SHADOW WAVES – ENTROPIES AND INTERACTIONS FOR DELTA AND SINGULAR SHOCKS

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Abstract. The paper deal with two key problems for delta (or singular) shock solutions to systems of conservation laws: entropy admissibility conditions for them (which is connected to the notorious uniqueness problem) and their interaction. A way we chose is to represent them by nets of piecewise constant (or constant with respect only to the space variable) functions, here called shadow waves. All the calculations then can be done on each element of such net using the usual Rankine-Hugoniot conditions only. The 3×3 pressureless gas dynamics model is the main example in the paper.

1. Introduction

Consider a following conservation law system

$$\partial_t f(U) + \partial_x g(U) = 0,\ U : \mathbb{R}^2_+ \to \Omega \subset \mathbb{R}^n, \tag{1.1}$$

where $f = (f^1, \ldots, f^n)$ and $g = (g^1, \ldots, g^n)$ are continuous mapping from Ω in $\mathbb{R}^n$. A name of f is *density function*, while g is called *flux function*. The functions f and g are continuous mappings from a physical domain $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$.

In a number of papers (some of them will be cited bellow) one can find examples of such systems having non-classical (singular) solutions for Riemann problem. The majority of them are delta (see [1], [2], [4], [10], [11], [13], etc) and singular shock solutions (see [12], [16], etc). These two solution types are the major objects in the paper. In the sense that we incorporate them in a new solution type – a shadow wave. Solutions concepts in these examples are different: measure theoretical method ([1], [2], [4], [11]), generalized variational principle (and measures [10]), use of smooth functions nets and weighted measure spaces ([12]), split delta functions ([15], [18]), Colombeau generalized functions ([16]), weak asymptotic method ([8], [9]),... Some of these concepts are limited by the form of a flux function, which has to be linear in one solution component (measure theoretical and split delta function methods, among others), or have some other special form as in weak asymptotic method. The others have a non-uniqueness problem in a way that there is a lot of possible choices for solution which cannot be collected in the same class (that is the case for Colombeau generalized functions). Entropy and entropy-flux functions for such systems usually do not have properties as nice as density and flux functions, and that could cause checking the usual Lax entropy condition to be impossible. That is one of the reasons why the authors of the above papers usually take some other admissibility criteria as overcompressibility. In fact, that condition is used in all the papers cited above. In most of them it is the only condition. Besides the uniqueness problem there is also a problem of treating an interaction problem which involves singular solutions. The result of such an interaction depends on a given system and chosen solution concept. For example, one can find some results in [8], [17], [18] and [23].

We shall try to overcome such problems by introducing so called *shadow wave solution* to systems of conservation laws in the present paper. A definition of a shadow wave is made to be as simple and robust as possible in order to get a chance to obtain a sort of uniqueness. Also, the definition is made to include delta and singular shocks as special cases as already said. Roughly speaking, we perturb a speed c of a wave from both sides by some small parameter ε so that the states U_0 and U_1 of a solution candidate U_ε are connected by three shocks. Two of them have perturbed speeds and the third one, in the middle, has a speed c. The equality is taken to be equality of distributional limit. The intermediate values, $U_{1,\varepsilon}$ on the left- and $U_{2,\varepsilon}$ on the right-hand side of the shock front, can tend to infinity as $\varepsilon \to 0$. Here, we use the following types of shadow waves.

- *Constant shadow wave* has constant $U_{1,\varepsilon}$ and $U_{1,\varepsilon}$ for each ε. If its speed is constant, it is called *simple*.
- *Weighted shadow wave* has $U_{1,\varepsilon}$ and $U_{1,\varepsilon}$ depending on t.

Thus, U_ε is still a piecewise constant function for each ε (or a piecewise constant for each t in the case of the weighted shadow wave) and one can use the usual Rankine-Hugoniot conditions and see whether the distributional limit equals zero as $\varepsilon \to 0$. Also, the usual entropy inequality can be easily checked regardless of the form of entropy and entropy-flux functions. The next advantage is a simplicity of treating an interaction problem involving a shadow wave (thus delta or singular shock, since they are included in the shadow wave definition as already said above). Let us give more detailed explanation of these advantages.

The starting point is that all delta and singular shock solutions in physical examples found in the references can be simply transferred into the shadow waves defined as above. One can then treat them is a uniform way with straightforward and simple calculations. The obtained results agree completely to the ones given in the original literature (see the references with their descriptions above). Note that some theoretical examples from [16] cannot be directly rewritten in the terms of shadow waves: A complete analysis for systems in such general form described in that paper is possible but it contains a lot of variants to be investigated and avoided here. One can find a brief description of a way to deal with such systems in Appendix.

It is possible to solve interaction problem involving one or two shadow waves in a pretty general form (see Theorem 7.1 bellow). Again the results of interactions recover the ones given in [18] for delta shock and in [17] for singular shock interactions.

Contrary to the papers cited above, an entropy admissibility condition can now be easily checked provided that there exists an entropy–entropy flux pair. That is of great importance when dealing with systems permitting delta or singular shocks: We are now in position to define a kind of uniqueness, so called *weak uniqueness* for shadow waves and all solutions containing it and other elementary waves (shocks, rarefactions, contact discontinuities and vacuum). Roughly speaking weak uniqueness means that all entropy shadow wave solutions to system (1.1) have a same distributional limit.

Even more, we proved that the overcompressibility is really equivalent to entropy conditions satisfied by all (weakly) convex entropies in a lot of cases from literature (see again the list of delta and singular shock papers above).

The definition of shadow waves resembles the wave front tracking procedure (see [3]) in a way that delta (or singular) shock is substituted by a fan of shocks depending on some small positive parameter ε which itself do not satisfy the Rankine-Hugoniot condition. A substitution of such a fan into the given system gives zero as a limit as ε goes to zero. That approximation procedure makes the interaction problem more treatable than before.

Let us mention that the perturbed speeds appeared in the papers [4], [5] and [14] in connection of hyperbolic perturbation of pressureless gas dynamic system (and some more general weakly hyperbolic systems in the paper [14]). A perturbation of the original system with vanishing (generalized) pressure term yields strictly hyperbolic genuinely nonlinear system. It posses classical solution and the solution consists of a shock followed by another shock for some initial data. It converges to a delta shock as perturbation parameter goes to zero. That situation looks exactly like the shadow wave solution with the equal intermediate states which solves the non-perturbed system in the sense given in the present paper.

The paper consists of four parts with an appendix. The plan of exposition is the following. We give a basic definition and a lemma which will be used trough the paper in the beginning.

The first part is devoted to shadow wave solutions to Riemann problem. The type used here is the most simple one, the simple constant shadow wave. We look first at Riemann problem for a system given in a general form. The result of that section is a set of formulas used afterward in specific cases. A set of all the states on the right which can be joined to a fixed state on the left by a shadow wave is called *shadow locus*. Also, one can find a general entropy condition for shadow waves by using convex or semi-convex entropy function with an appropriate entropy-flux for the given system. A subset of a shadow locus for which points the appropriate shadow wave solution is admissible is called *admissible shadow locus*. That was the contents of Sections 3 and 4.

In Section 5 we look at a case when the given system is linear in one component of unknown function. That case is well balanced between generality and usefulness of results. The main (model) example of the paper is 3×3 pressureless gas dynamics model derived from the Euler gas dynamics model (see [6], [7] and [21]) and it is described in Section 6. The entropy solution of that system obtained here is coherent to the known ones of 2×2 model given in [2], [4] or [10].

In the second part we use a constant shadow wave as a solution pattern. It is similar to the simple one with constant intermediate states but with non-zero initial mass. Also, its strength can be decreasing or constant function of time which can not be the true for the simple shadow wave. The main result is a theorem given in Section 7 about shadow waves in interactions and entropy conditions for them. It gives a way for joining incoming shadow wave(s) with an outgoing one. In Section 8 one can learn that our model example rarely posses such a solution to an interaction problem. That is the main reason for introducing the new type of shadow wave, the weighted one.

Before solving the interaction problem in our main example in the third part, we just recalculate a lot of shadow wave solutions to systems known to have delta or singular shock solution. It turns out that all of the results are practically recovered in the new setting. Even more, we are now in position to use entropy pairs to justify the obtained non-classical solutions. One will see that 2×2 systems are significantly simpler for analysis compared to higher dimensional ones. For example, shadow locus is expected to has 2-dimensional Lebesgue measure greater that zero, interaction of waves containing a shadow wave is much easier to handle, and so on.

The complete entropy solution to interaction problem involving shadow waves for our model problem can be found in the fourth part. Building blocks for such a construction are the weighted shadow waves. We use the joining theorem from Section 7 in the second part. Said in a very simplified way, for a system of dimension more than two a probability to find a constant shadow wave solution to the interaction problem is practically zero that is not the case for weighted shadow wave. It would be clear that the procedure described in the fourth part can be easily adopted for other (even for whole classes of) systems. Let us notice that one can always find a local solution (but not necessary entropy one) to general interaction problem.

The reader will soon be aware that we restrict ourselves with assumptions used in the definition of shadow waves despite a variety of other possibilities. That was done in order to explain a fairly general case of conservation law systems. Also, the main attention is focused on the delta shock case for the same reason. For example, systems admitting singular shocks can not be easily collected into well defined classes. Such assumptions are generally not needed when one deals with a concrete system and they can be more relaxed. In Appendix one will find two examples. The first one is shadow wave solution that violates the usual growth assumptions with respect to ε. Thus, we cannot use the basic formulas proved at the beginning. Nevertheless an overcompressive solution do exists and converges to δ' solution for the same system described in [20]. We have to mention that the entropy inequality holds only for a subclass of semi-convex entropy functions.

A singular shock given in its general form cannot always be directly treated as a singular shock (see [16]). So there is possibility that some systems given in a general form having a singular shock solution belonging to Colombeau space of generalized function do not admit a shadow wave solution. But one can use a bit different form of SDWs which easily incorporate these cases in Appendix (so called composite SDWs).

In the present paper we have used a majority of our attention for recovering previously known examples of delta and singular shock solutions now as entropic and weakly unique shadow waves. The definition of shadow waves could contain some other "objects" which also solve some systems without standard BV solutions as already mentioned δ' shocks. That was the first important left for the further investigation.

Also we just presented possibility for construction of a procedure for finding non-standard solutions to full Cauchy problems which would resembles wave front tracking procedure. That was the second problem left for a further investigation.

2. Basic formulas

The following notation will be used trough the paper. A parameter ε belongs to some interval $(0, \varepsilon_0)$ with ε_0 being as small as needed. Let a_ε be a net of reals and

u_ε be a net of locally integrable functions over some domain $\omega \subset \mathbb{R}^m$. We say that

$$a_\varepsilon \sim \varepsilon \text{ if there exists } A \in (0,\infty) \text{ such that } \lim_{\varepsilon\to 0} \frac{a_\varepsilon}{\varepsilon} = A,$$

and

$$u_\varepsilon \approx g \in \mathcal{D}'(\omega) \text{ if } \int_\omega u_\varepsilon \phi \to \langle g, \phi\rangle \text{ as } \varepsilon \to 0 \text{ for every test function } \phi \in \mathcal{C}_0^\infty(\omega).$$

The relation $u_\varepsilon \approx v_\varepsilon$ means $u_\varepsilon - v_\varepsilon \approx 0$, and we called it *distributional equality* or just equality if there is no chance for misunderstanding.

In the sequel, relations $\sim$, $\approx$, a "growth order", Landau symbols $\mathcal{O}(\cdot)$ and $o(\cdot)$ will always be used assuming $\varepsilon \to 0$. The half-space $\{(x,t) \in \mathbb{R} \times \mathbb{R}_+\}$ is denoted by $\mathbb{R}^2_+$.

All calculations in the paper are based on exploitation of the Rankine-Hugoniot conditions. We will obtain all results by the following basic lemma and its minor revisions.

Lemma 2.1. *Let $f, g \in \mathcal{C}(\Omega : \mathbb{R}^n)$ and $U : \mathbb{R}^2_+ \to \Omega \subset \mathbb{R}^n$ be a piecewise constant function given by*

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x < c(t) - a_\varepsilon(t) - x_{1,\varepsilon} \\ U_{1,\varepsilon}, & c(t) - a_\varepsilon(t) - x_{1,\varepsilon} < x < c(t) \\ U_{2,\varepsilon}, & c(t) < x < c(t) + b_\varepsilon(t) + x_{2,\varepsilon} \\ U_1, & x > c(t) + b_\varepsilon(t) + x_{2,\varepsilon} \end{cases}. \tag{2.1}$$

Here $x_{1,\varepsilon}, x_{2,\varepsilon} \sim \varepsilon$, while a_ε, b_ε are smooth functions equal zero at $t = 0$ with growth order less or equal to ε. Assume

$$\max_{i=1,2}\{\|f(U_{i,\varepsilon})\|_{L^\infty}, \|g(U_{i,\varepsilon})\|_{L^\infty}\} = \mathcal{O}(\varepsilon^{-1}). \tag{2.2}$$

Then

$$\begin{aligned} \partial_t f(U_\varepsilon) \approx & - c'(t)\Big(f(U_1) - f(U_0)\Big)\delta + \Big(a_\varepsilon'(t) f(U_{1,\varepsilon}) + b_\varepsilon'(t) f(U_{2,\varepsilon})\Big)\delta \\ & - c'(t)\Big((a_\varepsilon(t) + x_{1,\varepsilon}) f(U_{1,\varepsilon}) + (b_\varepsilon(t) + x_{2,\varepsilon}) f(U_{2,\varepsilon})\Big)\delta' \\ \partial_x g(U_\varepsilon) \approx & \Big(g(U_1) - g(U_0)\Big)\delta + \Big((a_\varepsilon(t) + x_{1,\varepsilon}) g(U_{1,\varepsilon}) + (b_\varepsilon(t) + x_{2,\varepsilon}) g(U_{2,\varepsilon})\Big)\delta', \end{aligned} \tag{2.3}$$

where δ and its derivative δ' are supported by the line $x = ct$.

Remark 2.1. The assumption (2.2) is not always necessary when one deals with a specific system of conservation laws. But it would be practically impossible to use a weaker assumption than it in order to deal with systems in a general form. At the end of the paper we present an example when a solution of a conservation law system of the form (1.1) which do not satisfy (2.2)

Remark 2.2. The constants $x_{i,\varepsilon}$, $i = 1, 2$ are useful when initial data contains delta function: If $\sigma := \lim_{\varepsilon\to 0} x_{1,\varepsilon} U_{1,\varepsilon} + x_{2,\varepsilon} U_{2,\varepsilon} \in \mathbb{R}^n$ exists, then the function U from (2.1) satisfies

$$U|_{t=0} = \begin{cases} U_0, & x < 0, \\ U_1, & x > 0 \end{cases} + \sigma \delta_{(0,0)}.$$

Proof. We shall use the Taylor expansion formula for a test function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}_+^2)$:

$$\phi(c(t)-a_\varepsilon(t)-x_{1,\varepsilon},t)=\phi(c(t),t)+\sum_{j=1}^m \partial_x^j\phi(c(t),t)\frac{(-a_\varepsilon(t)-x_{1,\varepsilon})^j}{j!}+\mathcal{O}(\varepsilon^{m+1})$$

$$\phi(c(t)+b_\varepsilon(t)+x_{2,\varepsilon},t)=\phi(c(t),t)+\sum_{j=1}^m \partial_x^j\phi(c(t),t)\frac{(b_\varepsilon(t)+x_{2,\varepsilon})^j}{j!}+\mathcal{O}(\varepsilon^{m+1}).$$

Due to the above growth assumptions on $a_\varepsilon,\ b_\varepsilon,\ f(U_{i,\varepsilon})$ and $g(U_{i,\varepsilon}),\ i=1,2$ one will see that it is enough to take $m=1$ in the above expansion, so

$$\begin{aligned}\phi(c(t)-a_\varepsilon(t)-x_{1,\varepsilon},t)&=\phi(c(t),t)-\partial_x\phi(c(t),t)(a_\varepsilon(t)+x_{1,\varepsilon})+\mathcal{O}(\varepsilon^2)\\ \phi(c(t)+b_\varepsilon(t)+x_{2,\varepsilon},t)&=\phi(c(t),t)+\partial_x\phi(c(t),t)(b_\varepsilon(t)+x_{2,\varepsilon})+\mathcal{O}(\varepsilon^2).\end{aligned}$$

Using the standard Rankine–Hugoniot shock calculations and the above approximations we have (up to terms less or equal to ε^2 to be precise)

$$\begin{aligned}\langle\partial_t f(U_\varepsilon),\phi\rangle\approx&-\int_0^\infty(c'(t)-a_\varepsilon'(t))\,(f(U_{1,\varepsilon})-f(U_0))\,\phi(c(t)-a_\varepsilon(t)-x_{1,\varepsilon},t)\,dt,\\ &-\int_0^\infty c'(t)\,(f(U_{2,\varepsilon})-f(U_{1,\varepsilon}))\,\phi(c(t),t)\,dt\\ &-\int_0^\infty(c'(t)+b_\varepsilon'(t))\,(f(U_1)-f(U_{2,\varepsilon}))\,\phi(c(t)+b_\varepsilon(t)+x_{2,\varepsilon},t)\,dt\\ \approx&-(f(U_{1,\varepsilon})-f(U_0))\int_0^\infty(c'(t)-a_\varepsilon'(t))\Big(\phi(c(t),t)-\partial_x\phi(c(t),t)\\ &\cdot(a_\varepsilon(t)+x_{1,\varepsilon})\Big)\,dt\\ &-(f(U_{2,\varepsilon})-f(U_{1,\varepsilon}))\int_0^\infty c'(t)\,\phi(c(t),t)\,dt\\ &-(f(U_1)-f(U_{2,\varepsilon}))\int_0^\infty(c'(t)+b_\varepsilon'(t))\Big(\phi(c(t),t)+\partial_x\phi(c(t),t)\\ &\cdot(b_\varepsilon(t)+x_{2,\varepsilon})\Big)\,dt.\end{aligned}$$

The assumptions from Lemma 2.1 imply

$$\begin{aligned}\langle\partial_t f(U_\varepsilon),\phi\rangle\approx&-(f(U_1)-f(U_0))\int_0^\infty c'(t)\phi(c(t),t)\,dt\\ &+\int_0^\infty\Big(a_\varepsilon'(t)f(U_{1,\varepsilon})+b_\varepsilon'(t)f(U_{2,\varepsilon})\Big)\phi(c(t),t)\,dt\\ &+\int_0^\infty c'(t)\Big((a_\varepsilon(t)+x_{1,\varepsilon})f(U_{1,\varepsilon})+(b_\varepsilon(t)+x_{2,\varepsilon})f(U_{2,\varepsilon})\Big)\\ &\cdot\partial_x\phi(c(t),t)dt\\ \approx&\Big\langle\Big(-c'(t)(f(U_1)-f(U_0))+a_\varepsilon'(t)f(U_{1,\varepsilon})+b_\varepsilon'(t)f(U_{2,\varepsilon})\Big)\\ &\cdot\delta(x-c(t)),\phi(x,t)\Big\rangle\\ &\Big\langle-c'(t)\Big((a_\varepsilon(t)+x_{1,\varepsilon})f(U_{1,\varepsilon})+(b_\varepsilon(t)+x_{2,\varepsilon})f(U_{2,\varepsilon})\Big)\\ &\cdot\delta'(x-c(t)),\phi(x,t)\Big\rangle.\end{aligned}$$

With the same type of reasoning, one sees that the space derivative is given by

$$
\begin{aligned}
\langle \partial_x g(U_\varepsilon), \phi\rangle \approx& (g(U_{1,\varepsilon}) - g(U_0)) \int_0^\infty \phi(c(t),t) - \partial_x\phi(c(t),t)\,(a_\varepsilon(t) + x_{1,\varepsilon})\,dt \\
&+ (g(U_{2,\varepsilon}) - g(U_{1,\varepsilon})) \int_0^\infty \phi(c(t),t)\,dt \\
&+ (g(U_1) - g(U_{2,\varepsilon})) \int_0^\infty \phi(c(t),t) + \partial_x\phi(c(t),t)\,(b_\varepsilon(t) + x_{2,\varepsilon})\,dt \\
\approx& (g(U_1) - g(U_0)) \int_0^\infty \phi(c(t),t)\,dt \\
&- \int_0^\infty \Big((a'_\varepsilon(t) + x_{1,\varepsilon}) g(U_{1,\varepsilon}) + (b'_\varepsilon(t) + x_{2,\varepsilon}) g(U_{2,\varepsilon})\Big)\partial_x\phi(c(t),t)\,dt \\
\approx& \Big\langle \Big(g(U_1) - g(U_0)\Big)\delta(x - c(t)), \phi(x,t)\Big\rangle \\
&+ \Big\langle \Big((a_\varepsilon(t) + x_{1,\varepsilon}) g(U_{1,\varepsilon}) + (b_\varepsilon(t) + x_{2,\varepsilon}) g(U_{2,\varepsilon})\Big) \\
&\cdot \delta'(x - c(t)), \phi(x,t)\Big\rangle.
\end{aligned}
$$

□

Remark 2.3. We used only constant mean-states $U_{1,\varepsilon}$, $U_{2,\varepsilon}$ and constant central SDW speed curve $(ct,t)_{t\geq 0}$ in (2.1). Such SDWs are not good enough for solving an SDW interaction problem for the main example in the paper, 3×3 pressureless system. The problem will be solved by introducing variable mean-states $U_{1,\varepsilon}(t)$ and $U_{2,\varepsilon}(t)$ in the fourth part. Lemma 10.1 will be a natural modification of the above assertion.

