# A note on airplane landing problem 

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#### Abstract

We study a motion of an airplane landing on a straight line and stretching a weightless viscoelastic fiber whose ends are anchored at points a given distance from the line. The constitutive model of the viscoelastic fiber comprises fractional derivatives of stress and strain and the restrictions on the coefficients that follow from Clausius Duhem inequality. We show that the dynamics of the problem is governed by a single integral equation involving Mittag-Leffler-type function. The existence of the solution was ensured by the Contraction Mapping Principle. The influence of the four constants describing the fiber properties on the landing track was examined by use of the first-order fractional difference approximation.


## 1 Introduction

The new tendency in engineering favors the design of slender structures incorporating new high performance materials. The use of such structures requires thorough knowledge of their physical properties, especially the study of viscoelastic response. This raises the problem of coupling geometric nonlinearity with either linear or nonlinear constitutive equations. According to Bagley [1] it seems that a generalized linear model of a viscoelastic body that contains fractional derivatives of stress and strain is capable of describing viscoelastic behavior of real materials in a more accurate way then nonlinear constitutive models with derivatives of integer order. Thus our intention here is to explore finite deformations coupled with so called standard fractional viscoelastic body. As stated by Seredyńska and Hanyga, papers on nonlinear fractional differential equations are rare, see [2], where studies of nonlinear pendulum and the Duffing equation, with a Caputo fractional derivative term replacing the usual damping, show striking differences between ordinary differential and fractional differential equations. In what follows we shall add one more example to the list of nonlinear fractional differential equations.

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## 2 The problem

We shall study a motion of an airplane landing on a straight line and stretching a weightless viscoelastic fiber whose ends are anchored at points a given distance from the line, see Fig. 1. Roughly speaking the landing script could be as follows. At the time $t=0$ the airplane of mass $m$, with velocity $v_{0}$, touches the flight deck and at the same moment, the weightless viscoelastic fiber, of length $2 h$, which was perpendicular to the line of landing. The stretching of the fibre will proceed until the airplane slow dawn. When its velocity become almost zero, say at $t=\bar{t}$, the airplane releases the fibre. Afterwords it will stop by use of the classical brake in neglected time.


Fig. 1. System under consideration.

The differential equation of motion of an airplane and the initial conditions read

$$
\begin{equation*}
m \xi^{(2)}=-2 f \sin \varphi, \quad \xi^{(1)}(0)=v_{0}, \quad \xi(0)=0, \quad f(0)=0, \tag{1}
\end{equation*}
$$

where we used $(\cdot)^{(k)}=d^{k}(\cdot) / d t^{k}$ to denote the $k$-th derivative with respect to time $t$, and where $\xi=\xi(t)$ and $f=f(t)$ stand for the coordinate and the contact force between the airplane and the fibre. It should be noted that large values of $\xi$ and $\varphi$ (the angle describing the fibre deformation) are allowed. Introducing the half measure of the isothermal uniaxial deformation of the fibre $x=x(t)$ we may take the relation between $f=f(t)$ and $x=x(t)$ (constitutive equation of the deformable fibre) in the following form

$$
\begin{equation*}
f+\tau_{f \alpha} f^{(\alpha)}=\frac{E_{\alpha} A}{h}\left(x+\tau_{x \alpha} x^{(\alpha)}\right), \tag{2}
\end{equation*}
$$

where $0<\alpha \leq 1, A$ is the area of the fibre cross-section, $E_{\alpha}$ is the modulus of elasticity, $\tau_{f \alpha}$ and $\tau_{x \alpha}$ are the constants of dimension [time] ${ }^{\alpha}$. In (2), for $0<\alpha<1$, we use
$(\cdot)^{(\alpha)}$ to denote the $\alpha-$ th derivative of a function $(\cdot)$ taken in the Riemann-Liouville form as $d^{\alpha}[g(t)] / d t^{\alpha}=g^{(\alpha)}=d\left[\Gamma^{-1}(1-\alpha) \int_{0}^{t} g(\xi)(t-\xi)^{-\alpha} d \xi\right] / d t$, where $\Gamma$ denotes the Euler Gamma function. In the special case when $\alpha=1$ equation (2) represents the standard model of linear viscoelastic solid with $\tau_{f 1}$ and $\tau_{x 1}$ known as the relaxation times. Note that there exists fundamental restrictions on the coefficients of the model, that follow from the second law of thermodynamics

$$
\begin{equation*}
E_{\alpha}>0, \tau_{f \alpha}>0, \tau_{x \alpha}>\tau_{f \alpha} \tag{3}
\end{equation*}
$$

