Discrete shock profiles as solutions of scalar conservation laws with discontinuous fluxes

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Abstract. The paper considers a scalar conservation law with a discontinuous flux F of the form $F(x,u) = H(x)g(u) + (1 - H(x))h(u)$ where $H(x)$ is the Heaviside function. Herein, the fluxes g and h are supposed to have one minimum and no maximum and at most one crossing in the interior of the domain of definition. The aim is to verify a weak solution of such a problem in the following way: We are looking for discrete shock profiles for its continuously differentiable perturbation with a parameter ε and Godunov's scheme for spatially varying flux functions. The obtained discrete shock profile satisfies a discrete entropy condition of Kruzhkov type and after letting $\varepsilon \to 0$, approaches an entropic weak solution of the original equation. Then we apply the obtained results to a special case of the above equation with the flux $F(x,u) = a(x)f(u)$, where a is piecewise constant.

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1. Introduction and Preliminaries

1.1. Scope of the paper

Conservation laws with discontinuous flux functions are found to be suitable models of many problems in engineering like traffic flows on highways with changing road conditions or the water flooding model in petroleum industry (to obtain a more detailed list of such applications see [13]). We consider a general form of scalar conservation law with a discontinuous flux

$$\begin{aligned} u_t + \big(H(x)g(u) + (1 - H(x))h(u)\big)_x &= 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{1.1}$$

where $u = u(x,t)$, $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ and $H(x)$ is the Heaviside function, together with the initial data

$$u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases} . \tag{1.2}$$

Such problems are often called "two-flux" problems.

In addition, we also consider the following special case of the above equation called the "multiplicative problem" of the form

$$u_t + (a(x)f(u))_x = 0, \tag{1.3}$$

where

$$a(x) = \begin{cases} a_l, & x < 0 \\ a_r, & x > 0 \end{cases} \tag{1.4}$$

together with the initial data (1.2).

Let us start with a brief literature overview. A Godunov-type algorithm for approximating solutions of scalar conservation laws of form (1.1) is presented in [1]. In that paper, the authors assume that the fluxes g and h have one minimum (maximum) and no maximum (minimum) in the domain of definition. Such fluxes are said to be fluxes of convex (concave) type. They also assume that the fluxes have at most one crossing in the observed domain. We will keep these assumptions herein. (However, our solution concept also applies to the case when one of the fluxes has one minimum and no maximum, and the other has one maximum and no minimum, as will be shown in our forthcoming paper.) In [3] equations of type (1.1) are considered under the same assumptions. The authors construct entropy solutions by using connections. Roughly speaking, a connection is a pair of reals (A, B) satisfying the Rankine-Hugoniot condition at $x = 0$, i.e. $h(A) = g(B)$. A more precise definition of connections will be given in Section 1.2. Using a specific interface entropy condition, the authors have constructed unique entropy solutions for each connection. In addition, they are able to find connections that are relevant for two-phase flows in heterogeneous porous media and call them "optimal connections" and the corresponding solutions "optimal entropy solutions". If both fluxes are of convex (concave) type, they define a connection (A, B) as a unique optimal connection if either A or B coincides with the argument of the bigger (smaller) of the local minima (maxima) of $g(u)$ and $h(u)$.

The authors of [5, 7] analysed a hyperbolic clarifier-thickener model. This model appears in the form of a combination of two-flux problems with three discontinuities. The flux of the model satisfies the so-called crossing condition, which geometrically means that the graphs of the left h and the right g part of the flux jump do not cross in the interior of the domain of definition, or if they do, then the graph of h lies above the graph of g to the left of any crossing point u_*. They claimed that for this model the physics requires that the proper solution is associated with the connection $(A, B) = (u_*, u_*)$. In [6] the same group of authors developed an Engquist-Osher-type algorithm to obtain approximations for entropic solutions of problem (1.1), (1.2) without imposing any assumptions on the type of flux crossing. The solution concept is similar as the one in [3], leading to unique entropic solutions for any connection (A, B).

In [10] the existence of the vanishing viscosity limit of (1.3) is proved by using the compensated compactness theory. In [6], the connection when A or B coincides with the argument of the bigger (smaller) of the local minima (maxima) of $a_r f(u)$ and $a_l f(u)$ is claimed to be the general accepted one for solving Eq. (1.3).

The entropy conditions in the literature differ from each other, since they are obtained through different approaches. According to [8], it seems that there is no universal entropy condition for the general conservation law (1.1). The explanation about differences in entropy criteria can be found in that paper. Our motivation is to consider (1.1) in the general case, i.e. we will not observe any particular physical problem that the equation is modelling. Our aim is to define entropy conditions by using discrete shock profiles (DSP for short) for perturbed equations. A DSP presents the exact solution of a numerical approximation of a conservation law. Equation (1.1) can be rewritten in the form

$$u_t + F(x,u)_x = 0,$$

where

$$F(x,u) = H(x)g(u) + (1 - H(x))h(u). \tag{1.5}$$

Take a conservation law of the form

$$u_t + (F_\varepsilon(x,u))_x = 0, \tag{1.6}$$

where F_ε is a continuously differentiable approximation of F, (see (2.1), (2.2)). Then Godunov's scheme for conservation laws with spatially varying flux [4] can be used. Among all DSPs we will choose the one that satisfies a discrete Kruzhkov type entropy inequality. We say that a shock wave solution of Eq. (1.1) is an entropy solution if a DSP for such a perturbation converges to it. In other words, the existence of an entropic DSP (a precise definition will be given in Sections 1.3 and 2) is taken to be an entropy criterion for shocks herein (like a well-known vanishing viscosity criterion).

Godunov's scheme and its several modifications are widely used for finding numerical solutions of conservation laws due to the fact that they are well adapted to entropy conditions. In the case of scalar conservation laws with continuous flux functions there always exists a DSP for Godunov's scheme if Oleinik's entropy condition is satisfied, (see [9]). Similarly, the case of more general flux functions has been proved recently in [15].

Our plan of exposition is as follows. After giving some introductory facts in Sections 1.2-1.5, we introduce a continuously differentiable perturbation of the flux in (1.1) and find a solution consisting of DSPs for the obtained perturbed equation in Section 2. Then we present an appropriate discrete version of Kruzkhov's entropy inequality in order to gain a unique entropy DSP among all obtained DSPs. In Section 2.1 we apply our results for obtaining entropy DSPs for the multiplicative equation (1.3). At the end, in

Section 3.2 we give a brief description about our further investigations in this area.

1.2. Initial assumptions, notations and definitions

Now, we will introduce some notations, give some definitions and state initial facts that we assume to hold in this paper. Let $I = [X, Y]$ be the domain of definition of the fluxes g and h, where $-\infty < X < Y < \infty$.

- **Assumption 1:** The fluxes $g, h \in C^1(I)$ and satisfy $g(X) = h(X)$ and $g(Y) = h(Y)$.

Definition 1.1. [13] Let $f \in C^1(I)$, then f is said to be a

- convex type flux if it has one minimum and no maximum in the interior of I,
- concave type flux if it has one maximum and no minimum in the interior of I.

- **Assumption 2:** The fluxes g and h are of convex type.
  Let us denote the unique minimum of g and h with θ_g and θ_h respectively, i.e.

$$g(\theta_g) = \min_{u \in I} g(u) \quad \text{and} \quad h(\theta_h) = \min_{u \in I} h(u).$$

  Note that the fluxes g and h need not be convex. The case when both fluxes are of concave type is very similar, so we omit it for simplicity.
- **Assumption 3:** The fluxes g and h intersect at most at one point in the interior of I. Let us denote the crossing point with u_*. The flux crossing is either regular, undercompressive or overcompressive (the appropriate definition is given below).

  According to [13], a flux crossing is said to be

a. Regular if either $h'(u_*), g'(u_*) \geq 0$ or $h'(u_*), g'(u_*) \leq 0$
b. Overcompressive if $h'(u_*) > 0$ and $g'(u_*) < 0$
c. Undercompressive if $h'(u_*) < 0$ and $g'(u_*) > 0$
d. Marginally under(over)compressive if it is not regular and either $h'(u_*) = 0$ or $g'(u_*) = 0$.