Definition 2.1. Functions of the form (2.1) are called *constant shadow waves* or constant SDW for short. We shall drop the word "constant" in the sequel if there is no chance for confusion. The value

$$\sigma_\varepsilon(t) := (a_\varepsilon(t) + x_{1,\varepsilon})U_{1,\varepsilon} + (b_\varepsilon(t) + x_{2,\varepsilon})U_{2,\varepsilon}$$

is called the *strength* and $c'(t)$ is called the *speed* of the shadow wave. We assume that $\lim_{\varepsilon\to 0}\sigma_\varepsilon(t) = \sigma(t) \in \mathbb{R}^n$ exists for every $t \geq 0$ and

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int U_\varepsilon(x,t)\phi(x,t)\,dx\,dt =& \langle \sigma(t)\delta(x - c(t)) + U_0 + [U]\theta(x - c(t)), \phi(x,t)\rangle \\
=& \int \sigma(t)\phi(c(t),t)dt + \int (U_0 + [U]\theta(x - c(t))\phi(x,t)dx\,dt,
\end{aligned}
$$

where θ is the Heaviside function and $[U] := U_1 - U_0$. The SDW *central line* is given by $x = c(t)$, while $x = c(t) - a_\varepsilon(t) - x_{1,\varepsilon}$ and $x = c(t) + b_\varepsilon(t) + x_{2,\varepsilon}$ are called the *external SDW lines*. The values $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$ are called the *shifts* while $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ are called the *intermediate states* of a given SDW.

Part 1. The Riemann problem

3. General formula

The following special case of (2.1)

$$U_\varepsilon(x,t)=\begin{cases} U_0, & x<(c-a_\varepsilon)t\\ U_{1,\varepsilon}, & (c-a_\varepsilon)t<x<ct\\ U_{2,\varepsilon}, & ct<x<(c+b_\varepsilon)t\\ U_1, & x>(c+b_\varepsilon)t \end{cases} \tag{3.1}$$

is general enough for solving Riemann problem as one could see bellow. We shall call it the *simple SDW*.

The formula (2.3) now has a simpler form

$$\begin{aligned} \partial_t f(U_\varepsilon)\approx&-c(f(U_1)-f(U_0))\delta-c(a_\varepsilon f(U_{1,\varepsilon})+b_\varepsilon f(U_{2,\varepsilon}))t\delta'\\ &+(a_\varepsilon f(U_{1,\varepsilon})+b_\varepsilon f(U_{2,\varepsilon}))\delta\\ \partial_x g(U_\varepsilon)\approx&(g(U_1)-g(U_0))\delta+(a_\varepsilon g(U_{1,\varepsilon})+b_\varepsilon g(U_{2,\varepsilon}))t\delta'. \end{aligned} \tag{3.2}$$

The support of δ (and δ' consequently) is the line $x=ct$.

A way to find a shadow wave solutions to a system of conservation laws (1.1) directly follows from Lemma 2.1. We use the following assumption to keep our discussion on a general level. An actual construction of SDW solution highly depends on a particular choice of f and g without it.

Assumption 3.1. All the components U_ε^i, $i=1,\dots,n$ of an SDW (2.1) satisfy

$$\|U_\varepsilon^i\|_{L^\infty}=\mathcal{O}(\varepsilon^{-1}),\ \text{if } f \text{ and } g \text{ are at most linear with respect to } i\text{-th variable}$$

or

$$\|U_\varepsilon^i\|_{L^\infty} \text{ has a growth order small enough for (2.2) to hold, otherwise.}$$

Definition 3.1. Components satisfying the first criteria are called the *major components* or ε^{-1}*-components*, while all other are called the *minor* ones.

A *delta shock* is a SDW associated with a δ distribution with all minor components having finite limits as $\varepsilon\to0$. If some of them are unbounded as $\varepsilon\to0$, then the wave is called *singular shock*.

The following definition contains an analogous notion to Hugoniot locus for shocks.

Definition 3.2. Let U_0 be fixed. The set of all $U_1\in\Omega$ such that there exists an SDW solution to (1.1) with the initial data

$$U|_{t=0}=\begin{cases} U_0, & x<0\\ U_1, & x>0 \end{cases}$$

is called the *shadow locus*. Points for which the above wave is admissible constitutes the it admissible locus. The admissibility will be defined trough entropy conditions given bellow. In the case when the SDW is delta (singular) shock, the above set is called *delta (singular delta) locus*.

Let us start a search for SDW solutions of 1.1. Substitution of the function U from (3.1) into the i-th equation in (1.1) yields

$$\begin{aligned}&\big(-c(f^i(U_1)-f^i(U_0))+a_\varepsilon f^i(U_{1,\varepsilon})+b_\varepsilon f^i(U_{2,\varepsilon})\big)\delta(x-ct)\\&-ct\big(a_\varepsilon f^i(U_{1,\varepsilon})+b_\varepsilon f^i(U_{2,\varepsilon})\big)\delta'(x-ct)+\big(g^i(U_1)-g^i(U_0)\big)\delta(x-ct)\\&+t\big(a_\varepsilon g^i(U_{1,\varepsilon})+b_\varepsilon g^i(U_{2,\varepsilon})\big)\delta'(x-ct)\approx 0.\end{aligned}$$

That implies

$$\begin{aligned}-c(f^i(U_1)-f^i(U_0))+a_\varepsilon f^i(U_{1,\varepsilon})+b_\varepsilon f^i(U_{2,\varepsilon})+g^i(U_1)-g^i(U_0)\approx 0\\ -c(a_\varepsilon f^i(U_{1,\varepsilon})+b_\varepsilon f^i(U_{2,\varepsilon}))+a_\varepsilon g^i(U_{1,\varepsilon})+b_\varepsilon g^i(U_{2,\varepsilon})\approx 0,\\ i=1,\ldots,n.\end{aligned}\tag{3.3}$$

Define

$$\kappa^i:=c(f^i(U_1)-f^i(U_0)-(g^i(U_1)-g^i(U_0))$$

to be so called *Rankine-Hugoniot deficit* (RH deficit for short) in the i-th equation. Now (3.3) reads as

$$\begin{aligned}a_\varepsilon f^i(U_{1,\varepsilon})+b_\varepsilon f^i(U_{2,\varepsilon})&\approx\kappa^i\\ a_\varepsilon g^i(U_{1,\varepsilon})+b_\varepsilon g^i(U_{2,\varepsilon})&\approx c\kappa^i,\ i=1,\ldots,n.\end{aligned}\tag{3.4}$$

That was the most general case with Assumption 3.1. Let us start our investigation of the above system for the the simplest, evolutionary case.

3.1. **Evolutionary systems.** If the system of conservation laws (1.1) is given in the evolutionary form $f^i(y)\equiv y^i$, $i=1,\ldots,n$, then the system (3.3) reduces to

$$\begin{aligned}-c(U_1^i-U_0^i)+a_\varepsilon U_{1,\varepsilon}^i+b_\varepsilon U_{2,\varepsilon}^i+g^i(U_1)-g^i(U_0)\approx 0\\ -c(a_\varepsilon U_{1,\varepsilon}^i+b_\varepsilon U_{2,\varepsilon}^i)+a_\varepsilon g^i(U_{1,\varepsilon})+b_\varepsilon g^i(U_{2,\varepsilon})\approx 0,\ i=1,\ldots,n.\end{aligned}\tag{3.5}$$

and the system (3.4) has now a simpler form

$$\begin{aligned}a_\varepsilon U_{1,\varepsilon}^i+b_\varepsilon U_{2,\varepsilon}^i&\approx\kappa^i\\ a_\varepsilon g^i(U_{1,\varepsilon})+b_\varepsilon g^i(U_{2,\varepsilon})&\approx c\kappa^i,\ i=1,\ldots,n.\end{aligned}\tag{3.6}$$

All the results in the present paper can be divided roughly into two types:

- general, descriptive results for systems given in a general form, and
- precise, concrete solutions to specific (physical) systems.

The following proposition belongs to the first class, and the phrase "...is contained in ..." instead of expected "...is ..." bellow explains what is difference between a "general result" and a "concrete solution".

Proposition 3.1. *Suppose that all the flux-functions in (1.1) are of at most linear growth with respect to k components, say $U^1,\ldots,U^k$ and superlinear with respect to others. Then a shadow locus to the system with $f(y)=y$ is contained in a $(k+1)$-dimensional manifold.*

Proof. We know that $U_j^{i,\varepsilon}$,$j=1,2$ and $i=k+1,\ldots,n$ has a growth order with respect to ε are as small as needed by Assumption 3.1. If the flux function has superlinear growth with respect to the i-th component U^i, then $U_{j,\varepsilon}^i=o(\varepsilon^{-1})$ by that assumption. That implies $\kappa^i=0$ for such i and the system (3.6) is now partially determined because of that ($(n-k)$ of $2n$ components are already satisfied). For a fixed left-hand state U_0 the following $n+1$ scalars: a speed c and a right-hand state U_1, have to satisfy the following $n-k$ equations

$$c = \frac{g^i(U_1) - g^i(U_0)}{u_1^i - u_0^i},\ i = k+1, \dots, n,$$

When the speed c is determined, the set of all possible values $U_1 = (U_1^1, \dots, U_1^n)$ solving the above system belongs to some intersection of $k+1$-dimensional manifold and the physical domain Ω. Its dimension would be maximal if it is possible to find all intermediate states $U_{1,\varepsilon}^i$ and $U_{2,\varepsilon}^i$ to solve (3.6) for each $i = 1, \dots, k$ once when c, U_0 and U_1 are known. Since $\kappa_{k+1} = \ldots = \kappa_n = 0$, the system (3.6) has $n+k$ non-trivial equations with $2n$ unknowns $U_{j,\varepsilon}^i$, $i = 1, \dots, n$, $j = 1, 2$.

$$\begin{aligned} a_\varepsilon U_{1,\varepsilon}^i + b_\varepsilon U_{2,\varepsilon}^i &\approx \kappa^i\ i = 1, \dots, k \\ a_\varepsilon g^i(U_{1,\varepsilon}) + b_\varepsilon g^i(U_{2,\varepsilon}) &\approx c\kappa^i,\ i = 1, \dots, k. \\ a_\varepsilon g^i(U_{1,\varepsilon}) + b_\varepsilon g^i(U_{2,\varepsilon}) &\approx 0,\ i = k+1, \dots, n. \end{aligned}$$

In general, such a system could have a solution (despite it is nonlinear). For example, problems might be caused by a special form of flux-functions (if $g^i \equiv g^j$, $i \neq j$) or by a fact that the solution for intermediate states should lie in a physical domain Ω. □

In the case of delta shocks the situation is simpler because we can assume $U_j^{i,\varepsilon} \to U_{s,j}^i \in \mathbb{R}$, $i = k+1, \dots, n$, $j = 1, 2$, and that the limits $\xi_1^i := \lim\limits_{\varepsilon \to 0} a_\varepsilon U_{1\varepsilon}^i$ and $\xi_2^i := \lim\limits_{\varepsilon \to 0} b_\varepsilon U_{2,\varepsilon}^i$, $i = 1, \dots, n$ exist. Suppose that $g_i(U) = \sum_{j=1}^k g_j^i(U^{k+1}, \dots, U^n) U^j$. Then the system (3.6) reduces to

$$\begin{aligned} \xi_1^i + \xi_2^i &= \kappa^i,\ i = 1, \dots, k \\ \sum_{j=1}^k g_j^i(U_{s,1}^{k+1}, \dots, U_{s,1}^n)\xi_1^j + \sum_{j=1}^k g_j^i(U_{s,2}^{k+1}, \dots, U_{s,2}^n)\xi_2^j &= c\kappa^i,\ i = 1, \dots, n, \\ \text{where } \kappa^i &= 0,\ i = k+1, \dots, n, \end{aligned}$$

and the analysis given in the proof of previous proposition is significantly simpler: The above system has $2k$ major intermediate states $U_{j,\varepsilon}^i$, $i = 1, \dots, k$, $j = 1, 2$ and $2(n-k)$ minor ones with limits $U_{s,j}^i$, $i = k+1, \dots, n$, $j = 1, 2$ as $\varepsilon \to 0$. The general idea for solving the system is to treat these limits as real parameters which are to be chosen such that the system has a solution ξ_j^i, $i = 1, \dots, k$, $j = 1, 2$.

The introductory part of the paper finishes with this section. We have transformed the question of SDW existence into the question of solving systems of equations in reals. Before analysis of some more specific cases (take, for example, systems linear in one variable), we shell do two important things:

- Give appropriate admissibility conditions via entropy – entropy flux pairs, and
- Give a method for continuation of SDW solutions after wave interactions take places.

Along with that plan we shall try to use these results in some specific systems of conservation laws.

4. Entropy conditions

Let $\eta(U)$ be a (strictly) convex or semi-convex entropy function for (1.1), with entropy-flux function $q(U)$. We shall use entropy condition in the following form. A solution U_ε to the system (1.1) with initial data $U|_{t=0} = U_{0,\varepsilon}$ is *admissible* if for every $T > 0$ we have

$$\underline{\lim}_{\varepsilon\to 0} \int_{\mathbb{R}} \int_0^T \eta(U_\varepsilon)\partial_t\phi + q(U_\varepsilon)\partial_x\phi \, dt \, dx + \int_{\mathbb{R}} \eta(U_{0,\varepsilon}(x,0))\phi(x,0)\, dx \geq 0, \quad (4.1)$$

for all non-negative test functions $\phi \in C_0^\infty(\mathbb{R}\times(-\infty,T))$.

Take a simple SDW U_ε from (3.1) and use the equality (2.3) from Lemma 2.1 with f substituted by η and g by q. As the delta function is a non-negative distribution, the first condition becomes

$$\overline{\lim}_{\varepsilon\to 0} - c(\eta(U_1) - \eta(U_0)) + a_\varepsilon\eta(U_{1,\varepsilon}) + b_\varepsilon\eta(U_{2,\varepsilon}) + q(U_1) - q(U_0) \leq 0 \quad (4.2)$$

But a derivative of the delta function has no constant sign and the second condition becomes

$$\lim_{\varepsilon\to 0} -c(a_\varepsilon\eta(U_{1,\varepsilon}) + b_\varepsilon\eta(U_{2,\varepsilon})) + a_\varepsilon q(U_{1,\varepsilon}) + b_\varepsilon q(U_{2,\varepsilon}) = 0. \quad (4.3)$$

Here, U_0, U_1, $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ are constants.

In the most of papers with delta or singular shock solution, the authors use overcompressibility as the admissibility condition: A wave is called the *overcompressive* one if all characteristics from both sides of the SDW line run into a shock curve, i.e.

$$\lambda_i(U_0) \geq c'(t) \geq \lambda_i(U_1),\ i = 1,\dots,n,$$

where c is a shock speed and $x = \lambda_i(U)t$, $i = 1,\dots,n$ are the characteristics of the system.

In the rest of the paper we shall present few examples when the entropy admissibility condition implies the overcompressibility admissibility condition.

The entropy condition is connected with a problem of uniqueness for a weak solution of a conservation law system. We give a definition of weak (distributional) uniqueness and some results about it afterward.

Definition 4.1. An SDW solution is called *weakly unique* if its distributional image is the unique. More precisely, a speed c of the wave has to be unique as well as the limit

$$\lim_{\varepsilon\to 0} a_\varepsilon U_{1,\varepsilon} + b_\varepsilon U_{2,\varepsilon}.$$

Let $i \in \{1,\dots,n\}$. If a limit $\lim\limits_{\varepsilon\to 0} a_\varepsilon U^i_{1,\varepsilon} + b_\varepsilon U^i_{2,\varepsilon}$ is unique, then we say that the i-th component is unique.

Note that all minor components of U_ε are unique by the above definition. The following proposition is a direct consequence of the SDW definition.

Proposition 4.1. *Suppose that (1.1) has an SDW solution.*

(a) *If there exists an equation of the system, say i-th one, such that a density function $f^i(U)$ is independent of major components of U, then a speed of the SDW is uniquely determined by the equation*

$$-c[f^i(U)] + [g^i(U)] = 0.$$

(b) *If there is an equation in the system, say i-th one, such that $f^i(U) = U^j$, where U^j is a major component, then it is uniquely determined by*

$$a_\varepsilon U^j_{1,\varepsilon} + b_\varepsilon U^j_{2,\varepsilon} = \kappa_i \in \mathbb{R}.$$

Consequently, if (a) holds and (b) holds for all major components, then a distributional limit of an SDW solution to (1.1) is unique. Specially, that is the case for a system given in evolutionary form.

Definition 4.2. We say that a solution to (1.1) is *weakly unique* if it consists from a unique combination of standard admissible elementary waves (shocks, rarefactions and contact discontinuities) and admissible SDW.

In the present paper, admissible solution means that it satisfy entropy inequality (4.1) for each entropy pair with convex (for strictly hyperbolic systems) or semi-convex entropy function (for weakly hyperbolic ones).

We shall prove a weak uniqueness of a solution to two systems: The first one is an 3×3 pressureless gas dynamics model with delta shocks, and the second one is a 2×2 system given in [12] with singular shocks.

5. Systems linear in one variable

When a given system (1.1) is linear in one component (say U^1 in the sequel), then we are in position to get additional results concerning the existence of shadow wave solutions to a Riemann problem. More precisely, we shall present some general results about delta shocks in the present section. That is an introduction for the next section where the main example is given. That is a well known 2×2 pressureless gas dynamics model given in [10] and [4] ("sticky particles" model in [2]), but now extended with the energy conservation law. It is important to add that the obtained results in two variables agree (their distributional limits at least) with the ones from the cited papers.

Let the system (1.1) be linear in U^1. Then the i-th equation of the system is

$$\partial_t\big(f_1^i(\underline{U})U^1 + f_2^i(\underline{U})\big) + \partial_x\big(g_1^i(\underline{U})U^1 + g_2^i(\underline{U})\big) = 0, \tag{5.1}$$

where f_i, g_i, $i = 1, 2$ are continuous functions with $\underline{U} := (U^2, \ldots, U^n)$. Set $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ as follows.

$$\underline{U}_{i,\varepsilon} := \underline{U}_{s,i} \in \mathbb{R}^{n-1},\ i = 1, 2,\ \lim_{\varepsilon\to 0} a_\varepsilon U^1_{1,\varepsilon} = \xi_1,\ \lim_{\varepsilon\to 0} b_\varepsilon U^1_{2,\varepsilon} = \xi_2,$$

where $\underline{U}_{s,i}$ and ξ_i, $i = 1, 2$ will be determined later. For an SDW U_ε given by (3.1) the difference $f(U_1) - f(U_0)$ is denoted by $[f(U_\varepsilon)]$.

From (5.1) one derives the following system of equations with respect to ξ_1 and ξ_2 for each $i = 1, \ldots, n$:

$$\begin{aligned} f_1^i(\underline{U}_{s,1})\xi_1 + f_1^i(\underline{U}_{s,2})\xi_2 &= \kappa_i \\ g_1^i(\underline{U}_{s,1})\xi_1 + g_1^i(\underline{U}_{s,2})\xi_2 &= c\kappa_i. \end{aligned} \tag{5.2}$$

Here $\kappa_i := c[f_1^i(\underline{U})U^1 + f_2^i(\underline{U})] - [g_1^i(\underline{U})U^1 + g_2^i(\underline{U})]$ as before.

The following theorem can be proved in such general case.