see [3]. We assume that $x(0)=0$. In the following we shall use the obvious geometrical relations $\sin \varphi=\xi /(h+x)$ and $\xi^{2}+h^{2}=(x+h)^{2}$. Introducing the dimensionless quantities $\bar{\xi}=\xi / h$, $\bar{x}=x / h, \bar{t}=t\left[2 E_{\alpha} A /(m h)\right]^{1 / 2}, \bar{\tau}_{x \alpha}=\tau_{x \alpha}\left[2 E_{\alpha} A /(m h)\right]^{\alpha / 2}, \bar{\tau}_{f \alpha}=\tau_{f \alpha}\left[2 E_{\alpha} A /(m h)\right]^{\alpha / 2}$, $\bar{f}=f / E_{\alpha} A$ and $\lambda=v_{0}\left[m /\left(2 E_{\alpha} A h\right)\right]^{1 / 2}$, one gets the following system describing the motion of the airplane at the landing phase

$$
\begin{gather*}
\xi^{(2)}=-f \frac{\xi}{\sqrt{1+\xi^{2}}}, \quad \xi^{(1)}(0)=\lambda, \quad \xi(0)=0, \quad f(0)=0,  \tag{4}\\
f+\tau_{f \alpha} f^{(\alpha)}=x+\tau_{x \alpha} x^{(\alpha)}, \tag{5}
\end{gather*}
$$

with

$$
\begin{equation*}
\xi^{2}+1=(1+x)^{2} \tag{6}
\end{equation*}
$$

In equations (4) - (6) the bar was omitted and the derivatives are taken with respect to dimensionless time. Also, the restrictions (3) $)_{2,3}$ in dimensionless form remain the same.

Our main concern is solution of (4) - (6). Before we proceed to it we make two remarks here. First, by differentiating (6) twice, variable $\xi$ could be eliminated, i.e., eq. (5) is to be solved together with nonlinear equation

$$
\begin{equation*}
x^{(2)}-\frac{\left[x^{(1)}\right]^{2}}{x(1+x)(2+x)}+f x \frac{(2+x)}{(1+x)^{2}}=0, \tag{7}
\end{equation*}
$$

with

$$
\begin{equation*}
x(0)=0, \quad x^{(1)}(0)=0, \quad\left(x^{(2)}(0)=\lambda^{2} \neq 0\right), \quad \text { and } \quad f(0)=0, \tag{8}
\end{equation*}
$$

but this form of the problem is not tractable enough. Secondly, we believe that the constitutive equation (5), so called the modified Zener model, is good enough to describe viscoelastic behaviour for wide class of real materials, metals, geological strata, glass, polymers for vibration control, even human root dentin, see [4] and [5] for example. When dealing with it, a special attention should be paid to thermodynamical restrictions that should be observed in determining parameters of the model from experimental results. However in some problems, despite the fact it violates thermodynamical constraint $\tau_{f \alpha}>0$, the term $\tau_{f \alpha}$ is small enough and could be neglected, for example see [6] where $\tau_{f \alpha}$ reads $0.69 \times 10^{-9} \sec ^{0.49}$. In such cases the problem (5), (7) reduces to single nonlinear fractional differential equation

$$
\begin{equation*}
x^{(2)}-\frac{\left[x^{(1)}\right]^{2}}{x(1+x)(2+x)}+\frac{\left(x^{2}+\tau_{x x} x x^{(\alpha)}\right)(2+x)}{(1+x)^{2}}=0, \tag{9}
\end{equation*}
$$

with initial conditions (8).

## 3 The solution

In order to solve the landing problem we shall apply the Laplace transform method. We shall show that the dynamics of the problem is governed by a single integral equation involving Mittag-Leffler-type function, whose solution is ensured by the Contraction Mapping Principle. Introducing $X=X(s)=\mathcal{L}\{x(t)\}=\int_{0}^{\infty} e^{-s t} x(t) d t$ and $F=F(s)=\mathcal{L}\{f(t)\}=\int_{0}^{\infty} e^{-s t} f(t) d t$, from (5) we get

$$
\begin{equation*}
F=\frac{1+\tau_{x \alpha} s^{\alpha}}{1+\tau_{f \alpha} s^{\alpha}} X \tag{10}
\end{equation*}
$$