A pair of reals (A, B) is called a connection if

$$h(A) = g(B),\ A \leq \theta_h,\ B \geq \theta_g. \tag{1.7}$$

1.3. Discrete shock profiles

After discretizing the $x-t$ plane with a uniform grid $\Delta x\mathbb{Z} \times \Delta t\mathbb{N}$ for $\Delta x > 0$, $\Delta t > 0$, with grid points labeled $(x_j, t_n) := (j\Delta x, n\Delta t)$ and the corresponding discrete unknowns u_j^n, the following numerical approximation of conservation law (1.6) can be obtained

$$u_j^{n+1} = u_j^n - \lambda(F^n_{j+1/2} - F^n_{j-1/2}), \tag{1.8}$$

where $x_{j-1/2} = (j-1/2)\Delta x$, $\lambda = \dfrac{\Delta t}{\Delta x}$, $F^n_{j-1/2} = \mathcal{F}(x_j, u^n_{j-1}, u^n_j)$, $F^n_{j+1/2} = \mathcal{F}(x_j, u^n_j, u^n_{j+1})$ and $\mathcal{F}$ is a numerical flux function. A discrete shock profile is a special solution of (1.8) of the form

$$u^n_j := U\left(\frac{x_j - st_n}{\Delta x}\right) = U(j - s\lambda n) \tag{1.9}$$

with the boundary conditions

$$U(-\infty) = u_l, \; U(+\infty) = u_r.$$

The discrete function U is called the shock profile, while s is the velocity of the wave. From (1.9) it is clear that for $\Delta x \to 0$ the solution of (1.8) approaches a shock wave of the form

$$u(x,t) = \begin{cases} u_l, & x < st \\ u_r, & x > st \end{cases}.$$

A minimal domain of U is the additive group $\mathbb{Z} + \lambda\mathbb{Z}$. When $s\lambda = r/l$ is rational, the discrete domain is $l^{-1}\mathbb{Z}$. On the other hand, when $s\lambda$ is irrational, $\mathbb{Z} + \lambda\mathbb{Z}$ becomes dense in $\mathbb{R}$. In that case, we have a so-called continuous profile, since we U is defined in the whole real line $\mathbb{R}$. Herein we assume that $s\lambda = r/l$ is rational. For more details about discrete shock profiles, we refer to [14, 15].

1.4. Godunov's method for conservation laws with spatially varying flux

In order to solve a conservation law with a spatially varying flux function of the form (1.6), we first have to discretize the flux function. For that purpose, we can choose, for example, the cell-centered flux discretization described in [4] and [12] where the flux function is discretized to yield a flux function $F_j(u)$ that holds throughout the j-th grid cell. The discretized flux function is defined simply by

$$F_j(u) = F_\varepsilon(x_j, u).$$

According to [4, 12], Godunov's method consists in solving the following Riemann problems of (1.6)

$$u_t + F_{j-1}(u)_x = 0, \quad u(x,0) = \begin{cases} u^n_{j-1}, & x < x_{j-1/2} \\ \tilde{u}^n_{l,j-1/2}, & x > x_{j-1/2} \end{cases}$$

and

$$u_t + F_j(u)_x = 0, \quad u(x,0) = \begin{cases} \tilde{u}^n_{r,j-1/2}, & x < x_{j-1/2} \\ u^n_j, & x > x_{j-1/2} \end{cases}$$

centred at $x_{j-1/2}$, where the states $\tilde{u}^n_{l,j-1/2}$ and $\tilde{u}^n_{r,j-1/2}$ are taken to be connected by a stationary shock, i.e.

$$F_{j-1}(\tilde{u}^n_{l,j-1/2}) = F_j(\tilde{u}^n_{r,j-1/2}), \tag{1.10}$$

and having the property that u^n_{j-1} can be connected to $\tilde{u}^n_{l,j-1/2}$ using only left-going waves, while $\tilde{u}^n_{r,j-1/2}$ can be connected to u^n_j using only right going waves. If the wave connecting $\tilde{u}^n_{l,j-1/2}$ and $\tilde{u}^n_{r,j-1/2}$ was not stationary, then a solution would not be bounded. See Section 16.4 in [12] for details.

Then we apply the flux differencing formula (1.8) by taking

$$\mathcal{F}^n_{j-1/2} = F_j(\tilde{u}^n_{r,j-1/2})$$
$$\mathcal{F}^n_{j+1/2} = F_j(\tilde{u}^n_{l,j+1/2}).$$

In the case of an autonomous conservation law, this coincides with the well-known standard Godunov's method.

1.5. The Riemann problem

Let us shortly explain the Riemann problem that we use for DSP construction. Consider problem (1.1), (1.2) under the initial assumptions given in Section 1.2.

We define $\bar{\bar{B}}$ to be a point satisfying

$$g(B) = g(\bar{\bar{B}}) \quad \text{and} \quad \bar{\bar{B}} < \theta_g. \tag{1.11}$$

In general, the Riemann problem (1.1), (1.2) has a solution in the form of left- and right-handed waves connected with a steady wave at $x = 0$ for each connection. (Note that the connection implies zero-speed Rankine-Hugoniot conditions.) But profiles, either discrete or viscous, are constructed for individual waves. For that purpose, we are looking for the case when there is only one travelling shock wave besides the steady one, say the right-handed one, and choose the left state in that manner. (Of course, one could do it reversely.) Such a solution exists for $u_l = A$ and $u_r \in (\bar{\bar{B}}, B)$:

$$u(x,t) = \begin{cases} u_l = A, & x < 0 \\ B, & 0 < x < \dfrac{g(B) - g(u_r)}{B - u_r} t \\ u_r, & x > \dfrac{g(B) - g(u_r)}{B - u_r} t \end{cases}. \tag{1.12}$$

Note that if $u_r < \bar{\bar{B}}$, the wave speed would be negative and the wave would interact with the steady one, which is not allowed. On the other hand, if $u_r > B$, the right going wave would be rather a rarefaction wave, since the shock does not satisfy Oleinik's entropy condition in that case. In other words,

$$\frac{g(u) - g(B)}{u - B} \geq s \geq \frac{g(u) - g(u_r)}{u - u_r}$$

is not true for all u between B and u_r. Since a DSP is defined only for shocks, we consider only the case when $u_r \in (\bar{\bar{B}}, B)$.

Note that by definition (1.7) $h'(A) \leq 0$, and since $u_l = A$, we have $h'(u_l) \leq 0$. A solution of the considered problem consisting only of a stationary and a right going shock wave also appears for any $u_l \in I$ satisfying $h'(u_l) > 0$, i.e. $u_l > \theta_h$ and $u_r \in (\bar{\bar{u}}_l, \bar{u}_l)$, where $\bar{u}_l$ and $\bar{\bar{u}}_l$ are given by

$$g(\bar{u}_l) = g(\bar{\bar{u}}_l) = h(u_l) \quad \text{and} \quad \bar{u}_l > \theta_g, \ \bar{\bar{u}} < \theta_g,$$

i.e.

$$u(x,t) = \begin{cases} u_l, & x < 0 \\ \bar{u}_l, & 0 < x < \dfrac{g(\bar{u}_l) - g(u_r)}{\bar{u}_l - u_r} t \\ u_r, & x > \dfrac{g(\bar{u}_l) - g(u_r)}{\bar{u}_l - u_r} t \end{cases} . \tag{1.13}$$

2. Existence of entropic DSPs for the two-flux equation

Let us consider the Riemann problem (1.1), (1.2) under the assumptions stated in Section 1.2. Our first aim is to obtain a DSP for a perturbation of the above problem for any admissible connection (A, B) (a precise definition will be given below) having a limit of the form (1.12). One part is very simple as we can use the results from [9] and find a DSP for the right-handed shock in the expected solution. So we have to deal only with a steady shock, i.e. to find a DSP for it.

