frac{1+\alpha}{2}}$ is sufficiently small, $A\lambda \geq 1$, $\rho_1 \leq \alpha^{\frac{1}{1-\alpha}}$ and (ρ_1, q_1) is lying on Γ_{ss}.

Proof. It is enough to prove that the second relation in (11) is satisfied.

(i) Since $e^{-2\lambda\rho_0^{-\frac{1+\alpha}{2}}} \to 0$ and $e^{-2\lambda(\rho_0^{-\frac{1+\alpha}{2}}+\rho_1^{-\frac{1+\alpha}{2}})} \to 0$ as $\rho_0 \to 0$, one gets $\hat{E}^1_\lambda \leq 0$, for ρ_0 sufficiently small, every $\rho_1 \geq 0$, $\lambda \geq 0$, and $\alpha \in (0,1)$.

(ii) One can conclude that $\rho_1^{\frac{1-\alpha}{2}} \to 0$ and $\rho_1^{-\frac{1+\alpha}{2}} \to \infty$ as $\rho_1 \to 0$. Using inequality (15), one gets

$$\begin{aligned}\hat{E}^1_\lambda \leq & \frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big)A^{-\frac{1}{1+\alpha}}\lambda^{-\frac{1}{1+\alpha}}\rho_0^{\frac{1-\alpha}{2}}\Big(e^{-2\lambda\rho_0^{-\frac{1+\alpha}{2}}} - e^{-2A\lambda\rho_0^{-\frac{1+\alpha}{2}}}\Big) \\ & - \sqrt{\alpha}\,\rho_0^{-\frac{\alpha}{2}}K_{\frac{\alpha}{1+\alpha}}\Big(2A\lambda\rho_0^{-\frac{1+\alpha}{2}}\Big) \leq 0,\end{aligned}$$

since $e^{-2\lambda\rho_0^{-\frac{1+\alpha}{2}}} \leq e^{-2A\lambda\rho_0^{-\frac{1+\alpha}{2}}}$.

(iii) Suppose that $\rho_0^{-\frac{1+\alpha}{2}}$ is sufficiently small. Using relations (14), (15) and (16) one gets

$$\begin{aligned}\hat{E}^1_\lambda \sim & \frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big)A^{-\frac{1}{1+\alpha}}\lambda^{-\frac{1}{1+\alpha}}(\rho_0^{\frac{1-\alpha}{2}}+\rho_1^{\frac{1-\alpha}{2}})\,e^{-2\lambda\rho_0^{-\frac{1+\alpha}{2}}} \\ & - \frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big)A^{-\frac{1}{1+\alpha}}\lambda^{-\frac{1}{1+\alpha}}\rho_0^{\frac{1-\alpha}{2}} - \sqrt{\alpha}\,\frac{1}{2}\Gamma\Big(\frac{\alpha}{1+\alpha}\Big)A^{-\frac{\alpha}{1+\alpha}}\lambda^{-\frac{\alpha}{1+\alpha}}\end{aligned}$$

$$
\begin{aligned}
&- K_{\frac{1}{1+\alpha}}\Big(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\Big)\rho_1^{-\frac{\alpha}{2}}\, e^{-2\lambda(\rho_0^{-\frac{1+\alpha}{2}}+\rho_1^{-\frac{1+\alpha}{2}})}\\
&+ \sqrt{\alpha}\, K_{\frac{\alpha}{1+\alpha}}\Big(2A\lambda\rho_1^{-\frac{1+\alpha}{2}}\Big)\rho_1^{-\frac{\alpha}{2}}\, e^{-2\lambda(\rho_0^{-\frac{1+\alpha}{2}}+\rho_1^{-\frac{1+\alpha}{2}})}\\
\le&\frac{1}{2}\Gamma\Big(\frac{1}{1+\alpha}\Big)A^{-\frac{1}{1+\alpha}}\lambda^{-\frac{1}{1+\alpha}}\rho_1^{\frac{1-\alpha}{2}}\, e^{-2\lambda\rho_0^{-\frac{1+\alpha}{2}}}\Big(1-e^{-2(1+A)\lambda\rho_1^{-\frac{1+\alpha}{2}}}\Big)\\
&- \sqrt{\alpha}\,\frac{1}{2}\Gamma\Big(\frac{\alpha}{1+\alpha}\Big)A^{-\frac{\alpha}{1+\alpha}}\lambda^{-\frac{\alpha}{1+\alpha}}\, e^{-2\lambda\rho_0^{-\frac{1+\alpha}{2}}}\Big(1-e^{-2(1+A)\lambda\rho_1^{-\frac{1+\alpha}{2}}}\Big).
\end{aligned}
$$