Theorem 5.1. *Assume that the density function f does not depend on U^1 in k equations of the system (5.1) (i.e. $f_1^i \equiv 0$, $i = i_1, \ldots, i_k$). Then the shadow locus is a subset of $n - k + 1$-dimensional manifold intersected by Ω.*

Proof. Suppose $f_1^{n-k+1} = \ldots = f_1^n = 0$. From the first equation in (5.2) it follows $\kappa_i = 0$ for each $i = n-k+1, \ldots, n$. Assume for a moment that $\underline{U}_{s,1}$ and $\underline{U}_{s,2}$ are known. If the left-hand side state U_0 is fixed, then the speed c and $U_1 = (U_1^1, \ldots, U_1^n)$ has to satisfy the following system

$$c = \frac{[g_1^i(\underline{U})U^1 + g_2^i(\underline{U})]}{[f_2^i(\underline{U})]}, \; i = n-k+1, \ldots, n. \tag{5.3}$$

There are k equations and $n+1$ scalar variables: $c, U_1^1, \ldots, U_1^n$, so we are free to chose $n-k+1$ of them provided that $\underline{U}_{s,1}$ and $\underline{U}_{s,2}$ are chosen in a good way.

Thus, the set of all possible values U_1 such that (5.3) is satisfied lies in an $n-k+1$-dimensional manifold (if the speed c was excluded from the above free choice).

Now we turn our attention to $U_{s,1}$ and $U_{s,2}$ and the first $n-k$ systems given by (5.2). Let $i \in \{1, \ldots, n-k\}$. Assuming

$$D_s^i(\underline{U}_{s,1}, \underline{U}_{s,2}) := \begin{vmatrix} f_1^i(\underline{U}_{s,1}) & f_1^i(\underline{U}_{s,2}) \\ g_1^i(\underline{U}_{s,1}) & g_1^i(\underline{U}_{s,2}) \end{vmatrix} \neq 0,$$

the solution (ξ_1, ξ_2) for each system (5.2) is given by

$$\begin{aligned} \xi_1^i(\underline{U}_{s,1}, \underline{U}_{s,2}) &= \frac{\kappa_i(g_1^i(\underline{U}_{s,2}) - cf_1^i(\underline{U}_{s,2}))}{D_s^i}, \\ \xi_2^i(\underline{U}_{s,1}, \underline{U}_{s,2}) &= \frac{\kappa_i(g_1^i(\underline{U}_{s,1}) - cf_1^i(\underline{U}_{s,1}))}{D_s^i}. \end{aligned} \tag{5.4}$$

A consistency for ξ_1 and ξ_2 found from each system produces the new one

$$\begin{aligned} \xi_1^1(\underline{U}_{s,1}, \underline{U}_{s,2}) &= \ldots = \xi_1^{n-k}(\underline{U}_{s,1}, \underline{U}_{s,2}) \\ \xi_2^1(\underline{U}_{s,1}, \underline{U}_{s,2}) &= \ldots = \xi_2^{n-k}(\underline{U}_{s,1}, \underline{U}_{s,2}) \end{aligned} \tag{5.5}$$

of $2(n-k-1)$ equations.

Let $i \in \{n-k+1, \ldots, n\}$. We already know that $f_1^i \equiv 0$, and substitution of ξ_1^1 and ξ_2^1 into the second equation in (5.2) for such i gives the following

$$g_1^i(\underline{U}_{s,1})\xi_1^1(\underline{U}_{s,1}, \underline{U}_{s,1}) + g_1^i(\underline{U}_{s,2})\xi_2^1(\underline{U}_{s,1}, \underline{U}_{s,1}) = 0. \tag{5.6}$$

So, there are k such equations, and the final conclusion is that the shadow locus is defined by (5.3) provided that there exist a solution $\underline{U}_{s,1}^2, \ldots, \underline{U}_{s,1}^n, \underline{U}_{s,2}^2, \ldots, \underline{U}_{s,2}^n$ to (5.5, 5.6) of $2n-k-2$ equations. Since there are $2n-2$ variables there is a chance for solving the system, and obtain a maximal dimension $n-k+1$ of shadow locus. □

Roughly speaking, each additional density function independent of U^1 reduces the dimension of the locus by 1.

The extreme case is when none of f_1 components vanishes (i.e. all density functions depend on U^1). We then solve (5.2) with respect to ξ_1 and ξ_2. For each $i = 1, \ldots, n$ all the solutions is given by (5.4) have to be the same, so we get the system

$$\begin{aligned} \xi_1^1 &= \xi_1^2 = \ldots = \xi_1^n \\ \xi_2^1 &= \xi_2^2 = \ldots = \xi_2^n \end{aligned}$$

of $2(n-1)$ equations with $2n-1$ unknowns: c, $\underline{U}_{s,j}^i$, $i = 2, \ldots, n$, $j = 1, 2$. Also, the condition $D_s^i \neq 0$, $i = 1, \ldots, n$ is assumed as above. There are no conditions on a speed c and right-hand values U_1 with fixed U_0. If the solution of the above really exists, then the shadow locus is whole Ω.

The other extreme case, $f_1^i \equiv 0$, $i = 2, \dots, n$ is described in Proposition 3.1. The dimension of a shadow locus is at most 2 in that case.

Remark 5.1. One can see in [16] that a dimension of a delta locus for 2×2 system linear in one variable was expected to be one when generalized functions were used. A delta locus obtained by using SDWs usually has a dimension equal two in such a case. Thus, a SDW delta locus form the present paper is much richer than delta locus from the cited paper. That is the answer to the problem of relatively small delta locus in general case for 2×2 system and problems with a construction of solution to arbitrary Riemann data posed in [16].

6. Entropy solutions to Riemann problem for pressureless gas dynamics model

One dimensional Euler gas dynamics system is given by

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) &= 0 \\ \partial_t(\rho u^2/2 + \rho e) + \partial_x((\rho u^2/2 + \rho e + p)u) &= 0, \end{aligned}$$

where ρ is a density, u is a velocity, p is a pressure and e is an internal energy of fluid particles. If the pressure is constant $p \equiv p_0 \geq 0$, then the system reduces to

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) &= 0 \\ \partial_t(\rho u^2/2 + \rho e) + \partial_x((\rho u^2/2 + \rho e + p_0)u) &= 0. \end{aligned} \tag{6.1}$$

For $p_0 = 0$ it is called pressureless gas dynamics (or sticky particles) model. The system has triple eigenvalue $\lambda_{1,2,3} = u$. In the pressureless case, $p_0 \equiv 0$, the Riemann problem

$$(\rho, u, e)|_{t=0} = \begin{cases} (\rho_0, u_0, e_0), & x < 0 \\ (\rho_1, u_1, e_1), & x > 0 \end{cases} \tag{6.2}$$

has a classical entropy solution consist of two contact discontinuities connected with the vacuum state ($\rho = 0$) if $u_0 \leq u_1$:

$$(\rho, u, e)|_{t=0} = \begin{cases} (\rho_0, u_0, e_0), & x < u_0 t \\ (0, \psi_2(x/t), \psi_3(x/t)), & u_0 t < x < u_1 t \\ (\rho_1, u_1, e_1), & x > u_1 t, \end{cases}$$

where $\psi_2(u_i) = u_i$, $\psi_3(u_i) = e_i$, $i = 0, 1$. ($\psi_2(y) \equiv y$, more precisely.)

We are now turning to the case $u_0 > u_1$ when there is no classical solution to the Riemann problem (6.1, 6.2).

6.1. **Entropy pairs and entropy solution.**

Construction of an SDW solution to (6.1) is the same for all non-negative reals p_0. But, there are differences between entropies in the cases $p_0 = 0$ and $p_0 \neq 0$.

An entropy pair (η, q) for a system in non-evolutionary form can be found in the following way (see [7], Chapter 3). We search for matrix B such that

$$\mathrm{D}\eta := \begin{bmatrix} \partial\eta/\partial\rho & \partial\eta/\partial u & \partial\eta/\partial e \end{bmatrix} = B\,\mathrm{D}f,\ \mathrm{D}q = B\,\mathrm{D}g. \tag{6.3}$$

We have

$$\mathrm{D}f := \begin{bmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ e+u^2/2 & \rho u & \rho \end{bmatrix},\ \mathrm{D}g := \begin{bmatrix} u & \rho & 0 \\ u^2 & 2\rho u & 0 \\ u(e+u^2/2) & \rho e + 3\rho u^2/2 + p_0 & \rho u \end{bmatrix}.$$

The general solution to (6.3) after inserting the above values for $\mathrm{D}f$ and $\mathrm{D}g$ is given by

$$\begin{aligned} \eta(\rho,u,e) =& \rho S(u, e + p_0/\rho) - \phi'(u) \\ q(\rho,u,e) =& \rho u S(u, e+p_0/\rho) - u\phi'(u) + \phi(u), \end{aligned} \tag{6.4}$$

where S and ϕ are some smooth functions.

Now we shall find necessary conditions for η to be semi-convex with respect to $(\rho, \rho u, \rho u^2/2 + \rho e)$ when $p_0 = 0$. The Hessian matrix of S with respect to these variables is positive semi-definite if $\partial_{12}S = 0$, $\partial_{11}S \geq 0$, $\partial_2 S \leq 0$, $\partial_{22}S \geq 0$. Take functions R and S to be smooth enough. We use the following entropy pair

$$\eta = \rho(R(u) + S(e)),\ q = \rho u(R(u)+S(e)),\ \text{where } R'' \geq 0,\ S' \leq 0,\ S'' \geq 0. \tag{6.5}$$

Let $p_0 \neq 0$, say $p_0 = 1$. Then

$$\eta = \rho S(u, e+1/\rho),\ q = u\eta,\ \partial_{11}S \geq 0,\ \partial_2 S \leq 0,\ \partial_{22}S \geq 0 \tag{6.6}$$

and η is semi-convex.

The obtained entropy pairs resembles the case of full gas dynamics system: It has an entropy function $\eta = \rho\, h(S)$, where $S = S(\rho,u)$ satisfies $\rho^2\partial_\rho S = -p\partial_e S$, $\partial_e S > 0$ (see [21], Lemma 4.8.2). The appropriate entropy flux equals $q = u\eta$ and if the entropy is convex, then h is decreasing.

Theorem 6.1. *Suppose that $u_0 > u_1$. Then there exists a unique shadow wave solution of the form (3.1) to the Riemann problem (6.1, 6.2) satisfying entropy the inequality*

$$\partial_t \eta(\rho,u,e) + \partial_x q(\rho,u,e) \leq 0,$$

where η and q are defined by (6.5) or (6.6).

Moreover, the validity of the above inequality for all semi-convex entropies η are equivalent to the overcompressibility of the SDW.

Proof. We use the results obtained in the previous section. In the case of the system (6.1), the relations (5.2) transform in the following three systems:

$$\begin{aligned} \xi_1 + \xi_2 &= \kappa_1 \\ u_{s,1}\xi_1 + u_{s,2}\xi_2 &= c\kappa_1, \end{aligned} \tag{6.7}$$

where $\kappa_1 := c[\rho] - [\rho u]$,

$$\begin{aligned} u_{s,1}\xi_1 + u_{s,2}\xi_2 &= \kappa_2 \\ u_{s,1}^2\xi_1 + u_{s,2}^2\xi_2 &= c\kappa_2 \end{aligned} \tag{6.8}$$

where $\kappa_2 := c[\rho u] - [\rho u^2]$, and

$$\begin{aligned} (u_{s,1}^2/2 + e_{s,1})\xi_1 + (u_{s,2}^2/2 + e_{s,2})\xi_2 &= \kappa_3 \\ (u_{s,1}^3/2 + e_{s,1}u_{s,1})\xi_1 + (u_{s,2}^3/2 + e_{s,2}u_{s,2})\xi_2 &= c\kappa_3 \end{aligned} \tag{6.9}$$

where $\kappa_3 := c[\rho u^2/2 + \rho e] - [\rho u^3/2 + \rho e u + p_0 u]$.

The system (6.7-6.9) is a degenerate case of (5.2) when it is impossible to find $U_{s,1}$ and $U_{s,2}$ such that there exists a non-negative solution (ξ_1, ξ_2) with $D_s^i =$

$u_{s,2} - u_{s,1} \neq 0$ (i.e. Theorem 5.1 can not be used) because the second equation in (6.7) and the first one in (6.8) imply $c\kappa_1 = \kappa_2$. After some straightforward calculations, one sees that the only possibility is to put one of the following

- $u_{s,1} = u_{s,2} = c$
- $u_{s,1} = c$, $u_{s,2} = \xi_2 = 0$ or $u_{s,2} = c$, $u_{s,1} = \xi_1 = 0$.

In all the cases the corresponding SDWs are practically the same (up to a choice of a_ε and b_ε) and one can treat them in an exactly same way. We choose the first one in the sequel.

If $\rho_0 \neq \rho_1$, then the relation $c\kappa_1 = \kappa_2$ determines the speed

$$c = \frac{\rho_1 u_1 - \rho_0 u_0 + |u_0 - u_1|\sqrt{\rho_0\rho_1}}{\rho_1 - \rho_0}. \tag{6.10}$$

As $\kappa_1 \geq 0$ (ξ_1 and ξ_2 represents densities and they are non-negative), we took the plus sign in the expression for c above. In the special case $\rho_0 = \rho_1$ we have $c = (u_0 + u_1)/2$. Once c is known all the RH deficits κ_i, $i = 1, 2, 3$ are also determined.

It is now clear that one of ξ_1 and ξ_2 may be chosen freely, say ξ_2. The next step is to equal both values for ξ_1 from (6.7) and (6.9), and solve the obtained equation

$$\kappa_1 - \xi_2 = \frac{2\kappa_3 - \kappa_2(2e_{s,2} + c^2)\xi_2}{2e_{s,1} + c^2}$$

for $e_{s,1}$, $e_{s,2}$. The choice of $e_{s,1}$ and $e_{s,2}$ will be made to satisfy the entropy condition. The obtained SDW solutions always satisfies the second entropy condition (4.3) for functions of the form (6.4) and any p_0 because of the following two reasons:

- Assumption 3.1 implies that $a_\varepsilon \rho_{1,\varepsilon}$ and $b_\varepsilon \rho_{2,\varepsilon}$ are bounded.
- The above SDW construction implies $u_{1,\varepsilon}, u_{2,\varepsilon} \to c$ as $\varepsilon \to 0$.

Let us now prove that (4.2) is also satisfied for appropriate $e_{s,1}$ and $e_{s,2}$.

Case I Take $p_0 \equiv 0$ and the entropy pair (6.5) first. Let us check the first entropy condition (4.2). Denote

$$\partial_t \eta(\rho, u, e) + \partial_x q(\rho, u, e) =: I_1 + I_2,$$

where

$I_1 = (u_1 - c)\rho_1\big(R(u_1) + S(e_1)\big) + (c - u_0)\rho_0\big(R(u_0) + S(e_0)\big)$ (Classical part)

$I_2 = (\xi_1 + \xi_2)R(c) + \xi_1 S(e_{1,s}) + \xi_2 S(e_{2,s})$ (Delta part).

Take for a moment $R(u) \equiv 0$, i.e. the entropy pair does not depend on u (that could be a real physical situation as described in [21], see the exact references above).

In a moment it will be clear why we take a particular solution for $e_{1,s}$ and $e_{2,s}$ in (6.9) to be

$$e_{1,s} = e_{2,s} = \frac{\kappa_3}{\kappa_1} - \frac{c^2}{2}.$$

The equation (6.7) now implies

$$I_2 = \kappa_1 S\Big(\frac{\kappa_3}{\kappa_1} - \frac{c^2}{2}\Big).$$

Using (6.10) and the assumption $u_0 \geq c \geq u_1$, $u_0 > u_1$ we have

$$c = \frac{\rho_1 u_1 - \rho_0 u_0 + (u_0 - u_1)\sqrt{\rho_0\rho_1}}{\rho_1 - \rho_0}.$$

Substituting values for κ_1 and κ_3 defined at the beginning of the proof, we have

$$\begin{aligned}\frac{\kappa_3}{\kappa_1}-\frac{c^2}{2}=&\frac{\rho_1(c-u_1)}{\rho_1(c-u_1)+\rho_0(u_0-c)}e_1+\frac{\rho_0(u_0-c)}{\rho_1(c-u_1)+\rho_0(u_0-c)}e_0\\&+\frac{c[\rho u^2/2]-[\rho u^3/2]}{\kappa_1}-\frac{c^2}{2}.\end{aligned}$$

We need to prove

$$A:=\frac{c[\rho u^2/2]-[\rho u^3/2]}{\kappa_1}-\frac{c^2}{2}\geq 0$$

in order to use semi-convexity of the function S. It is enough to prove

$$\begin{aligned}\kappa_1 A=&\frac{c}{2}(\rho_1u_1^2-\rho_0u_0^2)-\frac{1}{2}(\rho_1u_1^3-\rho_0u_0^3)\\&-\frac{c^3}{2}(\rho_1-\rho_0)+\frac{c^2}{2}(\rho_1u_1-\rho_0u_0)\\=&-\frac{\rho_1}{2}(c-u_1)^2(u_1+c)+\frac{\rho_0}{2}(u_0-c)^2(u_0+c)\geq 0,\end{aligned}$$

since $\kappa_1=(u_0-u_1)\sqrt{\rho_0\rho_1}\geq 0$. The speed c is uniquely determined by (6.10), so

$$\begin{aligned}(c-u_1)=&\frac{u_0-u_1}{\rho_1-\rho_0}\sqrt{\rho_0}(\sqrt{\rho_1}-\sqrt{\rho_0})\geq 0,\ \text{and}\\(u_0-c)=&\frac{u_0-u_1}{\rho_1-\rho_0}\sqrt{\rho_1}(\sqrt{\rho_1}-\sqrt{\rho_0})\geq 0.\end{aligned}$$

Substitution of these relations in the expression for $\kappa_1 A$ gives

$$\kappa_1 A=\frac{\rho_0\rho_1}{2}\Big(\frac{u_0-u_1}{\rho_1-\rho_0}(\sqrt{\rho_1}-\sqrt{\rho_0})\Big)^2(u_0-u_1),$$

and that is always non-negative since $u_0>u_1$.

If $u_0\geq c\geq u_1$, then $\rho_1(c-u_1)\geq 0$, $\rho_0(u_0-c)\geq 0$ and we can write

$$\frac{\kappa_3}{\kappa_1}-\frac{c^2}{2}=\alpha e_0+(1-\alpha)e_1+A,$$

where

$$\alpha=\frac{\rho_1(c-u_1)}{\rho_1(c-u_1)+\rho_0(u_0-c)}.$$

Then

$$I_2/\kappa_1=S(\alpha e_0+(1-\alpha)e_1+A)\leq\alpha S(e_0)+(1-\alpha)S(e_1)=-I_1/\kappa_1,$$

because $S'\leq 0$ and S is semi-convex. The last relation means $I_1+I_2\leq 0$ and the entropy condition is satisfied.

One can see that the assumption $e_{1,s}=e_{2,s}=\kappa_3/\kappa_1-c^2/2$ and the condition $u_0\geq c\geq u_1$ are necessary conditions for the entropy inequality (4.1) to be true for an arbitrary semi-convex η: The second condition simply means that the entropy SDW is overcompressive.

Let us return to the general case when $R=R(u)$ is semi-convex and $R''\geq 0$. We have to prove that $\tilde{I}_1+\tilde{I}_2\leq 0$, where

$$\begin{aligned}\tilde{I}_1=&(u_1-c)\rho_1R(u_1)+(c-u_0)\rho_0R(u_0)\\\tilde{I}_2=&(\xi_1+\xi_2)R(c)=\kappa_1R(c)\end{aligned}$$

Use of the semi-convexity of R gives

$$\begin{aligned} R(c) =& R\Big(\frac{\sqrt{\rho_0\rho_1}-\rho_0}{\rho_1-\rho_0}u_0+\frac{\rho_1-\sqrt{\rho_0\rho_1}}{\rho_1-\rho_0}u_1\Big)\\ \leq& \frac{\sqrt{\rho_0\rho_1}-\rho_0}{\rho_1-\rho_0}R(u_0)+\frac{\rho_1-\sqrt{\rho_0\rho_1}}{\rho_1-\rho_0}R(u_1)=:\tilde{I}_3. \end{aligned}$$

Since the first RH deficit satisfies $\kappa_1 \geq 0$, $\tilde{I}_1 = -\kappa_1\tilde{I}_3$, $\tilde{I}_2 = \kappa_1 R(c)$, we have $\tilde{I}_1+\tilde{I}_2 \leq 0$ that concludes the case $p_0 \equiv 0$.