We use a regularization of (1.5) $F_\varepsilon \in C^1(\mathbb{R} \times I)$ so that

$$F_\varepsilon(x,u) = F(x,u) \quad \text{for} \quad |x| \geq \varepsilon, \tag{2.1}$$

where ε is small enough, and use Godunov's scheme for conservation laws with spatially varying flux functions described in Section 1.4. But first, we require an additional condition to hold for the regularization in order to be admissible.

Definition 2.1. A regularization $F_\varepsilon \in C^1(\mathbb{R} \times I)$ given by (2.1) is said to be admissible if

$$\begin{aligned} F_\varepsilon(x,u) < F_\varepsilon(y,u) \quad &\text{if} \quad g(u) > h(u) \quad \text{and} \\ F_\varepsilon(x,u) > F_\varepsilon(y,u) \quad &\text{if} \quad g(u) < h(u) \end{aligned} \tag{2.2}$$

holds for every $x, y \in [-\varepsilon, \varepsilon]$ such that $x < y$ and every $u \in I$.

Remark 2.2. Assumption (2.2) ensures that $F_\varepsilon(x,u)$ and $F_\varepsilon(y,u)$ cross only at the edges of interval I and at a flux cossing $u_* \in I$ for all $x, y \in [-\varepsilon, \varepsilon]$.

Example. For example, a generalization of the regularization function presented in [2]

$$F_\varepsilon(x,u) = \begin{cases} h(u), & x \leq -\varepsilon \\ \dfrac{g(u) - h(u)}{2\varepsilon^2}(x+\varepsilon)^2 + h(u), & -\varepsilon \leq x \leq 0 \\ \dfrac{h(u) - g(u)}{2\varepsilon^2}(x-\varepsilon)^2 + g(u), & 0 \leq x \leq \varepsilon \\ g(u), & x \geq \varepsilon \end{cases} \tag{2.3}$$

is an admissible regularization. The discrete array of fluxes $\{F_j\}$ for (2.3) in the cases when g and h do not cross and when they have one crossing in the interior of I is shown in Fig. 1 A) and 1 B) respectively.

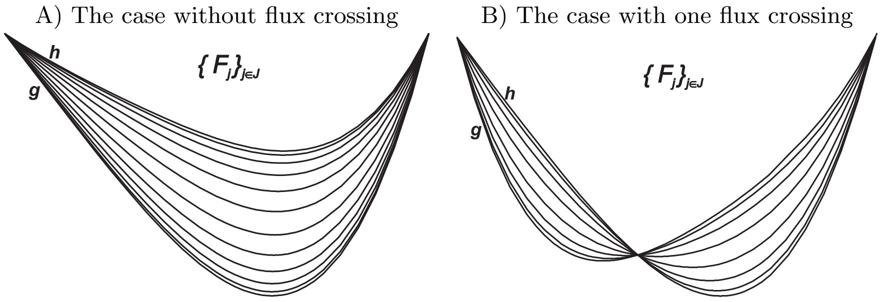

Figure 1. The array of discrete fluxes for an admissible regularization

After using the flux regularization (2.1) given by (2.2), Eq. (1.1) becomes

$$u_t + F_\varepsilon(x,u)_x = 0. \tag{2.4}$$

Now, let us select from the set of all connections those that are admissible with respect to the regularization, in the following manner.

Definition 2.3. A connection (A, B) is said to be admissible with respect to the regularization if equation

$$F_\varepsilon(x,u) = h(A)$$

has real solutions in I for all $x \in [-\varepsilon, \varepsilon]$.

In the sequel, we call such connections simply *admissible*. Fig. 2 shows examples of admissible and not admissible connections in various cases of flux crossing. As we can see, in the case when the fluxes g and h do not cross, or if the crossing is regular, Fig. 2 A) and B) respectively, every connection given by (1.7) is admissible with respect to the regularization. On the other hand, Fig. 2 C) indicates that in the case of an undercompressive flux crossing, the only admissible connection is $A = B = u_*$, as other connections, like (A_1, B_1) and (A_2, B_2) do not intersect F_k for every $k \in J$. In the case of an overcompressive flux crossing, all connections above the connection (A, B) where $h(A) = g(B) = h(u^*)$ are admissible, Fig. 2 D).

Theorem 2.4. *Let ε be small enough. Using the above assumptions and notations, Eq.* (2.4) *has a DSP for Godunov's scheme with the limit of the form*

- (1.12) *for any admissible connection and $u_l = A$, $u_r \in (\bar{\bar{B}}, B)$,*
- (1.13) *if $u_l > \theta_h$ and $u_r \in (\bar{\bar{u}}_l, \bar{u}_l)$.*

Proof. The solution of Eq. (1.1) is equivalent to the solution of the system

$$\begin{aligned} u_t + h(u)_x &= 0, x < 0 \\ u_t + g(u)_x &= 0, x > 0 \end{aligned} \tag{2.5}$$

and u satisfies the Rankine-Hugoniot condition $g(u^+) = h(u^-)$ at $x = 0$, where $u^+ = \lim_{x\to 0^+} u(x,t)$ and $u^- = \lim_{x \to 0^-} u(x,t)$. Our aim is to find DSPs for the right-handed shocks. For simplicity, we present the proof of the

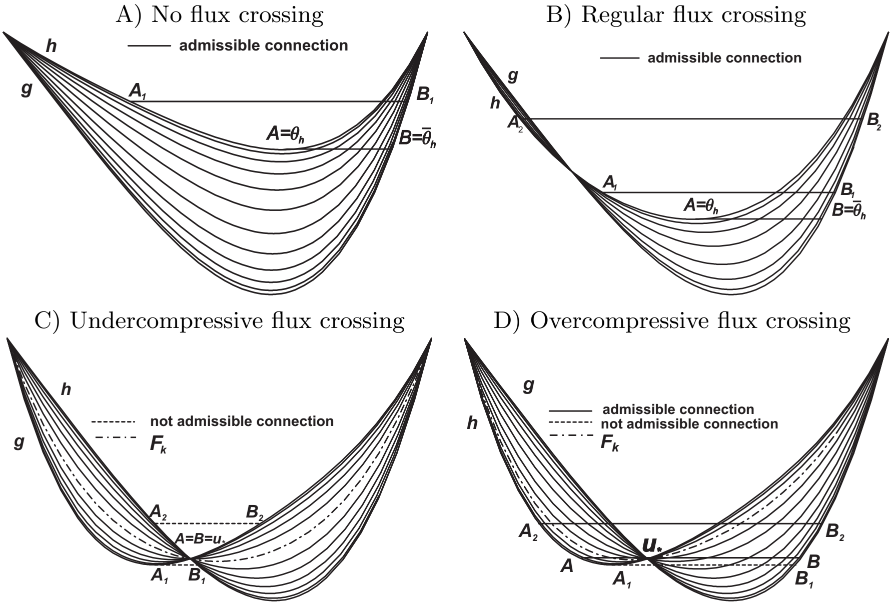

FIGURE 2. Admissibility of connections for various cases of flux crossing

existence only for DSPs with the limit of the form (1.12), i.e. for $A = u_l$, and $u_r \in (\bar{\bar{B}}, B)$, while the proof of the existence of DSPs with limits of the form (1.13) is very similar. Therefore, we have two problems to solve:
The first one is Eq. (2.4) together with

$$u_0(x) = \begin{cases} u_l = A, & x < 0 \\ B, & 0 < x \le \varepsilon \end{cases} . \tag{2.6}$$

The existence of a connection (A, B) here means that

$$F_\varepsilon(\varepsilon, B) = F_\varepsilon(-\varepsilon, A). \tag{2.7}$$

Since for $x \ge \varepsilon$ problem (2.5) is reduced to

$$u_t + g(u)_x = 0, \tag{2.8}$$

with the initial data

$$u_0(x) = \begin{cases} B, & x < \varepsilon, \\ u_r, & x > \varepsilon \end{cases} \tag{2.9}$$

this is the second problem for us to solve. It is clear that a DSP for our problem has two independent parts: a steady DSP starting from $x = 0$, and the right-hand sided one (DSP with positive speed) starting from $x = \varepsilon$. This permits us to split the proof in two parts. The first one is problem (2.4), (2.6). Let us discretize the problem on interval $[-\varepsilon, \varepsilon]$ where

$$\varepsilon = j^* \Delta x, \tag{2.10}$$

for some $j^* \in \mathbb{N}$. This gives the following sequence of Riemann problems

$$u_t + F_{j-1}(u)_x = 0, \quad u(x,0) = \begin{cases} u^n_{j-1}, & x < x_{j-1/2} \\ \tilde{u}_{l,j-1/2}, & x > x_{j-1/2} \end{cases}$$

and

$$u_t + F_j(u)_x = 0, \quad u(x,0) = \begin{cases} \tilde{u}_{r,j-1/2}, & x < x_{j-1/2} \\ u^n_j, & x > x_{j-1/2} \end{cases}.$$