If $A\lambda \ge 1$ and $\rho_1 \le \alpha^{\frac{1}{1-\alpha}}$, $\hat{E}^1_\lambda \le 0$ holds. □

Using the Proposition 1 and Example 2 one gets the following result.

Example 3. The relations in (11) for the second entropy pair (η_2, Q_2) holds if one of the following conditions is true

(i) ρ_1 is sufficiently small, (ρ_1, q_1) is lying on Γ_{ss},
(ii) ρ_0 is sufficiently small, (ρ_1, q_1) is lying on Γ_{ss},
(iii) $\rho_1^{-\frac{1+\alpha}{2}}$ is sufficiently small, $A\lambda \ge 1$, $\rho_0 \le \alpha^{\frac{1}{1-\alpha}}$ and (ρ_1, q_1) is lying on Γ_{ss}.

6. **Further research.** In this paper we focused our attention on making comparison between two conditions frequently used for admissibility check: convex entropy – entropy flux pair (called entropy condition further on) and overcompressibility. So far, we were not able to find the paper dealing with a problem where entropy condition is better than overcompressibility. So, we have proved that for each $\alpha \in (0,1)$ there exists a neighborhood of ρ_0 on some curve which is not unique and depends of α, where one can conclude that entropy condition is better than overcompressibility one for avoiding non-wanted week solutions. But we did not succeed to exclude them all. We were dealing with modified Bessel function of the second kind, which are not yet well explored and all suitable estimates we used were not enough to prove uniqueness (for estimates see (15), (16), (20)). However, through the above analysis we investigated several cases where solution to Riemann problem is unique i.e. the curve Γ_{ss} is entropic.

Next step in our research would be to prove global uniqueness. That investigation can go in several directions.

The first direction can also be a challenge for our colleagues who work with Bessel functions. In order to prove non-positivity of the entropy function one has to look for better approximations of the Bessel functions, since existing ones are not good enough to prove global non-positivity. Of course, one can find an alternative way to prove that the curve Γ_{ss} is entropic.

It seems (by all numerical experiments we have done) that (entropy) functions $\hat{E}^i_\lambda$, $i = 1,2$, with respect to λ always have only one extreme – a minimum, but we did not succeed to prove it. Using numerical algorithms one may get estimates for the minimum of the functions $\hat{E}^i_\lambda$, $i = 1,2$. For example, estimates for the minimum of $\hat{E}^1_\lambda$ as a function of $b = A\lambda\rho^{-\frac{1+\alpha}{2}}$, for $\rho_0 = \rho_1$ and $\alpha = \frac{1}{10}, \frac{1}{5}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{4}{5}$ are given by $b = 0.2814$, 0.3452, 0.4029, 0.4173, 0.4266, 0.4308, 0.4357 respectively.

Naturally, next step would be to prove existence of unique minimum of the function $\hat{E}^i_\lambda$, $i = 1,2$. In doing so, one can proceed with the use of the first and the second derivative of the observed function, as well as with the use of potentially new and better estimates for the modified Bessel functions of the second kind. Also, one can try to avoid use of derivatives, because of there complexity and focus attention on use of some other mathematical tool.

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