Case II Take $p_0 = 1$. Now, the entropy pair is given by (6.6), and

$$\begin{aligned} I_1 =& (u_1-c)\rho_1 S(u_1,e_1+1/\rho_1)+(c-u_0)\rho_0 S(u_0,e_0+1/\rho_0)\\ I_2 =& \xi_1 S(u_{s,1},e_{s,1})+\xi_0 S(u_{s,0},e_{s,0}). \end{aligned}$$

We already know that $u_{s,1} = u_{s,2} = c$. As in the previous case assume $u_0 > u_1$, $u_0 \geq c \geq u_1$ and put $e_{1,s} = e_{2,s} = (\kappa_3 - c^2\kappa_1/2)/\kappa_1$. Again, one will see that the choice is the unique if one wants (4.2,4.3) to hold for each entropy pair with semi-convex η. Now

$$I_2 = \kappa_1 S\Big(c,\frac{\kappa_3}{\kappa_1}-\frac{c^2}{2}\Big).$$

The equality

$$\frac{\sqrt{\rho_0\rho_1}-\rho_0}{\rho_1-\rho_0} = \frac{\rho_0 u_0 - c\rho_0}{\kappa_1}$$

implies

$$\begin{aligned} c =& \frac{\sqrt{\rho_0\rho_1}-\rho_0}{\rho_1-\rho_0}u_0+\frac{\rho_1-\sqrt{\rho_0\rho_1}}{\rho_1-\rho_0}u_1, \text{ and}\\ e_{s,1} =& \frac{\sqrt{\rho_0\rho_1}-\rho_0}{\rho_1-\rho_0}(e_0+1/\rho_0)+\frac{\rho_1-\sqrt{\rho_0\rho_1}}{\rho_1-\rho_0}(e_1+1/\rho_1)+A, \end{aligned}$$

where A is a non-negative constant from above.

Since S is semi-convex and $\partial_2 S \leq 0$, we have

$$\begin{aligned} S(c,e_{s,1}) \leq& S\Big(\frac{\sqrt{\rho_0\rho_1}-\rho_0}{\rho_1-\rho_0}u_0+\frac{\rho_1-\sqrt{\rho_0\rho_1}}{\rho_1-\rho_0}u_1,\\ & \frac{\sqrt{\rho_0\rho_1}-\rho_0}{\rho_1-\rho_0}(e_0+1/\rho_0)+\frac{\rho_1-\sqrt{\rho_0\rho_1}}{\rho_1-\rho_0}(e_1+1/\rho_1)\Big)\\ \leq& \frac{\sqrt{\rho_0\rho_1}-\rho_0}{\rho_1-\rho_0}S(u_0,e_0+1/\rho_0)+\frac{\rho_1-\sqrt{\rho_0\rho_1}}{\rho_1-\rho_0}S(u_1,e_1+1/\rho_1). \end{aligned}$$

Thus,

$$\frac{1}{\kappa_1}(I_1+I_2) \leq 0$$

and the proof is finished. Again, the SDW solution satisfies the entropy conditions (4.2, 4.3) for each semi-convex entropy η if and only if it is overcompressive.

Uniqueness. One could see from the above that c is always uniquely determined and that is the first condition for the weak uniqueness. That implies uniqueness of all RH deficits for fixed left- and right-handed states. Then, the first equation in (6.7) implies

$$\lim_{\varepsilon\to 0} a_\varepsilon\rho_{1,\varepsilon}+b_\varepsilon\rho_{2,\varepsilon} = \kappa_1,$$

for any representatives a_ε, b_ε, $\rho_{1,\varepsilon}$ and $\rho_{2,\varepsilon}$. That is precisely the second condition of the weak uniqueness (Definition 4.1). Thus the distributional images of all SDWs which solves the given system are the same and the obtained SDW solution is weakly unique. □

Remark 6.1. One can easily see that the obtained result is consistent with the ones obtained in [2], [4] and [10] in the sense that ρ_ε and u_ε weakly (in the distribution sense) converges to the solutions given in these papers.

Part 2. Interaction problem

Let us briefly list the main differences between a simple SDW and a constant SDW needed in an interaction problem. Everything is based on the fact that we now have a delta function added to the initial data.

- The shifts $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$ in (2.1) are needed when delta (singular) shock is involved in wave interaction. A sum of strengths of incoming SDWs is controlled by $x_{1,\varepsilon} + x_{2,\varepsilon}$.
- In the case of delta (singular) shocks of constant strength (see [17] about such waves) or in the case of delta contact discontinuities (see [18] for their definition) one has $a_\varepsilon(t) = -a_\varepsilon$ and $b_\varepsilon(t) = b_\varepsilon$ in (2.1), where a_ε and b_ε are some non-negative constants.
- In the case of delta (singular) shock with decreasing strength (see [17] again) the lines $a_\varepsilon(t) - x_{1,\varepsilon}$ and $b_\varepsilon(t) + x_{2,\varepsilon}$ meets the line ct at a point (X_1, T_1): $cT_1 = a_\varepsilon(T_1) - x_{1,\varepsilon} + o(\varepsilon)$ and $cT_1 = b_\varepsilon(T_1) + x_{2,\varepsilon} + o(\varepsilon)$. Then the relation (3.2) is valid for $t < T_1$. We can continue a solution after $t = T_1$ by solving Riemann problem again (translated to the point (X_1, T_1), of course).
- We have used only simple SDWs for the Riemann problem. Let us explain a genesis of SDWs with variable speed (2.1). Its intermediate states are constant for each ε, so the only way for speed to become non-constant is that U_0 or U_1 are non-constant. That would be the case when SDW interact with a rarefaction wave. If that interaction takes place, we divide the rarefaction wave into a fan of non-entropic shocks. Like in the wave front tracking algorithm (see [3]), substitution of the fan into system gives a term associated with zero. Then we shall look at interaction of the shadow wave with the elements of the fan. At least for some time (before the shadow wave changes its nature, i.e. before it ceases to be entropic), the approximated solution consists of a number of simple SDWs of the form (2.1) with a constant speed defined in small time intervals. Let $T = T_0 < T_1 < \ldots < T_N$ be the end points of that time intervals, and $c_1, \ldots, c_N$ be the resulting wave speeds there. Note that all of c_i and T_i, $i = 1, \ldots, N$ depend on ε. Then a continuous curve
$$c_\varepsilon(t) = \begin{cases} c_1 t + const_1, & t \in [T_0, T_1] \\ c_2 t + const_2, & t \in [T_1, T_2] \\ & \ldots \\ c_N t + const_N, & t \in [T_{N-1}, T_N] \end{cases}$$
is an approximation of the resulting wave trajectory. Taking a limit as the strength of non-entropic shocks goes to zero we get the function described by the form (2.1) where U_0 or U_1 is non-constant (depending on which side

of SDW is rarefaction). In a lot of systems a limit of such curves as $\varepsilon \to 0$ can be explicitly calculated (see [17] for singular shock and [18] for delta shocks).

So we start this part of the paper with the key assertion about SDWs continuation after interaction time. An important fact is that the assertion will be true for weighted SDWs used in the third part, too.

7. The basic formula for constant SDWs

The present section is devoted to finding a possibilities to join SDW(s) existing before the interaction time with a solution candidate afterward. That will give us conditions for SDW solution existence after the interaction.

Suppose that two SDW solutions to (1.1)

$$\tilde{U}_\varepsilon(x,t) = \begin{cases} U_0, & x - a < (\tilde{c} - \tilde{a}_\varepsilon)t \\ \tilde{U}_{1,\varepsilon}, & (\tilde{c} - \tilde{a}_\varepsilon)t < x - a < \tilde{c}t \\ \tilde{U}_{2,\varepsilon}, & \tilde{c}t < x - a < (\tilde{c} + \tilde{b}_\varepsilon)t \\ U_1, & x - a > (\tilde{c} + \tilde{b}_\varepsilon)t \end{cases}$$

and

$$\hat{U}_\varepsilon(x,t) = \begin{cases} U_1, & x - b < (\hat{c} - \hat{a}_\varepsilon)t \\ \hat{U}_{1,\varepsilon}, & (\hat{c} - \hat{a}_\varepsilon)t < x - b < \hat{c}t \\ \hat{U}_{2,\varepsilon}, & \hat{c}t < x - b < (\hat{c} + \hat{b}_\varepsilon)t \\ U_2, & x - b > (\hat{c} + \hat{b}_\varepsilon)t \end{cases}$$

meat each other (when $\tilde{c} > \hat{c}$ and $a < b$). Denote by (X,T) the intersection point of the external SDW lines $x = a + (\tilde{c} + \tilde{b}_\varepsilon)t$ and $x = b + (\hat{c} - \hat{a}_\varepsilon)t$, i.e. $X = a + (\tilde{c} + \tilde{b}_\varepsilon)T = b + (\hat{c} - \hat{a}_\varepsilon)T$ and $T = (b-a)/(\tilde{c} - \hat{c} + \hat{a}_\varepsilon + \tilde{b}_\varepsilon)$. At the time $t = T$ a distributional limit of solution is a sum of a classical piecewise constant function and a delta function supported by the interaction point. So, it is natural to ask ourselves a question: When the interaction produces a shadow wave solution for $t > T$? In order to answer that question we shall use the following assumptions.

Assumption 7.1. There exist a shadow wave solution

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x < (c - a_\varepsilon)t \\ U_{1,\varepsilon}, & (c - a_\varepsilon)t < x < ct \\ U_{2,\varepsilon}, & ct < x < (c + b_\varepsilon)t \\ U_2, & x > (c + b_\varepsilon)t \end{cases}$$

to (1.1) with the initial data

$$U(x,0) = \begin{cases} U_0, & x < 0 \\ U_2, & x > 0. \end{cases} \tag{7.1}$$

Denote by $\hat{\kappa}, \tilde{\kappa}, \kappa \in \mathbb{R}^n$ the Rankine-Hugoniot deficits corresponding to $\hat{U}$, $\tilde{U}$ and U, respectively. The relation (3.4) imply that $\tilde{a}_\varepsilon f(\tilde{U}_{1,\varepsilon})T + \tilde{b}_\varepsilon f(\tilde{U}_{2,\varepsilon})T \approx T\tilde{\kappa}$ and $\hat{a}_\varepsilon f(\hat{U}_{1,\varepsilon})T + \hat{b}_\varepsilon f(\hat{U}_{2,\varepsilon})T \approx T\hat{\kappa}$.

Assumption 7.2. It is possible to chose $\alpha \geq 0$ such that $\alpha\kappa = T(\tilde{\kappa} + \hat{\kappa})$.

Let us analyze these assumptions. Let $x_{1,\varepsilon}, x_{2,\varepsilon} = \mathcal{O}(\varepsilon)$ be non-negative numbers for ε small enough. We have seen that the system (1.1) has a SDW solution if (3.3) holds. Using Lemma 2.1 one can see that the function

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x < (c - a_\varepsilon)t - x_{1,\varepsilon} \\ U_{1,\varepsilon}, & (c-a_\varepsilon)t - x_{1,\varepsilon} < x < ct \\ U_{2,\varepsilon}, & ct < x < (c+b_\varepsilon)t + x_{2,\varepsilon} \\ U_2, & x > (c+b_\varepsilon)t + x_{2,\varepsilon} \end{cases}$$

is a SDW solution if

$$\begin{aligned} -c(f^i(U_2) - f^i(U_0)) + a_\varepsilon f^i(U_{1,\varepsilon}) + b_\varepsilon f^i(U_{2,\varepsilon}) + g^i(U_2) - g^i(U_0) &\approx 0 \\ -c(a_\varepsilon f^i(U_{1,\varepsilon}) + b_\varepsilon f^i(U_{2,\varepsilon})) + a_\varepsilon g^i(U_{1,\varepsilon}) + b_\varepsilon g^i(U_{2,\varepsilon}) &\approx 0 \\ -c(x_{1,\varepsilon} f^i(U_{1,\varepsilon}) + x_{2,\varepsilon} f^i(U_{2,\varepsilon})) + x_{1,\varepsilon} g^i(U_{1,\varepsilon}) + x_{2,\varepsilon} g^i(U_{2,\varepsilon}) &\approx 0. \end{aligned}$$

The first two equations in the above system are the same as in (3.3). Their left-hand sides are terms which multiply δ and $t\delta'$ in the i-th equation of the system (1.1). The left-hand side of the last equation above is a term which multiply δ' and it does not appear when SDW is of the form (3.1).

Since (3.3) is solvable by Assumption 7.1, the first two relations are satisfied. Also, one can see easily that $(x_{1,\varepsilon}, x_{2,\varepsilon}) = \alpha(a_\varepsilon, b_\varepsilon)$ solves the last equation for any real α in that case.

Assumption 7.2 could be relaxed when dealing with specific systems. The choice of $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$ need not always be proportional to a_ε and b_ε in that case. As many times in the paper, we put restrictive assumptions in order to treat fairly general systems.

Define

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x - X < (c-a_\varepsilon)(t-T) - x_{1,\varepsilon},\ t > T \\ U_{1,\varepsilon}, & (c-a_\varepsilon)(t-T) - x_{1,\varepsilon} < x - X < c(t-T),\ t > T \\ U_{2,\varepsilon}, & c(t-T) < x - X < (c+b_\varepsilon)(t-T) + x_{2,\varepsilon},\ t > T \\ U_2, & x - X > (c+b_\varepsilon)t + x_{2,\varepsilon},\ t > T. \end{cases} \tag{7.2}$$

The solution before interaction is given by

$$(\tilde{U}_\varepsilon \wedge \hat{U}_\varepsilon)(x,t) = \begin{cases} U_0, & x < (\tilde{c} - \tilde{a}_\varepsilon)t + a,\ t < T \\ \tilde{U}_{1,\varepsilon}, & (\tilde{c} - \tilde{a}_\varepsilon)t + a < x < \tilde{c}t + a,\ t < T \\ \tilde{U}_{2,\varepsilon}, & \tilde{c}t + a < x < (\tilde{c} + \tilde{b}_\varepsilon)t + a,\ t < T \\ U_1, & (\tilde{c} + \tilde{b}_\varepsilon)t + a < x < (\hat{c} - \hat{a}_\varepsilon)t + b,\ t < T \\ \hat{U}_{1,\varepsilon}, & (\hat{c} - \hat{a}_\varepsilon)t + b < x < \hat{c}t + b,\ t < T \\ \hat{U}_{2,\varepsilon}, & \hat{c}t + b < x < (\hat{c} + \hat{b}_\varepsilon)t + b,\ t < T \\ U_2, & x > (\hat{c} + \hat{b}_\varepsilon)t + b,\ t < T. \end{cases}$$

The anticipated solution V_ε is obtained by gluing the solution for $t < T$ (denoted by $\tilde{U}_\varepsilon \wedge \hat{U}_\varepsilon$) with the one defined by (7.2) for $t > T$ (denoted by U_ε):

$$V_\varepsilon(x,t) = \begin{cases} (\tilde{U}_\varepsilon \wedge \hat{U}_\varepsilon)(x,t), & t < T \\ U_\varepsilon(x,t), & t > T, \end{cases} \tag{7.3}$$

Theorem 7.1. *If Assumptions 7.1 and 7.2 hold, then*

$$\partial_t f(V_\varepsilon) + \partial_x g(V_\varepsilon) \approx 0,$$

where V_ε is defined in (7.3).

Proof. Denote by $\vec{n}_0 = (0,1)$ the unit normal to the line $t = T$ and define

$$\begin{aligned}
\gamma_1 &:= \{(x,t),\ t = T, x - a \leq (\tilde{c} - \tilde{a}_\varepsilon)T\} \\
\gamma_2 &:= \gamma_2' \cup \gamma_2'',\ \text{where} \\
&\quad \gamma_2' := \{(x,t),\ t = T, (\tilde{c} - \tilde{a}_\varepsilon)T \leq x - a \leq \tilde{c}T\} \\
&\quad \gamma_2'' := \{(x,t),\ t = T, \tilde{c}T < x - a < (\tilde{c} + \tilde{b}_\varepsilon)T\} \\
\gamma_3 &:= \gamma_3' \cup \gamma_3''\ \text{where} \\
&\quad \gamma_3' := \{(x,t),\ t = T, (\hat{c} - \hat{a})T \leq x - b \leq \hat{c}T\} \\
&\quad \gamma_3'' := \{(x,t),\ t = T, \hat{c}T \leq x - b \leq (\hat{c} + \hat{b}_\varepsilon)T\} \\
\gamma_4 &:= \{(x,t),\ t = T, x - b \geq (\hat{c} + \hat{b}_\varepsilon)T\} \\
\gamma_5 &:= \{(x,t),\ t = T, x \leq X - x_{1,\varepsilon}\} \\
\gamma_6 &:= \{(x,t),\ t = T, X - x_{1,\varepsilon} \leq x \leq X + x_{2,\varepsilon}\} \\
\gamma_7 &:= \{(x,t),\ t = T, x \geq X + x_{2,\varepsilon}\}.
\end{aligned}$$

The function $\tilde{U}_\varepsilon \wedge \hat{U}_\varepsilon$ is an approximate solution to (1.1) for $t < T$. Assumption 7.1 implies that U_ε is also an approximate solution to the same system for $t > T$.

Let $\gamma \subset \mathbb{R}^2_+$ be the union of discontinuity curves for both $\tilde{U}_\varepsilon \wedge \hat{U}_\varepsilon$ and U_ε, and $\omega = \mathbb{R}^2_+ \setminus \gamma$. The Divergence Theorem implies

$$\begin{aligned}
\int f(V_\varepsilon)\partial_t\phi + g(V_\varepsilon)\partial_x\phi\, dx = &\int_\gamma (g,f)(V_\varepsilon)\cdot \vec{n}\,\phi\, ds \\
&- \int_{(x,t)\in\omega, t<T} \partial_t f(\tilde{U}_\varepsilon \wedge \hat{U}_\varepsilon)\,\phi + \partial_x g(\tilde{U}_\varepsilon \wedge \hat{U}_\varepsilon)\,\phi\, dx \\
&- \int_{(x,t)\in\omega, t>T} \partial_t f(U_\varepsilon)\,\phi + \partial_x g(U_\varepsilon)\,\phi\, dx.
\end{aligned}$$

The integrals over ω and the line integrals over $\gamma \setminus \cup_{i=1}^7 \gamma_i$ converge to zero because both of $\tilde{U}_\varepsilon \wedge \hat{U}_\varepsilon$ and U_ε are approximated solutions to the system below and above the line $t = T$. The only fact which was left unproved is

$$\begin{aligned}
&\int_{\gamma_1} (g,f)(\tilde{U}_\varepsilon)\cdot\vec{n}_0\,\phi\, ds + \int_{\gamma_2} (g,f)(\tilde{U}_\varepsilon)\cdot\vec{n}_0\phi\, ds \\
&+ \int_{\gamma_3} (g,f)(\hat{U}_\varepsilon)\cdot\vec{n}_0\,\phi\, ds + \int_{\gamma_4} (g,f)(\hat{U}_\varepsilon)\cdot\vec{n}_0\,\phi\, ds \\
&- \int_{\gamma_5} (g,f)(U_\varepsilon)\cdot\vec{n}_0\,\phi\, ds - \int_{\gamma_6} (g,f)(U_\varepsilon)\cdot\vec{n}_0\,\phi\, ds - \int_{\gamma_7} (g,f)(U_\varepsilon)\cdot\vec{n}_0\,\phi\, ds \approx 0.
\end{aligned}$$

Note that $\tilde{U}_\varepsilon$ and U_ε are equal on the set $\gamma_1 \cap \gamma_5$, while $\hat{U}_\varepsilon$ and U_ε are equal on the set $\gamma_4 \cap \gamma_7$. Also, one-dimensional Lebesgue measure of the sets $(\gamma_1 \setminus \gamma_5) \cup (\gamma_5 \setminus \gamma_1)$ and $(\gamma_4 \setminus \gamma_7) \cup (\gamma_7 \setminus \gamma_4)$ tends to zero as $\varepsilon \to 0$ (because all of a_ε, b_ε, $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$ tend to zero as $\varepsilon \to 0$) while V_ε is bounded on these sets. Thus, one has only to prove that

$$\int_{\gamma_2} f(\tilde{U}_\varepsilon)\,\phi\, ds + \int_{\gamma_3} f(\hat{U}_\varepsilon)\,\phi\, ds \approx \int_{\gamma_6} f(U_\varepsilon)\,\phi\, ds.$$

But, that is a direct consequence of Assumption 7.2:

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\gamma_2} f(\tilde{U}_\varepsilon)\,\phi\,ds &= \int_{\gamma_2'} f(\tilde{U}_\varepsilon)\,\phi\,ds + \int_{\gamma_2''} f(\tilde{U}_\varepsilon)\,\phi\,ds \\ &= \lim_{\varepsilon\to 0}\big(\tilde{a}_\varepsilon f(\tilde{U}_{1,\varepsilon}) + \tilde{b}_\varepsilon f(\tilde{U}_{2,\varepsilon})\big)T\phi(X,T) = \tilde{\kappa}T\phi(X,T),\end{aligned}$$

and likewise

$$\lim_{\varepsilon\to 0}\int_{\gamma_3} f(\hat{U}_\varepsilon)\,\phi\,ds = \hat{\kappa}T\phi(X,T).$$

On the other hand

$$\lim_{\varepsilon\to 0}\int_{\gamma_0} f(\tilde{U}_\varepsilon)\,\phi\,ds = \lim_{\varepsilon\to 0}\big(x_{1,\varepsilon} f(U_{1,\varepsilon}) + x_{2,\varepsilon} f(U_{2,\varepsilon})\big)\phi(X,T) = \alpha\kappa\phi(X,T),$$

and put $\alpha = T(\hat{\kappa}+\tilde{\kappa})/\kappa$. That concludes the proof. □

Let us note that the main assumptions of the theorem are that there exists an SDW solution to (1.1) with the initial data (7.1), and that there exist shifts $x_{i,\varepsilon}$, $i = 1, 2$, described above. Let us explain what could happen if these assumptions are not satisfied.