For a steady DSP at $x = 0$, $u^{n+1}_j = u^n_j = u_j$ should hold for all j and all n. So, to solve the above two problems, we seek for states $\tilde{u}_{l,j-1/2}$ and $\tilde{u}_{r,j-1/2}$ satisfying (1.10) and having the property that u_{j-1} can be connected with $\tilde{u}_{l,j-1/2}$ using only left going waves, i.e.

$$c_{l,j} := \frac{F_{j-1}(\tilde{u}_{l,j-1/2}) - F_{j-1}(u_{j-1})}{\tilde{u}_{l,j-1/2} - u_{j-1}} \leq 0$$

or $\tilde{u}_{l,j-1/2} = u_{j-1}$ that corresponds to the case $F_{j-1}(\tilde{u}_{l,j-1/2}) = F_{j-1}(u_{j-1})$ and $c_{l,j} = 0$, and $\tilde{u}_{r,j-1/2}$ with u_j using only right going waves, i.e.

$$c_{r,j} := \frac{F_j(u_j) - F_j(\tilde{u}_{r,j-1/2})}{u_j - \tilde{u}_{r,j-1/2}} \geq 0$$

or $\tilde{u}_{r,j-1/2} = u_j$.

The fact $A = u_l$ implies that the first speed $c_{l,-j^*+1} = 0$. If any $c_{r,j} > 0$, $j < j^*$, the wave connecting u_j and $\tilde{u}_{r,j-1/2}$ would interact with the wave connecting $\tilde{u}_{l,j+1/2}$ and $\tilde{u}_{r,j+1/2}$ that has zero speed. This would contradict the relation $u^n_j = u_j$ for some n. In the case when $j = j^*$ one can see that $g(B) = F_{j^*}(u_{j^*}) = F_{-j^*}(u_{-j^*}) = h(A)$ due to (2.7). On the other hand, we have already seen that $F_{j^*-1}(\tilde{u}_{l,j^*-1/2}) = F_{j^*-1}(u_{j^*-1}) = ... = F_{-j^*}(u_{-j^*})$. This determines c_{r,j^*} to be zero as all velocities before. As a consequence we have

$$F_j(u_j) = F_{j-1}(u_{j-1}) = g(B) = h(A), \quad j = -j^* + 1, ..., j^*. \tag{2.11}$$

It is known that problem (2.8), (2.9) has a solution in the form of a shock wave with the velocity

$$s = \frac{g(u_r) - g(B)}{u_r - B}.$$

The existence of a DSP for a scalar conservation law with a continuous flux for an initial value discontinuity at $x = 0$, is proved in [9]. Note that our initial value problem is just translated into $x = \varepsilon$. Now we combine the right going wave and the steady one obtained above. The travelling DSP in this case has values equal to B for $\varepsilon < x < st + \varepsilon$, while the stationary DSP for (2.4), (2.6) has values equal to B for $x = \varepsilon$ as proved above. Since the velocity of the travelling DSP is positive, their union constitutes the new DSP for problem (2.4) with the initial data (1.2).

Note that the proof of the existence of DSPs with a limit of the form (1.13) would start by solving problem (2.4) together with

$$u_0(x) = \begin{cases} u_l, & x < 0 \\ \bar{u}_l, & 0 < x \leq \varepsilon \end{cases} \tag{2.12}$$

and (2.8) with

$$u_0(x) = \begin{cases} \bar{u}_l, & x < \varepsilon \\ u_r, & x > \varepsilon \end{cases} .$$

That proves the theorem. □

However, a solution of Eq. (2.11) is not unique, and therefore we impose an appropriate entropy condition below. In other words, our aim is to find out when there is a connection (A, B) for (2.11) that permits an entropy DSP for the perturbed problem (2.4) for different types of flux crossing for Godunov's scheme. As seen below, if such an entropic connection exists, it turns out to be unique. So, let us start by defining the discrete entropy condition for our problem. In [11], the authors observe a more general problem of the form

$$\begin{aligned} u_t + \left(F\left(\gamma(x), u\right)\right)_x &= D(u)_{xx} \\ u(x,0) &= u_0(x) \end{aligned}$$

and impose the following Kruzhkov type entropy inequality

$$\begin{aligned} &|u-c|_t + \big(\operatorname{sign}(u-c)\left(F(\gamma(x),u) - F(\gamma(x),c)\right)\big)_x \\ &\quad + |D(u) - D(c)|_{xx} + \gamma'(x)\operatorname{sign}(u-c)F_x(\gamma(x),c) \le 0 \end{aligned}$$

for all constant $c \in \mathbb{R}$ in the sense of distributions. For $D = 0$ and $\gamma(x) = x$, the above inequality reads

$$\begin{aligned} |u-c|_t + \big(\operatorname{sign}(u-c)\left(F_\varepsilon(x,u) - F_\varepsilon(x,c)\right)\big)_x& \\ + \operatorname{sign}(u-c)F_\varepsilon(x,c)_x \le 0.& \end{aligned} \tag{2.13}$$

Let us now find a correct discrete version of inequality (2.13). For that purpose, we make the following notation

$$G(x,u) := \operatorname{sign}(u-c)\left(F_\varepsilon(x,u) - F_\varepsilon(x,c)\right)$$

and let $\left(\frac{\partial G}{\partial x}\right)_{app}\left(x_j, u_j^n\right)$ be the approximation of $\frac{\partial G}{\partial x}\left(x_j, u_j^n\right)$. Then we should take

$$\frac{\partial G}{\partial x}\left(x_j, u_j^n\right) - \left(\frac{\partial G}{\partial x}\right)_{app}\left(x_j, u_j^n\right) \le 0, \tag{2.14}$$

since we would affect inequality (2.13) otherwise. Namely, by using the reverse inequality than (2.14), we would reduce the value of the partial derivative of G at every point (x_j, t_n) artificially. Such pointwise influences would accumulate by summing the discrete inequality over the spatial and time domain. On that way we could obtain a DSP that satisfies a discrete version of (2.13), but its limit does not satisfy (2.13) itself. In other words, condition (2.14) ensures that the limit of a DSP that satisfies a discrete version of (2.13) is entropic indeed.

From the Taylor series for functions of two variables we get

$$G(x_j \pm \Delta x, u_j^n \pm \Delta u_j^n) = G(x_j, u_j^n) \pm \Delta x \frac{\partial G}{\partial x}(x_j, u_j^n) \pm \Delta u_j^n \frac{\partial G}{\partial u}(x_j, u_j^n) + \mathcal{O}(k_{j,n}^2)$$

where $\Delta u_j^n = u_j^n - u_{j-1}^n$ and $k_{j,n} = \max\left(\Delta x, \Delta u_j^n\right)$. This implies

$$\frac{\partial G}{\partial x}(x_j, u_j^n) - \nabla[G_j] = -\mu_j \frac{\partial G}{\partial u}(x_j, u_j^n) + \mathcal{O}\left(k_{j,n}^2\right) \text{ if } u_{j-1}^n < u_j^n \tag{2.15}$$

$$\frac{\partial G}{\partial x}(x_j, u_j^n) - \nabla[G_j] = \mu_j \frac{\partial G}{\partial u}(x_j, u_j^n) + \mathcal{O}\left(k_{j,n}^2\right) \text{ if } u_{j-1}^n > u_j^n \tag{2.16}$$

$$\frac{\partial G}{\partial x}(x_j, u_j^n) - \Delta[G_j] = -\mu_{j+1} \frac{\partial G}{\partial u}(x_j, u_j^n) + \mathcal{O}\left(k_{j+1,n}^2\right) \text{ if } u_j^n < u_{j+1}^n \tag{2.17}$$

$$\frac{\partial G}{\partial x}(x_j, u_j^n) - \Delta[G_j] = \mu_{j+1} \frac{\partial G}{\partial u}(x_j, u_j^n) + \mathcal{O}\left(k_{j+1,n}^2\right) \text{ if } u_j^n > u_{j+1}^n \tag{2.18}$$

where

$$\nabla[G_j] = \frac{G(x_j, u_j^n) - G(x_{j-1}, u_{j-1}^n)}{\Delta x}, \Delta[G_j] = \frac{G(x_{j+1}, u_{j+1}^n) - G(x_j, u_j^n)}{\Delta x}$$

present the backward differencing and forward differencing approximation of the partial derivative of G at the point (x_j, u_j^n), respectively and $\mu_j = \dfrac{|\Delta u_j^n|}{\Delta x} \geq 0$.

