HIGHER ORDER SHADOW WAVES AND DELTA SHOCK BLOW UP IN THE CHAPLYGIN GAS

MARKO NEDELJKOV

Abstract. The introductory part of this paper contains an overview of known results about elementary and delta shock solutions to Riemann problem for well known Chaplygin gas model (nowadays used in cosmological theories for dark energy) in terms of entropic shadow waves. Shadow waves are introduced in [16] and they are represented by shocks depending on a small parameter ε with unbounded amplitudes having a distributional limit involving the Dirac delta function. In a search for admissible solutions to all possible cases of mutual interactions of waves arising from double Riemann initial data we found same cases that can not be resolved with already known types of elementary or shadow wave solutions. These cases are resolved by introducing a sequence of higher order shadow waves depending on integer powers of ε. It is shown that such waves has a distributional limit but only until some finite time T.

1. Introduction

Many conservation law systems posses formal delta (or singular) shock wave solutions but only some of them are physically relevant. The most common admissibility criteria used to select a proper solution of that type is overcompressibility. A wave is called overcompressive if all characteristics from both sides run into the shock. That resembles a mass concentration that goes to infinity at some points (see [3] for a real model of such a process). In paper [16], the author introduces shadow wave solution to conservation law systems. They are represented by nets of piecewise constant function for time variable t fixed parametrized by $\varepsilon > 0$. Some of these constants are of order ε^{-1} but supported by volumes of order $\varepsilon \ll 1$ so they stays bounded in L^1_{loc}-sense with respect to x-variable for each t. They contains delta and singular shocks as special but the most important examples. Their construction enables one to easily check admissibility of the obtained solution by using the Lax (semi)convex entropy – entropy flux pair for the given system by using standard Rankine–Hugoniot conditions. In almost all cases from literature where delta or singular shocks appear, these admissibility conditions are are proved to be good enough for selecting a unique (and physically relevant) delta shock (singular shock or shadow) wave solution (see [16]). Another strong point of the shadow wave theory is that the interaction problems can be easily approached. That is due to the fact that shadow waves resemble a well known and efficient Wave Front Tracking procedure (see [4] for example).

In this paper, we faces he following problem. After a successful use of both (equivalent in this case also) admissibility conditions for finding a unique solution to any Riemann problem. We look for a solution to a double Riemann problem (used for investigation of wave interactions). There are case with no admissible solution in the class of elementary and shadow waves. In fact, there exists a shadow wave weak solution but it is not admissible in the above sense. We tried to overcome that problem by introducing a sequence of shadow waves each new one parametrized by some new, smaller parameter that previous ones. Let us explain a bit that procedure: We fix ε (the initial shadow wave parameter) and solve the new initial problem obtained in such way by using elementary waves and shadow waves parametrized now by a new $\varepsilon_1 \ll \varepsilon$. New shadow wave is said to have order one. At a new interaction time we fix ε_1 and introduce a new parameter ε_2 for shadow waves of order two and solve the initial problem. And so on. The introduction of each new parameter increase the shadow wave order by one. Later on, it will be shown that it is enough to use a sequence of parameters $\varepsilon_i = \varepsilon^{i+1}$, $i = 1, 2, \ldots$. That procedure resembles well known weak asymptotic methods (let us just mention two in a huge set of papers dealing with similar objects, [12] and [7]).

The system that we investigate is the Chaplygin gas model. Some of the cosmology theories uses it as a model of the so called dark energy of the Universe. It models a compressible fluid with the pressure inversely proportional to the gas energy density, $p = -A/\rho$, for some $A > 0$ (see [10] for physical explanations). There are also more recent models with the pressure defined by $p = -A/\rho^\alpha$, $0 < \alpha \leq 1$, with the first one (up to our knowledge) introduced in [1], that are called generalized Chaplygin gas. In these models there is a significant mathematical difference between the cases $\alpha = 1$ and $\alpha \in (0.1)$. The first case is analyzed here, while the second one is considered in [17].