If Assumption 7.1 is not satisfied, then there are some other possibilities for solving the above interaction problem. Maybe it is possible to insert some intermediate states $\tilde{U}_{m,j}$, $j = 1, \ldots, n-1$. between U_0 and U_2 in the area above $t = T$ so that shadow wave follow or be followed by some other elementary wave(s). One can look in [17] for a similar situation when an interaction of two singular shocks produces one of the following three wave combinations: a single singular shock, an 1-rarefaction wave followed by a singular shock or a singular shock followed by a 2-rarefaction wave.

If a system is given in the evolution form with only one major component, say U^1, then there is a good chance to find α from Assumption 7.2: All RH-deficits, $\tilde{\kappa}^i$, $\hat{\kappa}^i$, and κ^i, $i = 2, \ldots, n$ are zero, one puts $\alpha = T(\tilde{\kappa}^1 + \hat{\kappa}^1)/\kappa^1$ provided $\operatorname{sign}(\tilde{\kappa}^1 + \hat{\kappa}^1)\operatorname{sign}(\kappa^1) = 1$. An example with a negative result one can find in [14] for a 2×2 generalized pressureless gas dynamics model.

The above theorem applies also to the case when one of the incoming waves $\tilde{U}$ or $\hat{U}$ is a shock. The proof is the same with some obvious changes (for example, $\hat{a}_\varepsilon = \hat{b}_\varepsilon = 0$ if the second wave is a shock). Obviously, the assertion also holds if a contact discontinuity is in the place of the shock.

Finally, the above theorem is useful for dealing with shadow and rarefaction wave interaction as already announced in the introduction of this part. When a rarefaction wave is substituted by a fan of non-entropy shocks of small strength (which solve the system in an approximated sense – see [3], for example), then the above theorem can be applied on interaction of an SDW and such non-entropy shock. After each such interaction we obtain a solution in the fan-form containing at least one SDW of the type (2.1) with $x_{1,\varepsilon}^2 + x_{2,\varepsilon}^2 > 0$ and the procedure can be continued. A trajectory of a resulting SDW is a broken line

$$\cup_{i=1}^m \{(c_i t + \alpha_i, t),\ t \in [T_{i-1}, T_i],\ \alpha_i \in \mathbb{R}\},$$

where T_i, $i = 1, \ldots, m$ are time coordinates of interaction points. One can find specific cases in [17] and [18] when it is possible to find an SDW central line $(c(t), t)$ as limit of the above trajectories by solving a governing ODE. Note that the resulting SDW can be of different nature, for example, with a constant or decreasing strength. That fact opens a door for solving the complete interaction problem in similar cases to the ones in that papers.

Remark 7.1. Suppose that the assumptions of the above theorem are true. It is easy to see what is the entropy condition for V_ε in that case. Let (η, q) be an entropy and entropy-flux pair for the system (1.1). Suppose that the functions $\tilde{U}_\varepsilon$, $\hat{U}_\varepsilon$ and U_ε satisfy the entropy inequality in their domains. Then one can see that the entropy condition is fulfilled if

$$\overline{\lim}_{\varepsilon\to 0}\ \Big(\int_{\gamma_6}\eta(U_\varepsilon)\phi - \int_{\gamma_2}\eta(\tilde{U}_\varepsilon)\phi - \int_{\gamma_3}\eta(\hat{U}_\varepsilon)\phi\Big) \le 0, \tag{7.4}$$

for every non-negative $\phi \in \mathcal{C}_0^\infty$. Due to the construction of V_ε we have

$$\eta(\tilde{U}_\varepsilon) = \begin{cases}\eta(\tilde{U}_{1,\varepsilon}), & (x,t)\in\gamma_2' \\ \eta(\tilde{U}_{2,\varepsilon}), & (x,t)\in\gamma_2''\end{cases},\ \eta(\hat{U}_\varepsilon) = \begin{cases}\eta(\hat{U}_{1,\varepsilon}), & (x,t)\in\gamma_3' \\ \eta(\hat{U}_{2,\varepsilon}), & (x,t)\in\gamma_3''\end{cases},$$

where $\gamma_2 = \gamma_2' \cup \gamma_2''$, $\gamma_3 = \gamma_3' \cup \gamma_3''$,

$$\mathrm{meas}(\gamma_2') = T\tilde{a}_\varepsilon, \mathrm{meas}(\gamma_2'') = T\tilde{b}_\varepsilon, \mathrm{meas}(\gamma_3') = T\hat{a}_\varepsilon, \mathrm{meas}(\gamma_3'') = T\hat{b}_\varepsilon.$$

Relation (7.4) is satisfied if and only if

$$\begin{aligned}&\overline{\lim}_{\varepsilon\to 0}\ (x_{1,\varepsilon}\eta(U_{1,\varepsilon}) + x_{2,\varepsilon}\eta(U_{2,\varepsilon})) \\ &\quad - \Big(\tilde{a}_\varepsilon\eta(\tilde{U}_{1,\varepsilon}) + \tilde{b}_\varepsilon\eta(\tilde{U}_{2,\varepsilon}) + \hat{a}_\varepsilon\eta(\hat{U}_{1,\varepsilon}) + \hat{b}_\varepsilon\eta(\hat{U}_{2,\varepsilon})\Big)T \le 0.\end{aligned} \tag{7.5}$$

Definition 7.1. The set of all states U_2 such that there exists (entropic) shadow wave solution to an interaction problem connecting the states U_0, U_1 and U_2 is called the *second (entropic) shadow locus*. We shall call it the *forward locus*, and the set of all states U_0 satisfying the above with U_1 and U_2 fixed will be then called the *backward shadow locus*.

7.1. **Pressureless gas dynamics.** Let us look at a shadow wave interaction problem for the system (6.1) and $p_0 \equiv 1$. The results for $p_0 \equiv 0$ are practically the same. Suppose that two shadow waves interact in the point (X, T). Entropy condition implies that that is possible only if $u_0 > u_1 > u_2$, where the first shadow wave joins (ρ_0, u_0, e_0) with (ρ_1, u_1, e_1), while the second joins (ρ_1, u_1, e_1) with (ρ_2, u_2, e_2).

We immediately see from Theorem 6.1 that (ρ_0, u_0, e_0) and (ρ_2, u_2, e_2) can be joined by an entropy shadow wave since $u_0 > u_2$. Thus Assumption 7.1 is satisfied. Let us look at Assumption 7.2 using the notation from Theorem 7.1.

In Theorem 6.1 we found that $u_{s,1} = u_{s,2} = c$, $\tilde{u}_{s,1} = \tilde{u}_{s,2} = \tilde{c}$ and $\hat{u}_{s,1} = \hat{u}_{s,2} = \hat{c}$. Denote $\lim_\varepsilon \tilde{a}_\varepsilon\tilde{\rho}_{1,\varepsilon} = \tilde{\xi}_1$, $\lim_\varepsilon \tilde{b}_\varepsilon\tilde{\rho}_{2,\varepsilon} = \tilde{\xi}_2$, $\lim_\varepsilon \hat{a}_\varepsilon\hat{\rho}_{1,\varepsilon} = \hat{\xi}_1$, $\lim_\varepsilon \hat{b}_\varepsilon\hat{\rho}_{2,\varepsilon} = \hat{\xi}_2$, $\lim_\varepsilon a_\varepsilon\rho_{1,\varepsilon} = \xi_1$, and $\lim_\varepsilon b_\varepsilon\rho_{2,\varepsilon} = \xi_2$. Also, the variables $e_{s,1} = e_{s,2}$, $\tilde{e}_{s,1} = \tilde{e}_{s,2}$ and $\hat{e}_{s,1} = \hat{e}_{s,2}$ are uniquely determined there.

The condition in Assumption 7.2 says that there exists a real α such that

$$\alpha\begin{bmatrix}\kappa_1 \\ c\kappa_1 \\ \kappa_3\end{bmatrix} = T\begin{bmatrix}\tilde{\kappa}_1 + \hat{\kappa}_1 \\ \tilde{c}\tilde{\kappa}_1 + \hat{c}\hat{\kappa}_1 \\ \tilde{\kappa}_3 + \hat{\kappa}_3\end{bmatrix}$$

From the first two equations of the above system one can see that the speed of the resulting wave is determined by the data in before-interaction waves:

$$c = \frac{\tilde{c}\tilde{\kappa}_1 + \hat{c}\hat{\kappa}_1}{\tilde{\kappa}_1 + \hat{\kappa}_1} = \frac{\tilde{c}(\tilde{\xi}_1 + \tilde{\xi}_2) + \hat{c}(\hat{\xi}_1 + \hat{\xi}_2)}{\tilde{\xi}_1 + \tilde{\xi}_2 + \hat{\xi}_1 + \hat{\xi}_2} =: \xi_\pi.$$

Using (6.10) we get the first condition for a second SDW locus (not involving variable e):

$$\frac{\rho_2 u_2 - \rho_0 u_0 + (u_0 - u_2)\sqrt{\rho_0 \rho_2}}{\rho_2 - \rho_0} = \xi_\pi, \ \rho_2 > 0.$$

Also, the constant α is determined by $\alpha := (\tilde{\kappa}_1 + \hat{\kappa}_1)T/\kappa_1$.

The third equation of the above system reduces to $\kappa_3(\tilde{\kappa}_1 + \hat{\kappa}_1) = \kappa_1(\tilde{\kappa}_3 + \hat{\kappa}_3)$, which is a linear equation with respect to e_2. So one can find easily a value for e_2 (we omit the exact formula since it does not carry any useful information). That was the second condition for the locus.

Thus, the second SDW locus is a just a curve and an SDW interaction problem cannot be solved with a usual SDWs in general. The complete solution will be given by using weighted SDWs in the fourth part.

Contrary to the above case, interaction problems for lot of 2×2 systems can be solved completely using the above notion of the second delta locus and constant SDWs. The original results are given in [17] and [18] and one can find a translation into SDW environment in the following part of the paper.

Part 3. **Some** 2×2 **systems revised**

In order to connect our results with the known ones about delta and singular shocks, we shall focus our attention on 2×2 systems. That means we look at systems having a delta or singular shock solution obtained by using different solution concepts (see the references given in the introduction). All the systems (except some artificial examples) from these papers have appropriate SDW solution converging to the known one in the distributional sense.

8. Delta shocks

8.1. **Non-evolutionary case.** Let a system be given in a general form (1.1) with $U = (u, v)$ where $\mathcal{C}^0$-functions f^i, g^i, $i = 1, 2$ are of at most linear growth with respect to variable v. We fix that notation trough this section. Assume that the following limits exist

$$\begin{aligned}
&\lim_{v\to\infty} \frac{f^i(u,v)}{v} = \overline{f}^i(u), \ \lim_{v\to-\infty} \frac{f^i(u,v)}{v} = \underline{f}^i(u), \\
&\lim_{v\to\infty} \frac{g^i(u,v)}{v} = \overline{g}^i(u), \ \lim_{v\to-\infty} \frac{g^i(u,v)}{v} = \underline{g}^i(u), \\
&\lim_{\varepsilon\to 0} a_\varepsilon v_{1,\varepsilon} = \xi_1, \ \lim_{\varepsilon\to 0} b_\varepsilon v_{2,\varepsilon} = \xi_2.
\end{aligned}$$

Also, assume $\xi_i \geq 0$, $i = 1, 2$. All other sign combinations of ξ_1 and ξ_2 can be treated in the same way as that one. Let us also assume that the limits $\lim_{\varepsilon\to 0} u_{j,\varepsilon} = u_{s,j}$ exist for $j = 1, 2$ and denote

$$f^i_j := \overline{f}^i(u_{s,j}), \ g^i_j := \overline{g}^i(u_{s,j}), \ \ i, j = 1, 2.$$

Now, the system (3.3) reduces to

$$\begin{aligned}
-c[f^1] + [g^1] + f^1_1\xi_1 + f^1_2\xi_2 &= 0 \\
-c[f^2] + [g^2] + f^2_1\xi_1 + f^2_2\xi_2 &= 0 \\
-c(f^1_1\xi_1 + f^1_2\xi_2) + g^1_1\xi_1 + g^1_2\xi_2 &= 0 \\
-c(f^2_1\xi_1 + f^2_2\xi_2) + g^2_1\xi_1 + g^2_2\xi_2 &= 0.
\end{aligned} \tag{8.1}$$

One can explicitly solve the first two equations in (8.1) with respect to ξ_1 and ξ_2:

$$\begin{aligned}\xi_1(u_{s,1},u_{s,2}) &= \frac{([g^1]-c[f^1])f_2^2+(c[f^2]-[g^2])f_1^2}{f_2^1f_1^2-f_1^1f_2^2}\\ \xi_2(u_{s,1},u_{s,2}) &= -\frac{([g^1]-c[f^1])f_1^2+(c[f^2]-[g^2])f_1^1}{f_2^1f_1^2-f_1^1f_2^2},\end{aligned}$$

provided $f_2^1f_1^2-f_1^1f_2^2\neq 0$. Then the last two equations in (8.1) are reduced to

$$\begin{aligned}c^2[f^1]-c[g^1] &= g_1^1\xi_1(u_{s,1},u_{s,2})+g_2^1\xi_2(u_{s,1},u_{s,2})\\ c^2[f^2]-c[g^2] &= g_1^2\xi_1(u_{s,1},u_{s,2})+g_2^2\xi_2(u_{s,1},u_{s,2}).\end{aligned}$$

Since there are two (nonlinear) equations with three unknowns, c, $u_{s,1}$ and $u_{s,2}$, one could expect that there is a solution to (8.1). Of course, existence depends on functions f^i and g^i, $i=1,2$, and the physical domain Ω as one will see during the presentation of particular cases bellow. Nevertheless, the shadow locus is usually not restricted to a curve, and possibilities for solving the system (1.1) by means of SDWs seams to be much richer than the procedure with generalized functions presented in [16]. In the latter case one could expect that all possible values for (u_2,v_2) belong to a curve.

Remark 8.1. Analogous situation appears in the case of $n\times n$ system when all of the functions f^i, g^i, $i=1,\dots,n$, are linear with respect to $V:=(U^2,\dots,U^n)$. The system (3.3) can be transferred into one very similar to (8.1): One eliminates variables $\xi_1^i:=\lim_{\varepsilon\to 0}a_\varepsilon U_{1,\varepsilon}^i$ and $\xi_2^i:=\lim_{\varepsilon\to 0}b_\varepsilon U_{2,\varepsilon}^i$, $i=2,\dots,n$, from the first $2(n-1)$ equations. After their substitution into the last two, one gets again two equations with three unknowns c, $U_{s,1}^1$ and $U_{s,2}^1$ which seems to be solvable in general.

8.2. **Evolutionary systems.** We shall use the notation from the previous case. Suppose now that $f^1(u,v)=u$ and $f^2(u,v)=v$. Instead of (8.1) we have

$$\begin{aligned}-c[f^1]+[g^1]&=0\\ -c[f^2]+[g^2]+\xi_1+\xi_2&=0\\ g_1^1\xi_1+g_2^1\xi_2&=0\\ -c(\xi_1+\xi_2)+g_1^2\xi_1+g_2^2\xi_2&=0.\end{aligned}$$

From the first equation in the system it follows $c=\dfrac{[g^1]}{[f^1]}$. From the second and third equation one can find

$$\begin{aligned}\xi_1(u_{s,1},u_{s,2}) &= -\frac{([g^2]-c[f^2])g_2^1}{g_2^1-g_1^1}\\ \xi_2(u_{s,1},u_{s,2}) &= \frac{([g^2]-c[f^2])g_1^1}{g_2^1-g_1^1}.\end{aligned}$$

The solution is unique providing $g_2^1\neq g_1^1$. Substitution of these values in the fourth equation gives

$$g_1^2\xi_1(u_{s,1},u_{s,2})+g_2^2\xi_2(u_{s,1},u_{s,2})=c^2[f^2]-c[g^2].$$

Thus, there is a single equation with two unknowns $u_{s,1}$ and $u_{s,2}$, so the chances to solve it seems to be good, except in some "degenerate cases" (for example, when g^1 is a nonzero constant).

8.3. **Relation with the known results.** Definition and assertions about existence of delta shock solutions to 2×2 conservation laws systems with Colombeau generalized functions are given in [16]. One can easily compare delta shocks in that and the present paper. Assume that there exists a generalized delta shock solution (u, v) to some conservation law system in that sense, where the delta function part is contained in the v-variable. Then there exists a shadow wave solution with $u_{1,\varepsilon} = u_0$, $u_{2,\varepsilon} = u_1$ and $v_{i,\varepsilon} = \alpha_i/\varepsilon$, $i = 1, 2$, for some constants α_1 and α_2.

Thus, all the examples from the cited paper can be directly recovered in this new framework. The opposite is not true. For example, Riemann problem for the system

$$\begin{aligned} \partial_t \rho + \partial_x(\rho g(u)) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u g(u)) &= 0 \end{aligned} \tag{8.2}$$

(so called "generalized pressureless gas dynamics model") has a solution in the form of delta shock when one uses Borel measure spaces (see [11]). A generalized function delta shock solution does not exist (nor two-sided delta shock solution). A singular shock wave solution does exist, but ρ is not non-negative – only its distributional limit is non-negative. One can show easily that is an SDW solution (delta shock, to be precise) to (8.2) converging to the one given in [11] like in Theorem 6.1.

Suppose that there exists a two-sided delta shock solution

$$u = \begin{cases} u_0, & x < ct \\ u_1, & x > ct \end{cases}, \qquad v = \begin{cases} v_0, & x < ct \\ v_1, & x > ct \end{cases} + \alpha_L \delta^- + \alpha_R \delta^+.$$

defined in [15] and [18] to some conservation law system linear in one of solution component, v for definiteness. A construction of an appropriate SDW solution is straightforward: Put

$$u_{1,\varepsilon} = u_0,\ u_{2,\varepsilon} = u_1,\ \lim_{\varepsilon \to 0} a_\varepsilon v_{1,\varepsilon} = \alpha_L,\ \lim_{\varepsilon \to 0} b_\varepsilon v_{2,\varepsilon} = \alpha_R.$$

The results with a solution containing Borel measure in v (given in [2], [4] and [11] among others) transfers into an SDW solution simply bu putting $u_{s,1} = u_{s,2} = u_s$ where u_s is a value of u on a delta shock curve, and $v_{1,\varepsilon} = v_{2,\varepsilon} = \alpha/\varepsilon$, where αt is a delta shock strength.

9. Singular shocks

While delta shocks usually appear in weakly hyperbolic systems, singular shocks are associated with strictly hyperbolic systems. Now we permit that a minor component of a solution satisfy both $|u_{i,\varepsilon}| \to \infty$ and $\varepsilon |u_{i,\varepsilon}| \to 0$. That is the main feature of singular shocks, as already mentioned.