From (2.14) and (2.15) we obtain the condition

$$\frac{\partial G}{\partial u}(x_j, u_j^n) \geq 0$$

which implies

$$\operatorname{sign}(u_j^n - c) \frac{\partial F_\varepsilon}{\partial u}\left(x_j, u_j^n\right) = \operatorname{sign}(u_j^n - c) F_j'(u_j^n) \geq 0 \tag{2.19}$$

It is easy to check that the case when $c \notin (u_{j-1}^n, u_j^n)$ is trivial, since the right hand side of the discrete entropy inequality obtained by using backward differencing (see (2.20) below) equals zero. So, for the case when $u_{j-1}^n < c < u_j^n$, (2.19) reads $F_j'(u_j^n) \geq 0$, since $\operatorname{sign}(u_j^n - c) > 0$ in this case. Similar, combining (2.14) and (2.16), we obtain $F_j'(u_j^n) \geq 0$, since $\operatorname{sign}(u_j^n - c) < 0$ in that case. In the same way, from (2.17) and (2.18), respectively, we obtain the condition $F_j'(u_j^n) \leq 0$.

Overall, we can conclude that in the case when $F_j'(u_j^n) \geq 0$, the correct approximation for $\frac{\partial G}{\partial x}(x_j, u_j^n)$ is obtained by backward differencing (since this avoids artificial reducing of the value of the partial derivative of G at any point (x_j, u_j^n) which would lead to an entropic DSP whose limit is not entropic at all, as already explained above) and we obtain the following discrete version of (2.13)

$$\begin{aligned} |u_j^{n+1} - c| - |u_j^n - c| + \lambda\big(\operatorname{sign}(u_j^n - c)\left(F_j(u_j^n) - F_j(c)\right) - \operatorname{sign}(u_{j-1}^n \\ -c)\left(F_{j-1}(u_{j-1}^n) - F_{j-1}(c)\right)\big) + \lambda \operatorname{sign}(u_j^n - c)\left(F_j(c) - F_{j-1}(c)\right) \leq 0. \end{aligned} \tag{2.20}$$

For the same reasons as for $F_j'(u_j^n) \geq 0$, the correct approximation for $\frac{\partial G}{\partial x}(x_j, u_j^n)$ in the case when $F_j'(u_j^n) \leq 0$ is obtained by forward differencing and we get

$$|u_j^{n+1}-c|-|u_j^n-c|+\lambda\Big(\operatorname{sign}(u_{j+1}^n-c)\left(F_{j+1}(u_{j+1}^n)-F_{j+1}(c)\right) \tag{2.21}$$
$$-\operatorname{sign}(u_j^n-c)\left(F_j(u_j^n)-F_j(c)\right)\Big)+\lambda\operatorname{sign}(u_j^n-c)\left(F_{j+1}(c)-F_j(c)\right)\le 0.$$

It is easy to check that both (2.20) and (2.21) are correct discretizations of inequality (2.13). For that purpose, take $\varphi(x,t)$ to be a test function with compact support, i.e.,

$$\varphi(x,t)=0,\ \text{for } t\ge T:=N\Delta t \text{ and } \varphi(x,t)=0,\ \text{for } x\notin[-\varepsilon,\varepsilon],$$

when T is large enough. If we sum (2.20) for every $j\in J=\{-j^*,...,j^*\}$, where j^* is given by (2.10), we get

$$\sum_{n=0}^{N}\sum_{j=-j^*}^{j^*}\varphi(x_j,t_n)(|u_j^{n+1}-c|-|u_j^n-c|)+\frac{\Delta t}{\Delta x}\sum_{n=0}^{N}\sum_{j=-j^*}^{j^*}\varphi(x_j,t_n)\big(Q(u^n;j)$$
$$-Q(u^n;j-1)+\operatorname{sign}(u_j^n-c)(F_j(c)-F_{j-1}(c))\big)\le 0,$$

where $Q(u^n;j)=\operatorname{sign}(u_j^n-c)(F_j(u_j^n)-F_j(c)),\ j\in J$. The summation by parts yields

$$-\sum_{j=-j^*}^{j^*}\varphi(x_j,0)|u_j^0-c|-\sum_{j=-j^*}^{j^*}\sum_{n=1}^{N}(\varphi(x_j,t_n)-\varphi(x_j,t_{n-1}))|u_j^n-c|$$
$$-\frac{\Delta t}{\Delta x}\sum_{n=0}^{N}\sum_{j=-j^*}^{j^*}(\varphi(x_{j+1},t_n)-\varphi(x_j,t_n))\big(Q(u^n;j)+\operatorname{sign}(u_j^n-c)F_j(c)\big)\le 0.$$

Rearranging, we find that

$$\Delta t\Delta x\sum_{n=1}^{N}\sum_{j=-j^*}^{j^*}\Big(\frac{\varphi(x_j,t_n)-\varphi(x_j,t_{n-1})}{\Delta t}|u_j^n-c|$$
$$+\frac{\varphi(x_{j+1},t_n)-\varphi(x_j,t_n)}{\Delta x}\big(Q(u^n;j)+\operatorname{sign}(u_j^n-c)F_j(c)\big)\Big)$$
$$+\Delta x\sum_{j=-j^*}^{j^*}\varphi(x_j,0)|u_j^0-c|\ge 0.$$

As this represents the Riemann sum for the weak formulation of (2.13), it just remains to let $\Delta x\to 0,\Delta t\to 0$. The proof that (2.21) is the correct discretization of (2.13) is similar. This gives us the following definition:

Definition 2.5. A DSP for Eq. (2.4) is said to be entropic if for every admissible regularization F_ε relation (2.20) holds if $F_j'(u_j)\ge 0$, and (2.21) holds if $F_j'(u_j)\le 0$ for each constant $c\in\mathbb{R}$ and every $j\in J$.

Now, we define the existence of entropic DSPs as a criterion for shocks.

Definition 2.6. A shock wave solution of Eq. (1.1) is said to be entropic if an entropic DSP of the perturbed Eq. (2.4) converges to it.