where we have used the standard expression for the Laplace transform of $z^{(\alpha)}$, given as $\mathcal{L}\left\{z^{(\alpha)}\right\}=$ $s^{\alpha} Z-\left[\left(\int_{0}^{t} z(\xi) d \xi /(t-\xi)^{\alpha}\right)\right]_{t=0}$, where $\mathcal{L}\{z(t)\}=Z=Z(s)$ and the term in brackets vanishes if $\lim _{t \rightarrow 0^{+}} z(t)$ is bounded, see [7]. The inversion of (10) yields the following relation between $f(t)$ and $x(t)$

$$
\begin{equation*}
f(t)=\frac{\tau_{x \alpha}}{\tau_{f \alpha}} x(t)+\frac{1}{\tau_{f \alpha}}\left(1-\frac{\tau_{x \alpha}}{\tau_{f \alpha}}\right) \int_{0}^{t} e_{\alpha, \alpha}\left(t-\xi, \frac{1}{\tau_{f \alpha}}\right) x(\xi) d \xi, \tag{11}
\end{equation*}
$$

where $e_{\alpha, \beta}(t ; \lambda)$ stands for the generalized Mittag-Leffler function $e_{\alpha, \beta}(t ; \lambda) \equiv E_{\alpha, \beta}\left(-\lambda t^{\alpha}\right) / t^{1-\beta}$ with $E_{\alpha, \beta}(t)=\sum_{n=0}^{\infty} t^{n} / \Gamma(\alpha n+\beta)$, see [8].

Next we use the geometrical relation (6). Namely, substituting $x=\sqrt{\xi^{2}+1}-1$ in (11) and the obtained function $f(t)$ into (4), the landing problem reduces to the following initial data problem

$$
\begin{align*}
& \xi^{(2)}=-\frac{\xi}{\sqrt{1+\xi^{2}}}\left\{\frac{\tau_{x \alpha}}{\tau_{f \alpha}}\left(\sqrt{1+\xi^{2}}-1\right)+\frac{1}{\tau_{f \alpha}}\left(1-\frac{\tau_{x \alpha}}{\tau_{f \alpha}}\right) \times\right. \\
& \left.\int_{0}^{t} e_{\alpha, \alpha}\left(t-\rho, \frac{1}{\tau_{f \alpha}}\right)\left[\sqrt{1+\xi^{2}(\rho)}-1\right] d \rho\right\}, \quad \xi(0)=0, \quad \xi^{(1)}(0)=\lambda \tag{12}
\end{align*}
$$

Thus, $\xi$ has to satisfy the following integral equation

$$
\begin{align*}
& \xi(t)= \int_{0}^{t}\{\lambda- \\
& \int_{0}^{s} \frac{\xi(u)}{1+\xi^{2}(u)}\left[\frac{\tau_{x \alpha}}{\tau_{f \alpha}}\left(\sqrt{1+\xi^{2}(u)}-1\right)+\frac{1}{\tau_{f \alpha}}\left(1-\frac{\tau_{x \alpha}}{\tau_{f \alpha}}\right) \times\right.  \tag{13}\\
&\left.\left.\int_{0}^{u} e_{\alpha, \alpha}\left(u-\rho, \frac{1}{\tau_{f \alpha}}\right)\left(\sqrt{1+\xi^{2}(\rho)}-1\right) d \rho\right] d u\right\} d s=\mathcal{M}(\xi(t)) .
\end{align*}
$$

Now, we are in the position to use the fixed point theorem argument. Let the $\xi$ be in Banach space $C^{1}((0, T))$, equipped with sup-norm, for some $T>0$. In the following, the sign $\|\cdot\|$ will be
used instead of $\|\cdot\|_{L^{\infty}((0, T))}$. Let us denote by $B$ the unit ball with center in $\left(\xi(0), \xi^{(1)}(0)\right)=(0, \lambda)$ in $C^{1}((0, T))$, and $B_{T}=B \times[0, T]$. We have

$$
\begin{equation*}
|\xi(t)-(0, \lambda)| \leq\left|\int_{0}^{t} \frac{\tau_{x \alpha}}{\tau_{f \alpha}}+\frac{1}{\tau_{f \alpha}}\right| \int_{0}^{T} e_{\alpha, \alpha}\left(s-\rho, \frac{1}{\tau_{f \alpha}}\right) d \rho|\xi(s) d s| \tag{14}
\end{equation*}
$$