In this section, we consider the system given in the following form

$$\begin{aligned} \partial_t u + \partial_x(g_{11}(u)v + g_{12}(u)) &= 0 \\ \partial_t v + \partial_x(g_{21}(u)v + g_{22}(u)) &= 0, \end{aligned}$$

where g_{ij}, $i,j=1,2$ are $\mathcal{C}^0$-functions. The system (3.5) now reduces to

$$\begin{aligned}
-c(u_1-u_0)+a_\varepsilon u_{1,\varepsilon}+b_\varepsilon u_{2,\varepsilon}&\\
+(g_{11}(u_1)v_1+g_{12}(u_1)-g_{11}(u_0)v_0-g_{12}(u_0))&\approx 0\\
-c(a_\varepsilon u_{1,\varepsilon}+b_\varepsilon u_{2,\varepsilon})+a_\varepsilon g_{11}(u_{1,\varepsilon})v_{1,\varepsilon}+b_\varepsilon g_{11}(u_{2,\varepsilon})v_{2,\varepsilon}&\\
+a_\varepsilon g_{12}(u_{1,\varepsilon})+b_\varepsilon g_{12}(u_{2,\varepsilon})&\approx 0\\
-c(v_1-v_0)+a_\varepsilon v_{1,\varepsilon}+b_\varepsilon v_{2,\varepsilon}&\\
+(g_{21}(u_1)v_1+g_{22}(u_1)-g_{21}(u_0)v_0-g_{22}(u_0))&\approx 0\\
-c(a_\varepsilon v_{1,\varepsilon}+b_\varepsilon v_{2,\varepsilon})+a_\varepsilon g_{21}(u_{1,\varepsilon})v_{1,\varepsilon}+b_\varepsilon g_{21}(u_{2,\varepsilon})v_{2,\varepsilon}&\\
+a_\varepsilon g_{22}(u_{1,\varepsilon})+b_\varepsilon g_{22}(u_{2,\varepsilon})&\approx 0
\end{aligned}$$

Using the assumptions on $(u_\varepsilon,v_\varepsilon)$, from the first equation we have

$$c=\frac{[g_{11}(u)v+g_{12}(u)]}{[u]}, \tag{9.1}$$

and from the third equation we have

$$a_\varepsilon v_{1,\varepsilon}+b_\varepsilon v_{2,\varepsilon}\approx c[v]-[g_{21}(u)v+g_{22}(u)]=:\kappa_2. \tag{9.2}$$

Finally, from the rest of the equations we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} a_\varepsilon(g_{11}(u_{1,\varepsilon})v_{1,\varepsilon}+g_{12}(u_{1,\varepsilon}))+b_\varepsilon(g_{11}(u_{2,\varepsilon})v_{2,\varepsilon}+g_{12}(u_{2,\varepsilon}))=0\\
&\lim_{\varepsilon\to 0} a_\varepsilon(g_{21}(u_{1,\varepsilon})v_{1,\varepsilon}+g_{22}(u_{1,\varepsilon}))+b_\varepsilon(g_{21}(u_{2,\varepsilon})v_{2,\varepsilon}+g_{22}(u_{2,\varepsilon}))=c\kappa_2.
\end{aligned} \tag{9.3}$$

It is impossible to do analysis of all possible cases for such a general form, so we analyze a special case, the system given in [12],

$$\begin{aligned}
\partial_t u+\partial_x(u^2-v)&=0\\
\partial_t v+\partial_x(u^3/3-u)&=0.
\end{aligned} \tag{9.4}$$

Here $g_{11}(u)=-1$, $g_{12}(u)=u^2$, $g_{21}(u)=0$ and $g_{22}(u)=u^3/3-u$. We assume $a_\varepsilon=b_\varepsilon=\varepsilon$ without a loss of generality. Then, from (9.1) and (9.2) we have

$$c=\frac{[u^2-v]}{[u]} \text{ and } \varepsilon(v_{1,\varepsilon}+v_{2,\varepsilon})=\kappa_2=c[v]-\Big[\frac{u^3}{3}-u\Big],$$

while (9.3) implies

$$\begin{aligned}
\lim_{\varepsilon\to 0}\varepsilon(-v_{1,\varepsilon}+u_{1,\varepsilon}^2-v_{2,\varepsilon}+u_{2,\varepsilon}^2)&=0\\
\lim_{\varepsilon\to 0}\varepsilon\Big(\frac{u_{1,\varepsilon}^3}{3}-u_{1,\varepsilon}+\frac{u_{2,\varepsilon}^3}{3}-u_{2,\varepsilon}\Big)=\lim_{\varepsilon\to 0}\varepsilon\Big(\frac{u_{1,\varepsilon}^3}{3}+\frac{u_{2,\varepsilon}^3}{3}\Big)&=c\kappa_2.
\end{aligned}$$

Let us put $u_{i,\varepsilon}=\bar u_{i,\varepsilon}+z_{i,\varepsilon}$ with $u_{i,\varepsilon}^2=v_i$, $z_{i,\varepsilon}=c$, $i=1,2$ and $\bar u_{1,\varepsilon}=-\bar u_{2,\varepsilon}$ (up to a term of growth rate $o(\sqrt[3]{\varepsilon})$ actually) with respect to ε. Then the first equation is obviously satisfied while

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon\Big(\frac{u_{1,\varepsilon}^3}{3}+\frac{u_{2,\varepsilon}^3}{3}\Big)\\
&=\lim_{\varepsilon\to 0}\Big(\frac13\bar u_{1,\varepsilon}^3+\bar u_{1,\varepsilon}^2c+\bar u_{1,\varepsilon}c^2+\frac13 c^3+\frac13\bar u_{2,\varepsilon}^3+\bar u_{2,\varepsilon}^2c+\bar u_{2,\varepsilon}c^2+\frac13 c^3\Big)\varepsilon\\
&=\lim_{\varepsilon\to 0} c(\bar u_{1,\varepsilon}^2+\bar u_{2,\varepsilon}^2)\varepsilon=c\kappa_2.
\end{aligned}$$

Also, RH deficit κ_2 defined in (9.2) has to be positive. In order to avoid problems with the entropy condition later, we put $v_{1,\varepsilon} = v_{2,\varepsilon} = \bar{u}^2_{1,\varepsilon}$. (It would be enough to take $v_{1,\varepsilon} \geq u^2_{1,\varepsilon}/2$ and $v_{2,\varepsilon} \geq u^2_{2,\varepsilon}/2$.)

The form of that solution resembles very much the one obtained in [12] (see also [16]). We have used the word "resemble" because the same distributional limit is not appropriate criteria. The main property which make difference between singular and delta shocks is a correction factor contained in minor components. Thus, it would not appear in distributional limit of a solution.

The system (9.4) is strictly hyperbolic, so we expect to find a entropy pair with a convex entropy function for it. That would give an admissibility criterion for shadow wave solutions.

Using the standard procedure (see, for example, [7] or [21]) one finds an entropy function η to be a solution to the equation

$$4\frac{\partial^2 \eta}{\partial x\, \partial y} + \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial y} = 0,$$

where $x = v - u^2/2 - u$ and $y = v - u^2/2 + u$. Convex entropy functions and their corresponding fluxes are given by

$$\begin{aligned} \eta(u,v) =& c e^{\gamma(v-u^2/2-u)} e^{-\gamma(v-u^2/2+u)/(1+4\gamma)}, \\ q(u,v) =& (u + 1 + \frac{1}{2\gamma})\eta(u,v),\ c>0,\ \gamma < -1/4. \end{aligned} \tag{9.5}$$

We can now check (4.2) and (4.3) for the above entropy pair and $a_\varepsilon = b_\varepsilon = \varepsilon$. Due to the construction of an SDW solution to (9.4) we have

$$\lim_{\varepsilon \to 0} v_{i,\varepsilon} - u^2_{i,\varepsilon}/2 \pm u_{i,\varepsilon} = +\infty,\ i = 1,2.$$

Convexity restrictions $\gamma < -1/4$ and $-\gamma/(1+4\gamma) < 0$ imply that both $\eta(u_{i,\varepsilon}, v_{i,\varepsilon})$ and $q(u_{i,\varepsilon}, v_{i,\varepsilon})$, $i = 1,2$ tend to zero faster than any power of ε as $\varepsilon \to 0$. Thus, the second entropy condition (4.3) is satisfied.

Let us check the first entropy condition. We have

$$\overline{\lim}_{\varepsilon \to 0} \varepsilon \eta(u_{1,\varepsilon}, v_{1,\varepsilon}) + \varepsilon \eta(u_{2,\varepsilon}, v_{2,\varepsilon}) = 0.$$

So it reduces to

$$-c(\eta(u_1,v_1) - \eta(u_0,v_0)) + q(u_1,v_1) - q(u_0,v_0) \leq 0.$$

Substituting the entropy pair (9.5) into that inequality we have

$$\eta(u_1,v_1)\Big(u_1 + 1 + \frac{1}{2\gamma} - c\Big) + \eta(u_0,v_0)\Big(c - \big(u_0 + 1 + \frac{1}{2\gamma}\big)\Big) \leq 0.$$

Since η is always positive for $c > 0$, the speed c has to satisfy

$$u_0 + 1 + \frac{1}{2\gamma} \geq c \geq u_1 + 1 + \frac{1}{2\gamma},\ \gamma \in (-\infty, -1/4).$$

That is

$$u_0 - 1 \geq c \geq u_1 + 1,$$

because $-2 < \dfrac{1}{2\gamma} < 0$. So we have proved that SDW is overcompressive if and only if it satisfy entropy condition. One can find a proof that singulars shock solution to (9.4) in [12] is unique when overcompressibility is used as an admissibility condition.

Let us now turn to the situation when two singular shocks interact at some point (X,T). There are three possible results of such an interaction: a single singular shock, a 1-rarefaction wave followed by a singular shock, and a singular shock followed by a 2-rarefaction wave (see [17]). To keep discussion simple we shall assume that the result of interaction is a single singular shock, i.e. there is a singular shock solution to (9.4) connecting the states (u_0,v_0) and (u_2,v_2). (In fact, we are using Assumption 7.1.)

The RH deficits of the incoming and outgoing waves are given by

$$\begin{aligned}\kappa &= \begin{bmatrix}\kappa_1\\ \kappa_2\end{bmatrix} = \begin{bmatrix}0\\ c(v_2-v_0)-\frac{1}{3}(u_2^3-u_0^3)+(u_2-u_0)\end{bmatrix}\\ \tilde{\kappa} &= \begin{bmatrix}0\\ \tilde{c}(v_1-v_0)-\frac{1}{3}(u_1^3-u_0^3)+(u_1-u_0)\end{bmatrix} \text{ and}\\ \hat{\kappa} &= \begin{bmatrix}0\\ \hat{c}(v_2-v_1)-\frac{1}{3}(u_2^3-u_1^3)+(u_2-u_1)\end{bmatrix}\end{aligned}$$

Also,

$$f(\tilde{U}_\varepsilon) = \begin{bmatrix}\varepsilon(\tilde{u}_{1,\varepsilon}+\tilde{u}_{2,\varepsilon})T\\ \varepsilon(\tilde{v}_{1,\varepsilon}+\tilde{v}_{2,\varepsilon})T\end{bmatrix} \approx \begin{bmatrix}0\\ (\tilde{\xi}_1+\tilde{\xi}_2)T\end{bmatrix}, \text{ and } f(\hat{U}_\varepsilon) \approx \begin{bmatrix}0\\ (\hat{\xi}_1+\hat{\xi}_2)T\end{bmatrix},$$

where $\xi_i := \lim_{\varepsilon\to 0}\varepsilon v_{i,\varepsilon}$, $\tilde{\xi}_i := \lim_{\varepsilon\to 0}\varepsilon \tilde{v}_{i,\varepsilon}$ and $\hat{\xi}_i := \lim_{\varepsilon\to 0}\varepsilon \hat{v}_{i,\varepsilon}$, $i=1,2$. (We have used the same notation as before.)

Assumption 7.2 holds since it is enough to put

$$\alpha := \frac{T}{\kappa_2}(\tilde{\kappa}_2+\hat{\kappa}_2),$$

because RH deficits for all the first equations are zero. That is, the result of interaction is really a singular shock.

Let us check the entropy condition. Due to (7.5), the following relation has to be satisfied

$$\begin{aligned}&\overline{\lim}_{\varepsilon\to 0}\ (x_{1,\varepsilon}\eta(u_{1,\varepsilon},v_{1,\varepsilon})+x_{2,\varepsilon}\eta(u_{2,\varepsilon},v_{2,\varepsilon}))\\ &-T\varepsilon(\tilde{\eta}(\tilde{u}_{1,\varepsilon},\tilde{v}_{1,\varepsilon})+\tilde{\eta}(\tilde{u}_{2,\varepsilon},\tilde{v}_{2,\varepsilon}))-T\varepsilon(\hat{\eta}(\hat{u}_{1,\varepsilon},\hat{v}_{1,\varepsilon})+\hat{\eta}(\hat{u}_{2,\varepsilon},\hat{v}_{2,\varepsilon}))\le 0,\end{aligned}$$

where $x_{1,\varepsilon}=x_{2,\varepsilon}=\alpha\varepsilon$, where the value of α is already defined above. Again, all the terms above tends to zero as $\varepsilon\to 0$ for both families of entropy pairs. Thus, the entropy condition for singular shock interaction solution defined in Theorem 7.1 is satisfied.

Remark 9.1. Note that the above conclusion holds when one of the incoming waves is just a shock: If a shock is on the right-hand side, then one can just put $\hat{a}_\varepsilon=\hat{b}_\varepsilon=0$, or $U_{i,\varepsilon}=U_0$ and $U_{2,\varepsilon}=U_1$. The complete interaction problem for elementary waves and singular shock ones is solved in [17] using a smooth approximations of waves. These results can be translated into the frame used through the present paper when a rarefaction wave is substituted by a fan of non-admissible shocks (like in Front Tracking Algorithm, [3]).

Part 4. **Weighted SDWs**

As one could see in Section 7.1 there is a need for some other type of solution to deal with a general SDW interaction problem. (It it is also true for 2×2 pressureless gas dynamic model). Thus, we shall start our investigation by that special case.

10. Interactions of SDWs in the case of pressureless gas dynamics model

As in the case of constant SDWs we have the following basic lemma.

Lemma 10.1. *Let $f, g \in \mathcal{C}(\Omega : \mathbb{R}^n)$ and $U : \mathbb{R}^2_+ \to \Omega \subset \mathbb{R}^n$ be a piecewise constant function for every $t \geq 0$:*

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x < c(t) - a_\varepsilon(t) \\ U_{1,\varepsilon}(t), & c(t) - a_\varepsilon(t) < x < c(t) \\ U_{2,\varepsilon}(t), & c(t) < x < c(t) + b_\varepsilon(t) \\ U_1, & x > c(t) + b_\varepsilon(t) \end{cases}. \tag{10.1}$$

The functions a_ε, b_ε are C^1-functions satisfying $a_\varepsilon(0) = x_{1,\varepsilon}$ and $b_\varepsilon(0) = x_{2,\varepsilon}$. Also, suppose that f and g satisfy (2.2). Then

$$\begin{aligned} \langle \partial_t f(U_\varepsilon), \phi \rangle \approx & \int_0^\infty \lim_{\varepsilon \to 0} \frac{d}{dt} \Big(a_\varepsilon(t) f(U_{1,\varepsilon}(t)) + b_\varepsilon(t) f(U_{2,\varepsilon}(t)) \Big) \phi(c(t),t)\, dt \\ & - \int_0^\infty c'(t) \Big(f(U_1) - f(U_0) \Big) \phi(c(t),t)\, dt \\ & + \int_0^\infty \lim_{\varepsilon \to 0} c'(t) \Big(a_\varepsilon(t) f(U_{1,\varepsilon}(t)) + b_\varepsilon(t) f(U_{2,\varepsilon}(t)) \Big) \partial_x \phi(c(t),t)\, dt \end{aligned} \tag{10.2}$$

and

$$\begin{aligned} \langle \partial_x g(U_\varepsilon), \phi \rangle \approx & \int_0^\infty \Big(g(U_1) - g(U_0) \Big) \phi(c(t),t)\, dt \\ & - \int_0^\infty \lim_{\varepsilon \to 0} \Big((a_\varepsilon(t) g(U_{1,\varepsilon}(t)) + (b_\varepsilon(t) g(U_{2,\varepsilon}(t)) \Big) \partial_x \phi(c(t),t)\, dt. \end{aligned} \tag{10.3}$$

The proof of the first relation in (2.3) from Lemma 2.1 can be easily adopted for (10.2) and we omit it here (one just have to take care that $U_{i,\varepsilon}$ depends on t, $i = 1, 2$). The proof of (10.3) is the same as the one given for that lemma.

One can see that Theorem 7.1 is still valid when U_ε defined by (7.2) is substituted by an appropriate weighted SDW. Assumption 7.1 is still needed but with "exists an SDW" substituted by "exists a weighted SDW". One will see bellow that existence of a weighted SDW is much easier to obtain than the existence of constant SDW to the same initial data.

10.1. **Two SDWs interaction.**
So, let us apply Lemma 10.1 for (6.1) in the situation when two SDWs interact.

Theorem 10.1. *A result of two SDW interaction for the pressureless system (6.1) is a weakly unique single entropic weighted SDW.*

Proof. Suppose that the SDWs interact in some point (X,T). We use the notation from Section 7.1. The sign $\tilde{\cdot}$ is reserved for data in the first (from the left) while the sign $\hat{\cdot}$ is reserved for the second SDW. (A speed of the first SDW is $\tilde{c}$ while a speed of the second one is $\hat{c}$, etc.) The first SDW connect the states $U_0 = \begin{bmatrix} \rho_0 \\ u_0 \\ e_0 \end{bmatrix}$ and $U_1 = \begin{bmatrix} \rho_1 \\ u_1 \\ e_1 \end{bmatrix}$ while the second one connect the states $U_1 = \begin{bmatrix} \rho_1 \\ u_1 \\ e_1 \end{bmatrix}$ and $U_2 = \begin{bmatrix} \rho_2 \\ u_2 \\ e_2 \end{bmatrix}$.

It is safe to transfer the interaction problem into the Cauchy one by the translation for vector (X, T)

$$\begin{aligned} &\partial_t f(U) + \partial_x g(U) = 0 \\ &U = \begin{cases} U_0, & x < 0 \\ U_2, & x > 0 \end{cases} + \sigma\delta_{(0,0)}. \end{aligned}$$

Here $\sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$ is a sum of strengths of incoming SDWs, $f(U) = \begin{bmatrix} \rho \\ \rho u \\ \rho\frac{u^2}{2} + \rho e \end{bmatrix}$ and $g(U) = \begin{bmatrix} \rho u \\ \rho u^2 \\ (\rho\frac{u^2}{2} + \rho e + p_0)u \end{bmatrix}$.

Let us use the notation similar to the one in the Theorem 6.1:

$$\begin{aligned} &\lim_{\varepsilon\to 0} a_\varepsilon(t)\rho_{1,\varepsilon}(t) =: \xi_1(t),\ \lim_{\varepsilon\to 0} b_\varepsilon(t)\rho_{2,\varepsilon}(t) =: \xi_2(t), \\ &\lim_{\varepsilon\to 0} u_{i,\varepsilon}(t) =: u_{s,i}(t),\ \lim_{\varepsilon\to 0} e_{i,\varepsilon}(t) =: e_{s,i}(t),\ i = 1, 2, \\ &[x]_1 := x_1 - x_0,\ [x]_2 := x_2 - x_1,\ \text{and } [x] := x_2 - x_0. \end{aligned}$$

Use of (10.2) and (10.3) gives the following system of differential equations:

$$\begin{aligned} (\xi_1(t) + \xi_2(t))' - c'(t)[\rho] + [\rho u] &= 0 \\ (\xi_1(t)u_{s,1}(t) + \xi_2(t)u_{s,2}(t))' - c'(t)[\rho u] + [\rho u^2] &= 0 \\ (\xi_1(t)(u_{s,1}^2(t)/2 + e_{s,1}(t)) + \xi_2(t)(u_{s,2}^2(t)/2 + e_{s,2}(t)))'& \\ -c'(t)[\rho(u^2/2 + e)] + [\rho(u^2/2 + e)u + p_0 u] &= 0 \\ -c'(t)\big(\xi_1(t) + \xi_2(t)\big) + \xi_1(t)u_{s,1}(t) + \xi_2(t)u_{s,2}(t) &= 0 \\ -c'(t)\big(\xi_1(t)u_{s,1}(t) + \xi_2(t)u_{s,2}(t)\big) + \xi_1(t)u_{s,1}^2(t) + \xi_2(t)u_{s,2}^2(t) &= 0 \\ -c'(t)\big(\xi_1(t)(u_{s,1}^2(t)/2 + e_{s,1}(t)) + \xi_2(t)(u_{s,2}^2(t)/2 + e_{s,2}(t))\big)& \\ +\xi_1(t)(u_{s,1}^2(t)/2 + e_{s,1}(t))u_{s,1}(t) + \xi_2(t)(u_{s,2}^2(t)/2 + e_{s,2}(t))u_{s,2}(t) &= 0 \end{aligned}$$

together with the initial data:

$$\begin{aligned} &(\xi_1 + \xi_2)(0) = \sigma_1 = T(\tilde{\xi}_1 + \tilde{\xi}_2 + \hat{\xi}_1 + \hat{\xi}_2) = T(\tilde{\kappa}_1 + \hat{\kappa}_1) > 0 \\ &(\xi_1 u_{s,1} + \xi_2 u_{s,2})(0) = \sigma_2 = T(\tilde{c}\tilde{\kappa}_1 + \hat{c}\hat{\kappa}_1) \\ &(\xi_1(u_{s,1}^2/2 + e_{s,1}) + \xi_2(u_{s,2}^2/2 + e_{s,2}))(0) = \sigma_3 = T(\tilde{\kappa}_3 + \hat{\kappa}_3) > 0, \\ &\text{where } \tilde{\kappa}_3 = \Big(\frac{\tilde{c}^2}{2} + \tilde{e}_s\Big)\tilde{\kappa}_1 \text{ and } \hat{\kappa}_3 = \Big(\frac{\hat{c}^2}{2} + \hat{e}_s\Big)\hat{\kappa}_1. \end{aligned}$$

We use the following simplifying assumption

$$\xi_1 = \xi_2 =: \xi/2,\ u_{s,1} = u_{s,2} =: u_s,\ e_{s,1} = e_{s,2} =: e_s$$

as in the case of Riemann problem (6.1,6.2). One will see bellow that it may be used without a loss in generality, since entropy conditions actually would imply $u_{s,1} \approx u_{s,2}$ and $e_{s,1} \approx e_{s,2}$.