Lemma 2.7. *Problem* (2.4), *together with* (2.6) *and* (2.7) *has a unique entropic stationary DSP for Godunov's scheme if and only if one of the following holds*

$$\begin{aligned} &F_j'(u_j^n) \geq 0 \textit{ for all } j \in J,\\ &F_j'(u_j^n) \leq 0 \textit{ for all } j \in J,\\ &\textit{or } u_j^n = u_* \textit{ for all } j \in J. \end{aligned} \tag{2.22}$$

Proof. Suppose that (2.22) holds for all $j \in J$. Let us first look at the case when $F_j'(u_j^n) \geq 0$ for all $j \in J$. Now

$$\operatorname{sign}(u_j^n - c) - \operatorname{sign}(u_{j-1}^n - c) = -2 \quad \text{and} \quad F_{j-1}(u_{j-1}^n) - F_{j-1}(c) \geq 0$$

for every constant c such that $u_j^n < c < u_{j-1}^n$, $j \in J$. Also

$$\operatorname{sign}(u_j^n - c) - \operatorname{sign}(u_{j-1}^n - c) = 2 \quad \text{and} \quad F_{j-1}(u_{j-1}^n) - F_{j-1}(c) \leq 0$$

for every constant c such that $u_{j-1}^n < c < u_j^n$, $j \in J$. This implies

$$\big(\operatorname{sign}(u_j^n - c) - \operatorname{sign}(u_{j-1}^n - c)\big)\big(F_{j-1}(u_{j-1}^n) - F_{j-1}(c)\big) \leq 0.$$

Taking into account (2.11) we have

$$\begin{aligned} 0 &\geq \big(\operatorname{sign}(u_j^n - c) - \operatorname{sign}(u_{j-1}^n - c)\big)\big(F_{j-1}(u_{j-1}^n) - F_{j-1}(c)\big)\\ &= \operatorname{sign}(u_j^n - c)\big(F_j(u_j^n) + F_j(c) - F_j(c) - F_{j-1}(c)\big)\\ &- \operatorname{sign}(u_{j-1}^n - c)\big(F_{j-1}(u_{j-1}^n) - F_{j-1}(c)\big)\\ &= \operatorname{sign}(u_j^n - c)\big(F_j(u_j^n) - F_j(c)\big) - \operatorname{sign}(u_{j-1}^n - c)\big(F_{j-1}(u_{j-1}^n)\\ &- F_{j-1}(c)\big) + \operatorname{sign}(u_j^n - c)\left(F_j(c) - F_{j-1}(c)\right) \end{aligned}$$

for every constant c from the interval (u_{j-1}^n, u_j^n). As stated before, the case when c is out of the interval (u_{j-1}^n, u_j^n) is trivial, since the right hand side of this inequality equals zero. Applying that $u_j^{n+1} = u_j^n$ holds for every j, n and $\lambda > 0$, one sees that (2.20) is true for every constant c.

In the case $F_j'(u_j^n) \leq 0$ one starts with

$$0 \geq \big(\operatorname{sign}(u_{j+1}^n - c) - \operatorname{sign}(u_j^n - c)\big)\big(F_{j+1}(u_{j+1}^n) - F_{j+1}(c)\big)$$

and by a similar calculation as in the case $F_j'(u_j^n) \geq 0$ the proof of (2.21) follows for every constant c.

Also if $u_j^n = u_*$ for every $j \in J$, we have

$$\operatorname{sign}(u_j^n - c) - \operatorname{sign}(u_{j-1}^n - c) = \operatorname{sign}(u_* - c) - \operatorname{sign}(u_* - c) = 0,$$

this implies

$$\big(\operatorname{sign}(u_j^n - c) - \operatorname{sign}(u_{j-1}^n - c)\big)\big(F_{j-1}(u_{j-1}^n) - F_{j-1}(c)\big) = 0.$$

Thus, (2.20) is true for every constant c. One can easily check that (2.21) also holds in this case, so by Definition 2.5, a DSP of the form $u_j^n = u_*$ is entropic.

On the other hand, let us now suppose that the obtained DSP is entropic. This means by Definition 2.5 that for any $j \in J$ the solution satisfies (2.20) for $F_j'(u_j) \geq 0$ or (2.21) for $F_j'(u_j) \leq 0$. Suppose that (2.20) holds for

$F_j'(u_j) \geq 0$ and every constant c. Using $u_j^{n+1} = u_j^n$ for very j and n, and $\lambda > 0$ we obtain

$$\begin{aligned}0 &\geq \operatorname{sign}(u_j^n - c)\big(F_j(u_j^n) - F_j(c)\big) - \operatorname{sign}(u_{j-1}^n - c)\big(F_{j-1}(u_{j-1}^n) \\ &\quad - F_{j-1}(c)\big) + \big(F_j(c) - F_{j-1}(c)\big)\operatorname{sign}(u_j^n - c) \\ &= \operatorname{sign}(u_j^n - c)\big(F_j(u_j^n) - F_{j-1}(c)\big) - \operatorname{sign}(u_{j-1}^n - c)\big(F_{j-1}(u_{j-1}^n) - F_{j-1}(c)\big).\end{aligned}$$

Taking into account (2.11), we obtain

$$\big(\operatorname{sign}(u_j^n - c) - \operatorname{sign}(u_{j-1}^n - c)\big)\big(F_{j-1}(u_{j-1}^n) - F_{j-1}(c)\big) \leq 0. \tag{2.23}$$

Suppose now that condition $F_j'(u_j^n) \geq 0$ is not satisfied for all $j \in J$. Then there exists an index $k \in J$ such that

$$F_k'(u_k^n) > 0 \quad \text{and} \quad F_{k-1}'(u_{k-1}^n) < 0$$

or

$$F_k'(u_k^n) < 0 \quad \text{and} \quad F_{k-1}'(u_{k-1}^n) > 0.$$

In both cases, when c is sufficiently close to u_{k-1}^n or u_k^n, inequality (2.23) does not hold, except in the case when $u_j^n = u_*$ for all $j \in J$.

The proof when (2.21) holds is analogous and system (2.11) has a unique solution satisfying (2.22) which is used for the construction of the unique entropic DSP for problem (2.4), (2.6). That proves the lemma. □

Similarly, for the case when $u_l > \theta_h$, we have the lemma bellow.

Lemma 2.8. *Problem* (2.4), *together with* (2.12) *has a unique entropic stationary DSP for Godunov's scheme if and only if*

$$F_j'(u_j^n) \geq 0 \text{ holds for all } j \in J. \tag{2.24}$$

Now we have the following theorem:

Theorem 2.9. *Consider Eq.* (2.4) *with the initial data* (1.2) *satisfying the above assumptions and having only shocks in a solution on the right-handed side. Godunov's scheme for solving this problem admits a unique entropic DSP whose limit is of the form*

- (a) (1.12) *for* $u_l = A$ *and* $u_r \in (\bar{\bar{B}}, B)$ *in the following cases:*
  - (i) *The flux crossing is regular or there is no crossing and* $h(\theta_h) > g(\theta_g)$. *The entropic connection is given by* $(A, B) = (\theta_h, \bar{\theta}_h)$, *where* $g(\bar{\theta}_h) = h(\theta_h)$, $\bar{\theta}_h > \theta_g$.
  - (ii) *The flux crossing is undercompressive. In that case* $A = B = u_*$ *defines an entropy connection.*
- (b) (1.13) *if* $h'(u_l) > 0$ *and* $u_r \in (\bar{\bar{u}}_l, \bar{u}_l)$ *in the following cases:*
  - (i) *There is a regular or no flux crossing, and* $h(\theta_h) > g(\theta_g)$.
  - (ii) *There is a regular or no flux crossing,* $h(\theta_h) < g(\theta_g)$ *and* $h(u_l) > g(\theta_g)$.
  - (iii) *The flux crossing is undercompressive and* $h(u_l) > h(u_*)$.
  - (iv) *The flux crossing is overcompressive and* $u_l > u_*$.

Remark 2.10. Consider case (a) of Theorem 2.9 and note that Godunov's scheme for conservation laws with spatially varying flux functions can also be used when $u_r > B$ for the same connections as given in the sub cases (i)-(ii). The limit of the obtained discrete solution contains then of a steady shock at $x = 0$ and a rarefaction wave at the right hand side. The same goes in case (b) for $u_r > \bar{u}_l$ under the conditions listed in sub cases (i)-(iv). But, since we are interested only in DSPs herein, we omit these cases.