By taking the supremum of (14) over the interval $[0, T]$, one can see that $\mathcal{M}(\xi) \in B_{T}$, if $\xi \in B_{T}$ for $T$ small enough. The first derivative can be estimated in the same way. Again, for $T_{0}<T$ small enough, one can see that $\mathcal{M}$ is a contractive mapping. Thus there exist a local solution (in the interval $\left[0, T_{0}\right]$ ). Let us now show the uniqueness of such a solution. Suppose that $\xi$ and $\xi_{1}$ are two solutions. Then

$$
\begin{equation*}
\left|\xi(t)-\xi_{1}(t)\right| \leq\left|\int_{0}^{t} \frac{\tau_{x \alpha}}{\tau_{f \alpha}}+\frac{1}{\tau_{f \alpha}}\right| \int_{0}^{T} e_{\alpha, \alpha}\left(s-\rho, \frac{1}{\tau_{f \alpha}}\right) d \rho| | \xi(s)-\xi_{1}(s)|d s| . \tag{15}
\end{equation*}
$$

Gronwall inequality then implies that $\left|\xi(t)-\xi_{1}(t)\right| \leq 0$, i.e. $\xi \equiv \xi_{1}$. It could be shown that the above solution is the global one, because $T_{0}$ does not depend on the initial data.

Finally by use of the first-order fractional difference approximation, we examine the influence of the four constants describing the fiber properties on the landing track. Introducing the time step $h,\left(t_{m}=m h, m=1,2, \ldots\right)$, we shall use standard difference approximations for the first and second derivatives. The fractional derivative $z_{m}^{(\alpha)}$ we take in the form $h^{-\alpha} \sum_{j=0}^{m} \omega_{j, \alpha} z_{m-j}$, with $\omega_{j, \alpha}$ calculated by the recurrence relationships $\omega_{0, \alpha}=1$ and $\omega_{j, \alpha}=(1-(\alpha+j) / j) \omega_{j-1, \alpha}, \quad(j=1,2,3, .$.$) , see [9]. Then, taking into account the geometri-$ cal relation (6), the discretization of (4) and (5) read

$$
\begin{align*}
& \xi_{m}=2 \xi_{m-1}-\xi_{m-2}-h^{2} f_{m-1} \xi_{m-1}\left(1+\xi_{m-1}\right)^{-1 / 2}, \quad m=3,4, \ldots  \tag{16}\\
& f_{m}=\left(\sqrt{\xi_{m}^{2}+1}-1\right) \frac{h^{\alpha}+\tau_{x \alpha}}{h^{\alpha}+\tau_{f \alpha}}+\frac{1}{h^{\alpha}+\tau_{f \alpha}} \sum_{j=1}^{m} \omega_{j, \alpha}\left[\tau_{x \alpha}\left(\sqrt{\xi_{m-j}^{2}+1}-1\right)-\tau_{f \alpha} f_{m-j}\right],
\end{align*}
$$

with $\xi_{0}=0, f_{0}=0$, and $\xi_{1}, \xi_{2}$ given as a solution of

$$
\begin{equation*}
\xi_{2}-2 \xi_{1}+h^{2} \frac{\xi_{1}}{\sqrt{1+\xi_{1}^{2}}} \frac{h^{\alpha}+\tau_{x \alpha}}{h^{\alpha}+\tau_{f \alpha}}\left(\sqrt{\xi_{1}^{2}+1}-1\right)=0, \quad-\xi_{2}+4 \xi_{1}-2 \lambda h=0 \tag{17}
\end{equation*}
$$

and with $f_{1}=\left(\sqrt{\xi_{1}^{2}+1}-1\right)\left(h^{\alpha}+\tau_{x \alpha}\right)\left(h^{\alpha}+\tau_{f \alpha}\right)^{-1}$ and $f_{2}=\left(\sqrt{\xi_{2}^{2}+1}-1\right)\left(h^{\alpha}+\tau_{x \alpha}\right) \times$ $\left(h^{\alpha}+\tau_{f \alpha}\right)^{-1}+\omega_{1, \alpha}\left[\tau_{x \alpha}\left(\sqrt{\xi_{1}^{2}+1}-1\right)-\tau_{f \alpha} f_{1}\right]\left(h^{\alpha}+\tau_{f \alpha}\right)^{-1}$.

## 4 Results

In order to illustrate the above results, we are going to present the numerical solution for the values $\alpha, E_{\alpha}, \tau_{f \alpha}$ and $\tau_{x \alpha}$ motivated by the paper of Fenander [6], where the railpad models
were investigated. Namely, for $\alpha=0.23$, and dimensionless values $\tau_{f \alpha}=0.004, \tau_{x \alpha}=1.183$, $\lambda=1$, the motion of the system is presented in Fig. 2.


Fig. 2. Motion of the airplane for $\alpha=0.23, \tau_{f \alpha}=0.004, \tau_{x \alpha}=1.183$ and $\lambda=1$.

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