The first thing one can notice is $u_s(t) = c'(t)$ and that the three out of six equations in the above system annihilate. So, we get the following 3×3 ODE system:

$$\begin{aligned} \xi'(t) =& u_s(t)[\rho] - [\rho u] \\ (\xi(t)u_s(t))' =& u_s(t)[\rho u] - [\rho u^2] \\ (\xi(t)u_s^2(t)/2 + \xi(t)e_s(t))' =& u_s(t)[\rho u^2/2 + \rho e] - [\rho u^3/2 + \rho e u + p_0 u]. \end{aligned} \tag{10.4}$$

The first two equations above are decoupled from the third one and can be written as

$$\begin{aligned} \xi'(t) =& \frac{y(t)[\rho]}{\xi(t)} - [\rho u] \\ y'(t) =& \frac{y(t)[\rho u]}{\xi(t)} - [\rho u^2], \text{with } y(t) = \xi(t)u_s(t), \end{aligned}$$

The initial data are $\xi(0) = \sigma_1 > 0$ and $y(0) = \sigma_2$. The standard ODE theory says that there is at least a local solution to the above system because ξ is positive at the initial time. We need some additional estimates for proving a global solution existence.

Initially, we have

$$u_s(0) = \frac{\sigma_2}{\sigma_1} = \frac{\tilde{c}\tilde{\kappa}_1 + \hat{c}\hat{\kappa}_1}{\tilde{\kappa}_1 + \hat{\kappa}_2} = \tilde{c} - \frac{(\tilde{c} - \hat{c})\tilde{\kappa}_1}{\tilde{\kappa}_1 + \hat{\kappa}_2} < \tilde{c} \le u_0,$$

due to the overcompressibility (implied by the entropy conditions for Riemann problem, see Theorem 6.1). Because of the same reason,

$$u_s(0) = \hat{c} + \frac{(\tilde{c} - \hat{c})\hat{\kappa}_1}{\tilde{\kappa}_1 + \hat{\kappa}_2}\hat{c} \ge u_2.$$

From the first two equations in (10.4) we obtain

$$u_s'(t) = -\frac{1}{\xi(t)}([\rho]u_s^2(t) - 2[\rho u]u_s(t) + [\rho u^2]).$$

Assume $[\rho] \neq 0$, first. Denote by $A_1 < A_2$ roots of the right-hand side of above ODE,

$$A_{1,2} = \frac{[\rho u] \pm |u_0 - u_2|\sqrt{\rho_0 \rho_2}}{[\rho]}.$$

The value of $-1/\xi(t)$ is always negative when exists since $\xi(0) = \sigma_1 > 0$. If $[\rho] > 0$, them $u_s(t)$ increases when it is between A_1 and A_2 and decreases when it is less than A_1 or greater than A_2. The opposite is true if $[\rho] < 0$. There are two possible cases:

- If $\rho_2 > \rho_0$, then $A_1 \le u_2 \le A_2 \le u_0$. If $u_s(0) \in (u_2, A_2)$, then $u_s(t)$ increases but stays bellow A_2. If $u_s(0) \in (A_2, u_0)$, then $u_s(t)$ decreases but stays above A_2.
- If $\rho_0 > \rho_2$, then $u_2 \le A_1 \le u_0 \le A_2$. Again, if $u_s(0) \in (u_2, A_1)$, then $u_s(t)$ increases. If $u_s(0) \in (A_1, u_0)$, then $u_s(t)$ decreases.

That proves $u_0 > u_s(t) > u_2$, for each $t \geq 0$ in both of the cases (so called the overcompressibility condition is satisfied) if a solution to (10.4) exists. We are going to use it immediately to estimate $\xi(t)$. We have

$$\xi'(t) = [\rho]u_s(t) - [\rho u] \geq [\rho]u_2 - [\rho u] = \rho_0(u_0 - u_2) > 0$$

if $\rho_2 > \rho_0$. Also

$$\xi'(t) = [\rho]u_s(t) - [\rho u] \geq [\rho]u_0 - [\rho u] = \rho_2(u_0 - u_2) > 0,$$

if $\rho_0 > \rho_2$ So, $\xi(t)$ is not decreasing which implies the global existence of solution $(\xi(t), u_s(t))$.

The third equation in (10.4) is linear in e_s, and it is easy to find its solution $e_s(t)$ since $\xi(t)$ and $u_s(t) = y(t)/\xi(t)$ are already known. We just have to verify that it is non-negative (that is also a physical reason – internal energy cannot be negative).

The third equation in (10.4) imply

$$\begin{aligned}(\xi(t)e_s(t))' =& \frac{1}{2}([\rho]u_s(t) - [\rho u])u_s^2(t) - ([\rho u]u_s(t) - [\rho u^2])u_s(t) \\ &+ \frac{1}{2}([\rho u^2]u_s(t) - [\rho u^3]) + [\rho e]u_s(t) - [\rho ue] - p_0[u].\end{aligned}$$

After some elementary calculations we get

$$\begin{aligned}&\frac{1}{2}([\rho]u_s(t) - [\rho u])u_s^2(t) - ([\rho u]u_s(t) - [\rho u^2])u_s(t) + \frac{1}{2}([\rho u^2]u_s(t) - [\rho u^3]) \\ =& \frac{1}{2}\rho_2(u_s(t) - u_2)^3 + \frac{1}{2}\rho_0(u_0 - u_s(t))^3 \geq 0\end{aligned}$$

(we already have proved that $u_0 \geq u_s(t) \geq u_2$ provided a solution exists) and

$$[\rho e]u_s(t) - [\rho ue] - p_0[u] = \rho_2 e_2(u_s(t) - u_2) + \rho_0 e_0(u_0 - u_s(t)) + p_0(u_0 - u_2) \geq 0.$$

That proves $(\xi(t)e_s(t))' \geq 0$ and

$$(\xi(t)e_s(t))' \geq \rho_2 e_2(u_s(t) - u_2) + \rho_0 e_0(u_0 - u_s(t)) \tag{10.5}$$

in addition (since $p_0(u_0 - u_2) \geq 0$). Initially, $e_s(0) = \frac{\sigma_3}{\sigma_1} - \frac{1}{2}\left(\frac{\sigma_2}{\sigma_1}\right)^2 = \frac{\sigma_3}{\sigma_1} - \frac{1}{2}u_s^2(0)$. Substitution of the values found above and a direct calculation gives

$$e_s(0) = \frac{\tilde{\kappa}_1\hat{\kappa}_1}{2(\tilde{\kappa}_1 + \hat{\kappa_1})^2}(\tilde{c} - \hat{c})^2 + \frac{\tilde{\kappa}_1\tilde{e}_s + \hat{\kappa}_1\hat{e}_s}{\tilde{\kappa}_1 + \hat{\kappa_1}} \geq 0.$$

Thus, $e_s(t) \geq 0$, $t > 0$ since we already know that $\xi(t) \geq 0$, $t > 0$.

We have assumed $[\rho] \neq 0$ above. Suppose now that $\rho_0 = \rho_2$. Then the first equation in (10.4) imply

$$\xi(t) = -[\rho u]t + \xi(0) = \rho_0(u_0 - u_2)t + \xi(0) > 0,$$

and from the second one we have

$$\begin{aligned}u_s'(t) =& -\frac{2\rho_0(u_0 - u_2)}{\rho_0(u_0 - u_2)t + \xi(0)}u_s(t) + \frac{\rho_0(u_0 - u_2)(u_0 + u_2)}{\rho_0(u_0 - u_2)t + \xi(0)} \\ &\begin{cases} < 0, & u_s(t) > (u_0 + u_2)/2 \\ = 0, & u_s(t) = (u_0 + u_2)/2 \\ > 0, & u_s(t) < (u_0 + u_2)/2. \end{cases}\end{aligned}$$

Using elementary properties of the above ODE one can see that

- If $u_s(0) > (u_0 + u_2)/2$, then $u_s(t) \in [(u_0 + u_2)/2, u_s(0)]$.

- If $u_s(0) < (u_0+u_2)/2$, then $u_s(t) \in [u_s(0), (u_0+u_2)/2]$.

That implies the existence of a global solution to (10.4) in this case, too. The third equation of the system can be solved exactly in the same way as above.

Remark 10.1. The obtained solution is overcompressive as have been seen above, and one can say that it satisfy admissible conditions used in most of the papers about delta and singular shocks. In the present paper, overcompressibility was consequence of (4.1) and we continue to check that condition.

Now, we have to prove that the above solution satisfies entropy condition like an SDW solution to Riemann problem and that it is the only such a solution.

Let η, q be an entropy pair for (1.1) which admits a solution of the form (10.1). For a solution in the form of weighted SDW the conditions (4.2,4.3) are substituted by

$$\begin{aligned} &-c'(t)(\eta(U_2)-\eta(U_0)) + (q(U_2)-q(U_0)) \\ &\quad + \lim_{\varepsilon\to 0}\frac{d}{dt}(\eta(U_{1,\varepsilon}(t))a_\varepsilon + \eta(U_{2,\varepsilon}(t))b_\varepsilon) \leq 0 \end{aligned} \tag{10.6}$$

and

$$\begin{aligned} &\lim_{\varepsilon\to 0} c'(t)(\eta(U_{1,\varepsilon}(t))a_\varepsilon + \eta(U_{2,\varepsilon}(t))b_\varepsilon) \\ &\quad -q(U_{1,\varepsilon}(t))a_\varepsilon(t) - q(U_{2,\varepsilon}(t))b_\varepsilon(t) = 0. \end{aligned} \tag{10.7}$$

(That is a simple consequence of (10.2) and (10.3).) In the special case of 3×3 pressureless gas dynamics, (10.7) is equivalent to

$$\lim_{\varepsilon\to 0}(c'(t)-u_{1,\varepsilon}(t))\,\eta(U_{1,\varepsilon}(t))\,a_\varepsilon + (c'(t)-u_{2,\varepsilon}(t))\,\eta(U_{2,\varepsilon}(t))\,b_\varepsilon = 0,$$

for every semi-convex η. That it is true if and only if $u_{1,\varepsilon}(t) \approx u_{2,\varepsilon}(t) \approx u_s(t)$ as it was already assumed during the construction of solution.

In the sequel we shall take only the case $p_0 = 0$ for simplicity. The procedure is similar for other values of pressure. The semi-convex entropy pair for (6.1) is now given by

$$\eta(\rho,u,e) = \rho(R(u)+S(e)),\ q(\rho,u,e) = u\eta(\rho,u,e) = \rho u(R(u)+S(e))$$

with $R'' \geq 0$, $S' \leq 0$, $S'' \geq 0$. The domain of R is the set of all reals and the domain of S is the set of all non-negative reals.

Let us first choose $R = 0$ and prove that (10.6) is also satisfied, i.e.

$$\begin{aligned} I(t) :=& -(u_0-u_s(t))\rho_0 S(e_0) - (u_s(t)-u_2)\rho_2 S(e_2) + (\xi(t)S(e_s(t)))' \\ =& -(u_0-u_s(t))\rho_0 S(e_0) - (u_s(t)-u_2)\rho_2 S(e_2) \\ &+ \xi'(t)S(e_s(t)) + \xi(t)S'(e_s(t))e_s'(t) \leq 0. \end{aligned}$$

Let t be fixed for a moment and denote $\beta := S'(e_s(t)) \leq 0$ and $\alpha := S(e_s(t)) - \beta e_s(t)$. Then $S(e) = \alpha + \beta e + S^*(e)$, where $S^*(e)$ is a convex function such that $S^*(e_s(t)) = (S^*)'(e_s(t)) = 0$ (due to the choice of α and β). These equalities and convexity of $S^*(e)$ implies $S^*(e) \geq 0$.

Using the above decomposition of $S(e)$ we have

$$\begin{aligned}I(t)=&-(u_0-u_s(t))\rho_0(\alpha+\beta e_0+S^*(e_0))-(u_s(t)-u_2)\rho_2(\alpha+\beta e_2+S^*(e_2))\\&+\xi'(t)(\alpha+\beta e_s(t)+S^*(e_s(t)))+\xi(t)e_s'(t)(\beta+(S^*)'(e_s(t)))\\=&\alpha(-(u_0-u_s(t))\rho_0-(u_s(t)-u_2)\rho_2+\xi'(t))\\&+\beta(-(u_0-u_s(t))\rho_0e_0-(u_s(t)-u_2)\rho_2e_2+(\xi(t)e_s(t))')\\&+(-(u_0-u_s(t))\rho_0S^*(e_0)-(u_s(t)-u_2)\rho_2S^*(e_2))\\=&\alpha I_1+\beta I_2+I_3.\end{aligned}$$

The first equation in (10.4) implies $I_1=0$. Relation (10.5) gives $I_2\geq 0$, so $\beta I_2\leq 0$. Finally, it follows from the non-negativity of $S^*(e)$ and relation $u_0\geq u_s(t)\geq u_2$ that $I_3\leq 0$. Thus, $I(t)\leq 0$ and the first part of entropy inequality proof is finished.

Now we prove the entropy inequality for the general case of entropy pair $\eta=\rho(R(u)+S(e))$, $q=u\eta$, $R''\geq 0$, $S'\leq 0$ and $S''\geq 0$. It is enough to prove

$$\begin{aligned}J(t):=&-(u_0-u_s(t))\rho_0R(u_0)-(u_s(t)-u_2)\rho_2R(u_2)\\&+\xi'(t)R(u_s(t)))+\xi(t)R'(u_s(t)))u_s'(t)\leq 0\end{aligned}$$

in addition to already proved inequality for S. Obviously, $J(t)=0$ for any affine R due to (10.4), so we may assume $R(u_s(t))=R'(u_s(t))=0$ for each $t\geq 0$. Then $R(u)\geq 0$ by convexity assumption and

$$J(t)\leq -(u_0-u_s(t))\rho_0R(u_0)-(u_s(t)-u_2)\rho_2R(u_2)\leq 0.$$

That proves the full entropy inequality in the case $p_0\equiv 0$ when two SDWs interact giving a single weighted SDW. Now, the weak uniqueness follows immediately. □

10.2. **An SDW interaction with a classical BV wave pattern.** Take $p_0=0$. Let us examine what is a result of an SDW interaction with a wave consists from two contact discontinuities connected with a vacuum state (CD+Vac+CD for short). That wave combination occurs for the initial data (6.2) with $u_0\leq u_1$ (equality means that wave is a single CD). For simplicity we shall look only the case when SDW is on the left of a centered wave combination CD+Vac+CD (the SDW is faster than the CD+Vac+CD). That situation is a result of a solution to (6.1) with the initial data

$$(\rho,u,e)|_{t=0}=\begin{cases}(\rho_0,u_0,e_0), & x<-a^2\\(\rho_1,u_1,e_1), & -a^2<x<0\\(\rho_2,u_2,e_2), & x>0\end{cases}$$

where $u_0>u_1$ and $u_1\leq u_2$ and a is some nonzero real.

We have proved that a weighted SDW is entropic if and only if it is overcompressive in the previous theorem, so we use it in the present interaction analysis.

The first interaction point (X,T) is determined by the intersection of central before-interaction SDW curve, $x+a^2=(\tilde{c}+b_\varepsilon)t$ and the right-handed CD curve $x=u_1t$. For a moment, let assume that the wave combination is divided into a fan of small amplitude non-entropic waves. It means that the right-hand side of (X,T)

may be taken to be $(0, u_r, e_r)$ with $u_r > u_1$ and $u_r - u_1$ small enough such that the following system

$$\begin{aligned}\xi'(t) &= -\rho_0 u_s(t) + \rho_0 u_0 = \rho_0(u_0 - u_s(t))\\ (\xi(t)u_s(t))' &= -\rho_0 u_0 u_s(t) + \rho_0 u_0^2 = \rho_0 u_0(u_0 - u_s(t)) = u_0 \xi'(t), \text{ or}\\ u_s'(t) &= \frac{\rho_0}{\xi(t)}(u_s(t) - u_0)^2.\end{aligned} \tag{10.8}$$

substitutes the first two equations from (10.4) for t in some interval $[T, T_r]$. The initial data for the above system are given by

$$\begin{aligned}\xi(T) &= \sigma_1 = \tilde{\kappa}_1 T = T(u_0 - u_1)\sqrt{\rho_0\rho_1},\\ u_s(T) &= \frac{\sigma_2}{\sigma_1} = \frac{\tilde{c}\tilde{\kappa}_1}{\tilde{\kappa}_1} = \tilde{c} = \frac{[\rho u]_1 - [u]_1\sqrt{\rho_0\rho_1}}{[\rho]_1}.\end{aligned}$$

A short analysis shows that again $\xi(t) > 0$ for $t > T$. Also, $\xi(t)$ and $u_s(t)$ are non-decreasing functions in that interval.

Obviously, the second equation in (10.8), $u_0\xi'(t) = (\xi(t)u_s(t))'$, implies

$$\mathbb{R} \ni c_1 = u_0\xi(t) - \xi(t)u_s(t) = (u_0 - u_s(t))\xi(t).$$

The constant c_1 is determined by putting $t = T$ above:

$$c_1 = \sigma_1(u_0 - u_s(T)) = \frac{(u_0 - u_s)^2 \rho_1 \sqrt{\rho_0}}{\sqrt{\rho_0} + \sqrt{\rho_1}} T > 0.$$

After a change of variables $w(t) = u_0 - u_s(t)$, system (10.8) has a global solution because

$$\xi(t) = \frac{c_1}{w(t)} \text{ and } w'(t) = -\frac{\rho_0}{c_1} w^3$$

imply

$$w(t) = \frac{1}{\sqrt{\sigma_1^2/c_1^2 + 2\rho_0(t-T)/c_1}} \to 0, \text{ as } t \to \infty.$$

That means $u_s \to u_0$, as $t \to \infty$ while non-negativity of w implies $u_s(t) < u_0$ in addition.

Thus, the first two components $\rho(t)$ and $u_s(t)$ are known for all $t > T$ and the third equation in (10.4) is linear with respect to e_s so it has a global solution, too.

One could see that we have never used values u_r and e_r above, since vacuum state $\rho_r \equiv 0$ dominates them. Therefore, the obtained solution can be prolonged after the time $t = T_r$ provided $\rho = 0$, i.e. we can safely take $u_r = u_2$ and $e_r = e_2$.

Let us check when the resulting SDW is overcompressive during its propagation trough the vacuum state. The central curve of the after interaction SDW is given by

$$\Gamma := \{(x,t) : \ x - X = \int_T^t u_s(r)\,dr, \ t \geq T\},$$

where (X, T) is the first interaction point. The value of u on the left of Γ equals u_0. The above solution $u_s(t)$ satisfy $u_s(t) < u_0$ so the first part of overcompressibility condition is fine. On the right of Γ we have $u = x/t$ (in the vacuum area). Using the formula for Γ we have

$$\frac{x}{t} = \frac{X + \int_T^t u_s(r)\,dr}{t} = \frac{u_1 T + \int_T^t u_s(r)\,dr}{t} \leq \frac{u_s(t)T + u_s(t)(t-T)}{t} = u_s(t).$$

We have used that the before-interaction SDW is overcompressive and $u_s(T) = \tilde{c} > u_1$. Also, the function u_s is proved to be non-decreasing and $u_s(t) > u_1$. The relation $x/t < u_s(t)$ on the curve Γ proves that the after-interaction SDW is overcompressive during its propagation trough the vacuum state.

The speed of right-handed (second) CD in the above wave combination equals u_2 and we have the following possibilities for the behavior of after-interaction SDW.