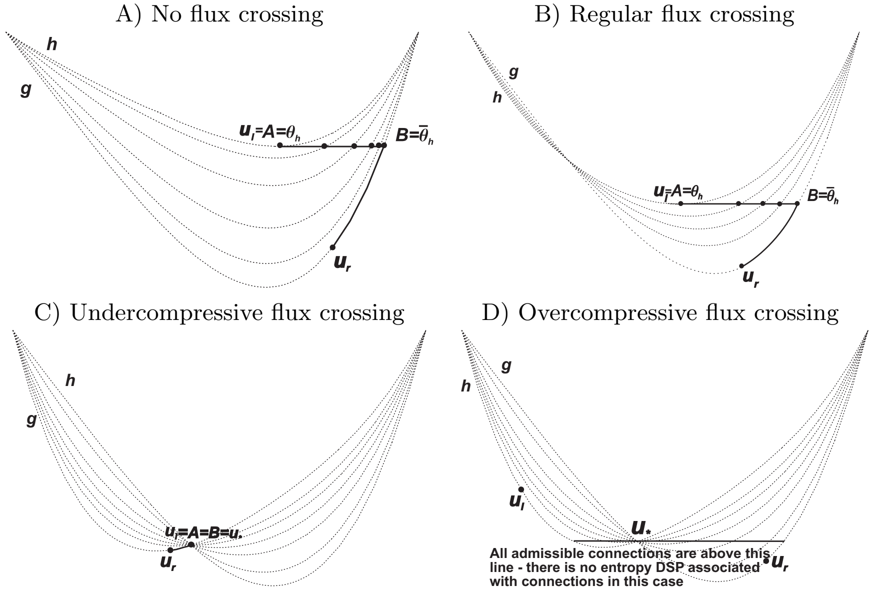

FIGURE 3. The structure of entropy DSPs associated with connections for various cases of flux crossings and $u_l = A$

Proof. (a)(i) Let $g(\theta_g) < h(\theta_h)$. There are three possible cases:

(i.1) $h'(u_*) < 0$, $g'(u_*) < 0$,
(i.2) $h'(u_*) \geq 0$, $g'(u_*) \geq 0$,
(i.3) there is no intersection.

(i.1) In this case we have to prove that the line of the connection $l := [(\theta_h, h(\theta_h)), (\bar{\theta}_h, g(\bar{\theta}_h))]$ intersects the curve $\omega := (u, F_\varepsilon(x, u))$ in a point $u_{x,\varepsilon}$ such that $F'(x, u_{x,\varepsilon}) \geq 0$ for every $x \in [-\varepsilon, \varepsilon]$. It is clear that ω is non-decreasing at the interval $(\theta_{x,\varepsilon}, Y)$, where $\theta_{x,\varepsilon}$ represents the minimum of $F_\varepsilon(x, u)$ for a fixed $x \in [-\varepsilon, \varepsilon]$. Here (2.2) implies $h(\theta_h) > F_\varepsilon(x, \theta_{x,\varepsilon}) > g(\theta_g)$. This implies $F'(x, u_{x,\varepsilon}) \geq 0$ for all $x \in [-\varepsilon, \varepsilon]$. Now, let us suppose that the entropy connection $(A, B) = (\theta_h, \bar{\theta}_h)$ is not unique. Then there exists a connection (A_1, B_1), $A_1 \neq A$, $B_1 \neq B$ for which the line $l_1 := [(A_1, h(A_1)), (B_1, g(B_1))]$ intersects the curve ω in a point $\hat{u}_{x,\varepsilon}$ such

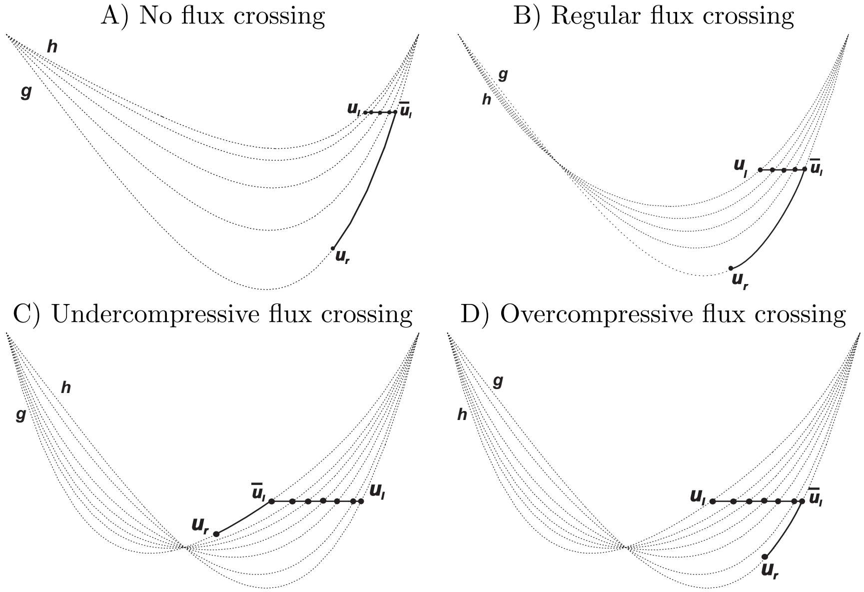

FIGURE 4. The structure of entropy DSPs for various cases of flux crossings for $h'(u_l) > 0$

that $F'(x, \hat{u}_{x,\varepsilon}) \geq 0$ for every $x \in [-\varepsilon, \varepsilon]$. This contradicts the fact that $h'(A_1) \leq 0$, $g'(B_1) \geq 0$ and that $A = \theta_h$ is the argument of unique minimum of the flux h. Hence, the uniqueness of the entropy connection follows. The cases (i.2) and (i.3) can be proved in a similar way.

(ii) The fact $F(x, u_*) = F(y, u_*)$ for all $x, y \in [-\varepsilon, \varepsilon]$ implies $F_j(u_*) = F_i(u_*)$, for all $i, j \in J$ which gives $u_j^n = u_*$ for all $j \in J$, so (2.22) holds. This time, the uniqeness of the entropy connection follows from the fact that $A = B = u_*$ is the only admissible connection. The structure of entropy DSPs for case (a) can be seen in Fig. 3.

(b)(i)Let us suppose that the fluxes have no crossing or have a regular crossing at (X, Y) and $h(\theta_h) > g(\theta_g)$. Using a similar notation as above, i.e. $\omega = (u, F_\varepsilon(x, u))$, $\theta_{x,\varepsilon}$ is its unique minimum at a fixed $x \in [-\varepsilon, \varepsilon]$ and $l := [(u_l, h(u_l)), (\bar{u}_l, g(\bar{u}_l))]$, we prove that ω intersects l at a point $u_{x,\varepsilon}$ such that $F'_\varepsilon(x, u_{x,\varepsilon}) \geq 0$ for every $x \in [-\varepsilon, \varepsilon]$. From (2.2) it is clear that $h(\theta_h) > F_\varepsilon(x, \theta_{x,\varepsilon}) > g(\theta_g)$. This implies that for every $x \in [-\varepsilon, \varepsilon]$ the point $(\theta_{x,\varepsilon}, F_\varepsilon(x, \theta_{x,\varepsilon}))$ is below the line l. Taking into account that $h'(u_l) > 0$, we can conclude that for every $x \in [-\varepsilon, \varepsilon]$ the curve ω intersects l at a point $u_{x,\varepsilon}$ such that $F'_\varepsilon(x, u_{x,\varepsilon}) \geq 0$. Suppose now that this entropy DSP is not unique. Then there exist two points $A, B \in I$ such that $A \neq u_l$, $B \neq \bar{u}_l$ and $h(A) = g(B)$. The entropy condition (2.22) implies $h'(A), g'(B) \geq 0$ which produces a wave (u_l, A) of positive speed that interacts with the steady one (A, B), which is not allowed. Hence, the uniqueness follows.

(ii) In this case (2.2) gives $F_\varepsilon(x,\theta_{x,\varepsilon}) < g(\theta_g)$. Again, taking into account that $h'(u_l) > 0$, we can conclude that the assertion follows. The uniqueness can be proved analogously as in (i).

Let us define a ponit $\bar{u}_* \in I$ such that $h(\bar{u}_*) = h(u_*)$, $h'(u_*) > 0$ in the case when u_* is an undercompressive flux crossing, and $g(\bar{u}_*) = g(u_*)$, $g'(u_*) > 0$ in the overcompressive case. Using the fact that in the case of an over- or undercompressive flux crossing, for every $x \in [-\varepsilon, \varepsilon]$ the point $(\theta_{x,\varepsilon}, F_\varepsilon(x,\theta_{x,\varepsilon}))$ is below the line $l := [(u_*, h(u_*)), (\bar{u}_*, h(\bar{u}_*))]$, (iii) and (iv) can be proved very similarly as in the previous cases. The structure of entropy DSPs for $h'(u_l) > 0$ for various cases of flux crossing discussed above is shown in Fig. 4. □

Finally, according to Definition 2.6, it just remains to let $\varepsilon \to 0$ to recover entropic shock wave solutions for Eq. (1.1).