If $u_2 \geq u_0$, then its speed is always less or equal to the speed of the second CD and Γ never leaves the vacuum area. That is the final answer to the interaction problem in that case.

f $u_0 > u_2$, then there is a time when u_s became greater than u_2 (since $u_s(t) \to u_0$ as shown above). Thus, Γ leaves the vacuum area after some time, say T_1. Denote the intersection point of Γ and $x = u_2 t$ by (X_1, T_1). We stop the time and we have again initial data containing a delta function for system (6.1)

$$(\rho, u, e)|_{t=T_1} = \begin{cases} (\rho_0, u_0, e_0), & x < X_1 \\ (\rho_2, u_2, e_2), & x > X_1 \end{cases} + \sigma_1 \delta_{(X_1, T_1)}$$

That problem has a unique weighted SDW solution since $u_0 > u_2$. The proof is the same as the one for Theorem 10.1. That concludes analysis of the last case.

One can use the similar arguments when CD+Vac+CD is on the left of SDW.

Part 5. Other possibilities

11. δ'-shocks

Assumption 3.1 was needed to keep a discussion on a general level – to consider systems in as much as possible unrestricted form. It may be avoided for specific problems. Here we present the hyperbolic system from [20]

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0 \\ \partial_t v + \partial_x (f'(u) v) &= 0 \\ \partial_t w + \partial_x (f''(u) v^2 + f'(u) w) &= 0, \; f''(u) > 0 \end{aligned} \tag{11.1}$$

and particularly its special case $f(u) = u^2$ (also see [22] for a vanishing viscosity approach). The above system is the main example of the one having so called δ'-shock solution.

Let us find an SDW solution to (11.1) with an arbitrary initial data

$$(u, v, w)|_{t=T_1} = \begin{cases} (u_0, v_0, w_0), & x < 0 \\ (u_2, v_2, w_2), & x > 0. \end{cases}$$

For that purpose we have to extend the results of Lemma 2.1 (3.2, more precisely) to one more derivative. That can be done by expanding a test function up to the second term. Like in the lemma one can prove the following relation

$$\begin{aligned} \partial_t U_\varepsilon \approx & (-c[U] + \lim_{\varepsilon \to 0} (\varepsilon U_{1,\varepsilon} + \varepsilon U_{2,\varepsilon}))\delta \\ & + (-c \lim_{\varepsilon \to 0} (\varepsilon U_{1,\varepsilon} + \varepsilon U_{2,\varepsilon}) + \lim_{\varepsilon \to 0} (\varepsilon^2 U_{1,\varepsilon} - \varepsilon^2 U_{2,\varepsilon})) t\delta' \\ & - \frac{c}{2} \lim_{\varepsilon \to 0} ((\varepsilon^2 U_{1,\varepsilon} - \varepsilon^2 U_{2,\varepsilon})) t^2 \delta'' \\ \partial_x U_\varepsilon \approx & [U]\delta + \lim_{\varepsilon \to 0} (\varepsilon U_{1,\varepsilon} + \varepsilon U_{2,\varepsilon}) t\delta' + \frac{1}{2} \lim_{\varepsilon \to 0} (\varepsilon^2 U_{1,\varepsilon} - \varepsilon^2 U_{2,\varepsilon}) t^2 \delta'', \end{aligned} \tag{11.2}$$

where U_ε is of the form (2.1) (without Assumption 3.1) and the supports of δ and its derivatives are $x = ct$ as before. We have used $a_\varepsilon = b_\varepsilon = \varepsilon$ for simplicity.

Solution to the first two equations is already known. For $u_0 > u_1$ it is a delta shock

$$(u,v)(x,t) = \begin{cases} (u_0, v_0), & x < (c-\varepsilon)t \\ (u_{1,\varepsilon}, v_{1,\varepsilon}), & (c-\varepsilon)t < x < ct \\ (u_{2,\varepsilon}, v_{1,\varepsilon}), & ct < x < (c+\varepsilon)t \\ (u_1, v_1), & x > (c+\varepsilon)t, \end{cases}$$

where the speed is given by $c = [f(u)]/[u]$, $\lim_{\varepsilon\to 0} u_{i,\varepsilon} = u_{s,i} \in \mathbb{R}$ and $\lim_{\varepsilon\to 0} \varepsilon v_{i,\varepsilon} = \xi_i \in \mathbb{R}$, $i = 1,2$. (The other three equations reduces to identities). All the above values are determined by six equations generated from the first two equations in system (11.1),

$$\begin{aligned} &-c[u] + [f(u)] = 0 \text{ determines the speed, while} \\ &\xi_1 + \xi_2 = c[v] - [f'(u)v] =: \kappa_2 \\ &f'(u_{s,1})\xi_1 + f'(u_{s,2})\xi_2 = c\kappa_2 \text{ determines } \xi_1, \xi_2, \text{ provided } u_{s,1} \neq u_{s,2}. \end{aligned}$$

From the third equation in (11.1) we get

$$\begin{aligned} -c[w] + \varepsilon w_{1,\varepsilon} + \varepsilon w_{2,\varepsilon} + [f''(u)v^2 + f'(u)w] &\approx 0 \\ -c(\varepsilon w_{1,\varepsilon} + \varepsilon w_{2,\varepsilon}) + \varepsilon^2 w_{1,\varepsilon} - \varepsilon^2 w_{2,\varepsilon} & \\ +\varepsilon\big(f''(u_{1,\varepsilon})v_{1,\varepsilon}^2 + f'(u_{1,\varepsilon})w_{1,\varepsilon} + f''(u_{2,\varepsilon})v_{2,\varepsilon}^2 + f'(u_{2,\varepsilon})w_{2,\varepsilon}\big) &\approx 0 \\ -\frac{c}{2}(\varepsilon^2 w_{1,\varepsilon} - \varepsilon^2 w_{2,\varepsilon}) & \\ +\frac{1}{2}\varepsilon^2\big(f''(u_{1,\varepsilon})v_{1,\varepsilon}^2 + f'(u_{1,\varepsilon})w_{1,\varepsilon} - f''(u_{2,\varepsilon})v_{2,\varepsilon}^2 - f'(u_{2,\varepsilon})w_{2,\varepsilon}\big) &\approx 0. \end{aligned} \tag{11.3}$$

From the second equation in (11.3) one can see that $\max\{w_1, w_2\} \sim \varepsilon^{-2}$ since $v_i^2 \sim \varepsilon^{-2}$. Together with the first equation in the system one can see that $\varepsilon^2 w_i \approx \alpha_i$, $i = 1,2$ with $\alpha_1 = -\alpha_2 =: \alpha$. Denote by $\beta_i = \lim_{\varepsilon\to 0} \varepsilon(w_i - \alpha_1\varepsilon^{-2})$ (and assume that the above limit exists) for $i = 1,2$. Then the first equation reduces to

$$\beta_1 + \beta_2 = c[w] - [f''(u)v^2 + f'(u)w] =: \kappa_3. \tag{11.4}$$

The second equation splits into two ones, one for each power of ε,

$$f''(u_{s,1})\xi_1^2 + f''(u_{s,2})\xi_2^2 + (f'(u_{s,1}) - f'(u_{s,2}))\alpha = 0$$

(for ε^{-1}) which determines α as a function of $(u_{s,1}, u_{s,2})$,

$$\alpha = \frac{f''(u_{s,1})\xi_1^2 + f''(u_{s,2})\xi_2^2}{f'(u_{s,2}) - f'(u_{s,1})} = \frac{f''(u_{s,1})(f'(u_{s,2}) - c)^2 + f''(u_{s,2})(c - f'(u_{s,1}))^2}{(f'(u_{s,2}) - f'(u_{s,1}))^3}\kappa_2^2$$

and

$$f'(u_{s,1})\beta_1 + f'(u_{s,2})\beta_2 = c\kappa_3 - 2\alpha \tag{11.5}$$

for ε^0. like in the case of unknowns (ξ_1, ξ_2) the system (11.4, 11.5) has a unique solution (β_1, β_2) provided $u_{s,1} \neq u_{s,2}$. (And one can see now that there are no solutions at all for $u_{s,1} = u_{s,2}$.)

The third equation in (11.3) gives

$$\frac{1}{2}\big(f''(u_{s,1})\xi_1^2 - f''(u_{s,2})\xi_2^2\big) + \frac{\alpha}{2}\big(f'(u_{s,1}) + f'(u_{s,2})\big) = \alpha c.$$

After substitutions of known values, we get the following equation with two unknowns $(u_{s,1}, u_{s,2})$

$$f''(u_{s,1})(f'(u_{s,2}) - c)^3 - f''(u_{s,2})(c - f'(u_{s,1}))^3 = 0. \tag{11.6}$$

Thus, a δ'-shock solution to (11.1) exists if and only if (11.6) has a solution $(u_{s,1}, u_{s,2})$ with $u_{s,1} \neq u_{s,2}$. In general, one can expect that such a solution exists since we have only one equation with two unknowns.

The distributional limit of such a solution is

$$\begin{aligned}
u &\approx u_0 + (u_1 - u_0)\theta(x - ct)\\
v &\approx v_0 + (v_1 - v_0)\theta(x - ct) + \kappa_2 t\delta(x - ct)\\
w &\approx w_0 + (w_1 - w_0)\theta(x - ct) + \kappa_3 t\delta(x - ct) + \alpha t^2\delta'(x - ct),
\end{aligned}$$

where θ is the Heaviside function. The weak uniqueness of a δ'-shock holds if and only if α is the same for all solutions $(u_{s,1}, u_{s,2})$ to (11.6).

Even for a special choice of $f(u) = u^2$ that does not hold: Equation (11.6) reduces to $u_{s,1} + u_{s,2} = c$, $\xi_1 = \xi_2 = \dfrac{\kappa_2}{2}$ and $\alpha = \dfrac{\xi_1^2 + \xi_2^2}{u_{s,2} - u_{s,1}} = \dfrac{\kappa_2^2}{2(u_{s,2} - u_{s,1})}$. Consequently the δ'-shock solution is not weakly unique.

Following formula (11.2), entropy conditions (4.1) are now

$$\begin{aligned}
-c[\eta] + \overline{\lim}_{\varepsilon\to 0}\varepsilon\eta(U_{1,\varepsilon}) + \varepsilon\eta(U_{2,\varepsilon}) + [q(U)] &\leq 0\\
-c(\varepsilon\eta(U_{1,\varepsilon}) + \varepsilon\eta(U_{2,\varepsilon})) + \varepsilon^2\eta(U_{1,\varepsilon}) - \varepsilon^2\eta(U_{2,\varepsilon}) + \varepsilon q(U_{1,\varepsilon}) + \varepsilon q(U_{2,\varepsilon}) &\approx 0\\
-\frac{c}{2}\big(\varepsilon^2\eta(U_{1,\varepsilon}) - \varepsilon^2\eta(U_{2,\varepsilon})\big) + \frac{1}{2}\big(\varepsilon^2 q(U_{1,\varepsilon}) - \varepsilon^2 q(U_{2,\varepsilon})\big) &\approx 0.
\end{aligned}$$

for an entropy pair (η, q).

Now we give the entropy arguments in the special case $f(u) = u^2$. (In that case there also exists an artificial viscosity limit to δ'-shock solution as one could see in [22].) Entropy pairs for system (11.1) are given in [19] and for $f(u) = u^2$ we have

$$\begin{aligned}
\eta(u, v, w) &= G'(u)v^2 + F(u)v + H(u) + G(u)w\\
q(u, v, w) &= 2G(u)uw + 2G'(u)uv^2 + 2G(u)v^2 + 2F(u)uv + 2\int uH'(u)\,du,
\end{aligned}$$

where $\int$ denotes a primitive function and F, G and H are arbitrary twice differentiable functions. The entropy function η is semi-convex if and only if $\eta(u, v, w) = H(u) + l(u, v, w)$, $H''(u) \geq 0$ and l is a linear function. For the linear part, the entropy condition is satisfied with equality due to the given system, so we have to prove it taking $\eta = H(u)$ and $q = 2\int uH'(u)\,du$.

The inequality reduces to

$$-c(H(u_1) - H(u_0)) + 2\int_{u_0}^{u_1} uH'(u)\,du \leq 0.$$

Using $c = u_0 + u_1$ and integration by parts the above condition becomes

$$\frac{H(u_0) + H(u_1)}{2} \geq \frac{1}{u_0 - u_1}\int_{u_1}^{u_0} H(u)\,du.$$

It is true for $u_0 > u_1$ and any semi-convex H and the proof is completed.

12. Composite SDWs

An SDW from Definition 2.1 is defined via one left- and one right-handed infinitesimal cone. We now generalize the definition with SDW with N infinitesimal cones on each side. The new wave is called composite SDW or N-SDW. The generalization of the usual SDW is straightforward. and one of the reasons to present it here is to recover all types of singular shocks described in [16]. The second reason is to improve chances for SDWs to be an approximate solution and to satisfy the entropy criterion (4.2,4.3).

The composite SDWs are not given in the main part of the paper because

- We want to keep discussion in as simple as possible form.
- The "usual" SDWs are good enough for all concrete examples found in the literature. Also, their use do not improve the results in the main example of 3×3 pressureless gas dynamics model.

Let us define them. Put

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x < (c + a_{-N,\varepsilon})t \\ U_{-N,\varepsilon}, & (c + a_{-N,\varepsilon})t < x < (c + a_{-N+1,\varepsilon})t \\ \vdots & \\ U_{0,\varepsilon}, & (c + a_{0,\varepsilon})t < x < (c + a_{1,\varepsilon})t \\ \vdots & \\ U_{N-1,\varepsilon}, & (c + a_{N-1,\varepsilon})t < x < (c + a_{N,\varepsilon})t \\ U_1, & x > (c + a_{N,\varepsilon})t \end{cases} \tag{12.1}$$

The simplest choice is $a_{i,\varepsilon} = i\varepsilon$ (homogeneous), $i = -n, \dots, n$. A more flexible one is $a_{i,\varepsilon} = ib_i\varepsilon$ (semi-homogeneous case), where $b_i \in \mathbb{R}$, $i = -n, \dots, n$ are chosen in an appropriate way.

Using the standard assumption $|U_{i,\varepsilon}| = \mathcal{O}(\varepsilon^{-1})$, $i = -N, \dots, N-1$, we calculate the t-derivative of composite SDW like before:

$$\begin{aligned}
\langle \partial_t U_\varepsilon, \phi \rangle = & \int_0^\infty \sum_{i=-N}^{N-2} -(c + a_{i+1,\varepsilon})\, (U_{i+1,\varepsilon} - U_{i,\varepsilon})\, \phi((c + a_{i+1,\varepsilon})t, t)\, dt \\
& - \int_0^\infty (c + a_{-N,\varepsilon})\, (U_{-N,\varepsilon} - U_0)\, \phi((c + a_{-N,\varepsilon})t, t)\, dt \\
& - \int_0^\infty (c + a_{N,\varepsilon})\, (U_1 - U_{N-1,\varepsilon})\, \phi((c + a_{N,\varepsilon})t, t)\, dt \\
\approx & \int_0^\infty \sum_{i=-N}^{N-2} -(c + a_{i+1,\varepsilon})\, (U_{i+1,\varepsilon} - U_{i,\varepsilon})\, (\phi(ct, t) + a_{i+1,\varepsilon} t \partial_x \phi(ct, t))\, dt \\
& - \int_0^\infty (c + a_{-N,\varepsilon})\, (U_{-N,\varepsilon} - U_0)(\phi(ct, t) + a_{-N,\varepsilon} t \partial_x \phi(ct, t))\, dt \\
& - \int_0^\infty (c + a_{N,\varepsilon})\, (U_1 - U_{N-1,\varepsilon})(\phi(ct, t) + a_{N,\varepsilon} t \partial_x \phi(ct, t))\, dt \\
= & \int_0^\infty \Big(\sum_{i=-N}^{N-1} (a_{i+1,\varepsilon} - a_{i,\varepsilon}) U_{i,\varepsilon} - c(U_1 - U_0) \Big)\, \phi(ct, t)\, dt \\
& + \int_0^\infty c \sum_{i=-N}^{N-1} (a_{i+1,\varepsilon} - a_{i,\varepsilon}) U_{i,\varepsilon}\, t \partial_x \phi(ct, t)\, dt.
\end{aligned}$$

That is

$$\begin{aligned}
\partial_t U_\varepsilon = & \Big(\sum_{i=-N}^{N-1} (a_{i+1,\varepsilon} - a_{i,\varepsilon}) U_{i,\varepsilon} - c(U_1 - U_0) \Big)\, \delta(x - ct) \\
& - c \sum_{i=-N}^{N-1} (a_{i+1,\varepsilon} - a_{i,\varepsilon}) U_{i,\varepsilon}\, t\delta'(x - ct).
\end{aligned}$$

In the same way as above, one finds x-derivative of a composite SDW:

$$\begin{aligned}
\langle \partial_x U_\varepsilon, \phi \rangle \approx & (U_1 - U_0) \int_0^\infty \phi(ct, t)\, dt - \int_0^\infty \sum_{i=-N}^{N-1} (a_{i+1,\varepsilon} - a_{i,\varepsilon}) U_{i,\varepsilon}\, t \partial_x \phi(ct, t)\, dt \\
= & (U_1 - U_0) \langle \delta(x - ct), \phi \rangle + \sum_{i=-N}^{N-1} (a_{i+1,\varepsilon} - a_{i,\varepsilon}) U_{i,\varepsilon} \langle t\delta'(x - ct), \phi \rangle.
\end{aligned}$$

Suppose that f and g are continuous mappings from the domain Ω into $\mathbb{R}^n$. Take, for example, the semi-homogeneous case where we choose the perturbation $a_{i,\varepsilon}$, $i = -N, \dots, N$ such that

$$b_i := \frac{a_{i+1,\varepsilon} - a_{i,\varepsilon}}{\varepsilon} \in \mathbb{R},\ i = -N, \dots, N-1.$$

Then

$$\begin{aligned}\partial_t f(U_\varepsilon) \approx&\Big(-c(f(U_1)-f(U_0)) + \sum_{i=-N}^{N-1} b_i \varepsilon f(U_{i,\varepsilon})\Big)\delta(x-ct)\\ &- c\sum_{i=-N}^{N-1} b_i\varepsilon\, f(U_{i,\varepsilon})\, t\,\delta'(x-ct)\\ \partial_x g(U_\varepsilon) \approx&(g(U_1)-g(U_0))\delta(x-ct) + \sum_{i=-N}^{N-1} b_i\varepsilon\, g(U_{i,\varepsilon})\, t\,\delta'(x-ct).\end{aligned}$$

Thus, a composite SDW is a approximate solution to (1.1) if

$$\varepsilon\sum_{i=-N}^{N-1} b_i f(U_{i,\varepsilon}) \approx \kappa, \text{ and } \varepsilon\sum_{i=-N}^{N-1} b_i g(U_{i,\varepsilon}) \approx c\kappa, \tag{12.2}$$

where $\kappa = (\kappa^1, \dots, \kappa^n)$ is a vector of RH deficits.

If the system (1.1) is linear in first component, then

$$\xi_i := \lim_{\varepsilon\to 0} \varepsilon U^1_{i,\varepsilon} \in \mathbb{R},\ i = -N, \dots, N-1,$$

while all other variables and constants have a same meaning as in Section 6. Then (12.2) has a simpler form

$$\sum_{i=-N}^{N-1} b_i f(\underline{U}_{s,i})\xi_i = \kappa, \text{ and } \sum_{i=-N}^{N-1} b_i g(\underline{U}_{s,i})\xi_i = c\kappa.$$

Next, assume that there exists a entropy pair η, q for a system which a SDW U_ε of the form (12.1). Then the limits in the distributional sense has to satisfy

$$\begin{aligned}\overline{\lim}_{\varepsilon\to 0} - c(\eta(U_1)-\eta(U_0)) + \sum_{i=-N}^{N-1}\varepsilon b_i\,\eta(U_{i,\varepsilon}) + q(U_1) - q(U_0) \leq& 0\\ \lim_{\varepsilon\to 0}\sum_{i=-N}^{N-1}\varepsilon b_i\,(q(U_{i,\varepsilon}) - c\eta(U_{i,\varepsilon})) =& 0.\end{aligned} \tag{12.3}$$

That substitutes entropy conditions (4.2,4.3) for usual SDW.

Now, one can see that choosing the constants b_i, $i = -N, \dots, N-1$ during a solution construction gives us a better opportunity to satisfy the second equation in (12.3) provided that the number of useful entropy pairs is bounded.

Acknowledgment

The author is grateful to the referee for his/her valuable comments and corrections of the paper.

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