2.1. A special, "multiplicative" case

Let us now consider problem (1.3), (1.2). One can easily see that this problem is a special case of problem (1.1), (1.2) where $h(u)$ is substituted by $a_l f(u)$ and $g(u)$ by $a_r f(u)$ for some f having one minimum and no maximum at the interior of I and satisfying $f(X) = f(Y)$. In this case, the appropriate perturbed equation is of the form

$$u_t + (a_\varepsilon(x) f(u))_x = 0,$$

where

$$a_\varepsilon(x) = a(x) \quad \text{for} \quad |x| \geq \varepsilon$$

is a continuously differentiable approximation of $a(x)$. It is easy to check that $a_\varepsilon f(u)$ is an admissible regularization of $f(u)$ in the sense of (2.2) if $a_\varepsilon(x)$ is monotone. So, we make this assumption here. In addition, we take that $a_l > a_r > 0$. In this case, the fluxes $a_l f$ and $a_r f$ achieve their unique minimum at the same point, denoted with θ. In other words, $\theta = \theta_h = \theta_g$. This is the case when the fluxes $h = a_l f$ and $g = a_r f$ do not cross, so according to the above theory, there exists a unique connection $(A, B) = (\theta, \bar{\theta})$, where $a_r f(\bar{\theta}) = a_l f(\theta)$, $f'(\bar{\theta}) > 0$, satisfying (2.22), i.e.

$$f'(u_j^n) \geq 0 \quad \forall j \in J \tag{2.25}$$

building a steady and a right going DSP, if $u_l = A$ and $u_r \in (\bar{\bar{B}}, B)$, where $\bar{\bar{B}}$ is given by (1.11). If $u_l > \theta$, we have a solution of the form (1.13). Note that (2.25) is equivalent to

$$\operatorname{sign}(u_{j-1}^n - \theta) \operatorname{sign}(u_j^n - \theta) \geq 0 \quad \forall j \in \{-j^* + 1, ..., j^*\}.$$

According to Lemma 2.7, the steady DSP satisfies the following Kruzhkov-type inequality

$$|u_j^{n+1} - c| - |u_j^n - c| + \lambda\big(a_j \operatorname{sign}(u_j^n - c)(f(u_j^n) - f(c))$$
$$-a_{j-1} \operatorname{sign}(u_{j-1}^n - c)(f(u_{j-1}^n) - f(c))\big) + \lambda \operatorname{sign}(u_j^n - c)(a_j - a_{j-1}) f(c) \leq 0,$$

for every constant c, where a_j present the discrete values of $a_\varepsilon(x)$. Again, the travelling DSP is entropic due to Oleinik's entropy condition.

3. Conclusions and further research

In this final section, we will review the research contributions of this paper and discuss the directions for future research.

3.1. Contributions of our research

The acceptability of shock waves as solutions of conservation laws is not uniformly resolved. Besides entropy pair conditions (Lax, Liu, Oleinik), there are viscous entropy conditions requiring that an entropy solution presents a limit of a viscous profile. The main contribution of our research is the use of DSPs instead of viscous profiles. Using a discrete version of Kruzkhov's entropy inequality for a perturbed version of Eq. (1.1) we determine appropriate entropy conditions for DSPs. For different types of flux crossings, we have obtained unique entropic DSPs by using Godunov's method for spatially varying flux. Our entropy condition (2.22) is obtained from the request for the steady DSP to be entropic, so although we consider only steady DSPs combined with a right going DSP, our theory can also be applied for combining steady DSPs with discrete rarefaction waves. For example, such a solution occurs in the case when the pair (B, u_r) or $(\bar{u}_l, u_r)$ does not satisfy Oleinik's condition. This means that we can apply our solution concept to a wider range of initial data than we observed herein.

3.2. Further investigations

For simplicity, herein we do not consider (1.1), (1.2) in the so-called convex-concave case, i.e. when one of the fluxes g or h has one minimum and no maximum, and the other has one maximum and no minimum. In our forthcoming research, we shall consider the above presented theory in the context of the convex-concave case. Furthermore, it is known that in the convex-concave type flux case, there are certain choices of initial data for which there is no weak solution satisfying the Rankine-Hugoniot condition at $x = 0$. The same problem occurs in the case when the fluxes do not cross in the domain of definition. In such cases the only meaningful solutions appear in the form of generalized solutions satisfying a weaker formulation of the Rankine-Huginot condition. Our aim is to extend the solution concept presented herein to these cases, too.

References

[1] A. Adimurthi, J. Jaffre, G. D. V. Gowda, *Godunov-type methods for conservation laws with a flux function discontinuous in space*, SIAM J. Numer. Anal. **42** (2004) 179–208.

[2] Bernard S., Colombeau J. F., *Conservation laws with discontinuous coeffitients,* Journal of mathematical analysis and applications, **258**, (2001), 63–86.

[3] A. Adimurthi, S. Mishra, G. D. V. Gowda, *Optimal entropy solutions for conservation laws with discontinuous flux-functions,* J. Hyperbolic Differ. Equ. **2** (2005) no.4 783–837.

[4] D. Bale, R. J. LeVeque, S. Mitran, J. A. Rossmanith, *A wave propagation method for conservation laws and balance laws with spatially varying flux function,* SIAM J.Sci. Comput. **24** (2002) no.3 955–978.

[5] R. Bürger, K. H. Karlsen, J. D. Towers, *A model of continuous sedimentation of flocculated suspensions in clarifier-thickener units,* SIAM J.Appl. Math. **vol 65** (2005) no.3 882–940.

[6] R. Bürger, K. H. Karlsen, J. D. Towers, *An Engquist-Osher-type scheme for conservation laws with discontinuous flux adapted to flux connections,* SIAM J. Numer. Anal. **47**(2009), no.3, 1684–1712.

[7] R. Bürger, K. H. Karlsen, N. H. Risebro, J. D. Towers, *Well-posedness in* BV_t *and convergence of a difference scheme for continuous sedimentation in ideal clarifier-thickener units,* Numer. Math. **97** (2004) 25–65.

[8] R. Bürger, K. H. Karlsen, S. Mishra, J.D. Towers, *On conservation laws with discontinuous flux,* in Trends in applications of mathematics to mechanics, Ber. Math., Shaker Verlag, Aachen, (2005) pp. 75–84.

[9] H. Fan, *Existence of discrete shock profiles of a class of monotonicity preserving schemes for conservation laws,* Math. Comp. **70** (2000) no. 235, 1043–1069.

[10] K. H. Karlsen, N. H. Risebro, J. D. Towers, *On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient,* Electron. J. Differential Equations, **93** (2002), 23 pp. (electronic).

[11] K. H. Karlsen, N. H. Risebro, J. D. Towers, *Upwind difference approximations for degenerate parabolic convection-diffusion equations with a discontinuous coefficient,* IMA J. Numer. Anal.,**22** (2002), no.4, 623–664.

[12] R. J. LeVeque, *Finite volume methods for hyperbolic problems,* Cambridge Texts in Applied Mathematics, Cambridge, United Kingdom, (2004).

[13] S. Mishra, *Analysis and numerical approximation of conservation laws with discontinuous coefficients,* PH.D. thesis, Indian Institute of Science Bangalore, (2005).

[14] D. Serre, *Discrete shock profiles: existence and stability,* in *Hyperbolic systems of balance laws*, 79–158, Lecture Notes in Math., 1911, Springer, Berlin, (2007).

[15] D. Serre, L^1*-stability of nonlinear waves in scalar conservation laws,* in *Evolutionary equations* vol. I, 473–553, Handb. Differ. Equ., North-Holland, Amsterdam, (2004).

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