The system modeling Chaplygin gas consists of mass and momentum conservation laws

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x\Big(\rho u^2 - \frac{A}{\rho}\Big) &= 0, \end{aligned}$$

where u denotes the velocity of the gas. In this paper we shall fix $A = 1$ and use the momentum variable $q = \rho u$,

$$\begin{aligned} \partial_t \rho + \partial_x q &= 0 \\ \partial_t q + \partial_x\Big(\frac{q^2 - 1}{\rho}\Big) &= 0. \end{aligned} \tag{1.1}$$

The physical domain for the system is the hyperplane $\{(\rho, q) |\ \rho > 0\}$ since a pressure in the vacuum state would be infinite otherwise. The sound speed of the system tends to zero as $\rho \to \infty$. That property allows the mass concentration in a finite time and one can expect delta shock or some other wave with similar properties to be a part of a solution at least for some initial data.

Important clues for our investigation can be found in the classical pressureless gas dynamics model ($p = 0$) where the delta shock solutions are well known and examined (see [3], [5], [8], for instance). Moreover, the existence of delta shocks for system (1.1) was already proved in [2]. A detailed analysis of the system properties is presented there and we will state only some important facts here. The system is strictly hyperbolic with the characteristics $\lambda_1(\rho, q) = \frac{q-1}{\rho} < \lambda_2(\rho, q) = \frac{q+1}{\rho}$, and both fields are linearly degenerate (let us note that it is not the case for the generalized Chaplygin gas). So, there are only contact discontinuities as the elementary wave solutions for the Riemann data

$$(\rho,q)=\begin{cases}(\rho_0,q_0), & x<0\\ (\rho_1,q_1), & x>0.\end{cases} \tag{1.2}$$

The contact discontinuity curves are given by

$$\mathrm{CD}_1(\rho_0,q_0):\ q_1=1+\frac{q_0-1}{\rho_0}\rho_1,\ \mathrm{CD}_2(\rho_0,q_0):\ q_1=-1+\frac{q_0+1}{\rho_0}\rho_1 \tag{1.3}$$

Using the standard methods for finding entropies (see [6] for example), one finds that the system posses an infinite number of convex entropies. The general form of an entropy function for (1.1) is

$$\eta=\frac{\rho}{2}\Big(F\Big(\frac{q-1}{\rho}\Big)+G\Big(\frac{q+1}{\rho}\Big)\Big) \tag{1.4}$$

with the entropy-flux function given by

$$Q=\frac{1}{2}\Big((q+1)F\Big(\frac{q-1}{\rho}\Big)+(q-1)G\Big(\frac{q+1}{\rho}\Big)\Big). \tag{1.5}$$

The entropy function η is convex if and only if both F and G are convex. The most important additional conservation law is the energy conservation (see [2])

$$\partial_t\Big(\frac{q^2+1}{\rho}\Big)+\partial_x\Big(\frac{q}{\rho}\frac{q^2-1}{\rho}\Big)=0.$$

It is straightforward to see from (1.3) that the states (ρ_0,q_0) and (ρ_1,q_1) in (1.2) can be connected by the elementary waves if and only if

$$\begin{aligned}&\lambda_2(\rho_1,q_1)=\frac{q_1+1}{\rho_1}>\frac{q_0-1}{\rho_0}=\lambda_1(\rho_0,q_0),\ \text{i.e. when}\\ &\rho_0(q_1+1)-\rho_1(q_0-1)>0.\end{aligned} \tag{1.6}$$

The solution then consists of two contact discontinuities connected by a constant state

$$(\rho_s,q_s)=\Big(\frac{2}{\lambda_1(\rho_1,q_1)-\lambda_2(\rho_0,q_0)},\frac{\lambda_2(\rho_1,q_1)+\lambda_1(\rho_0,q_0)}{\lambda_2(\rho_1,q_1)-\lambda_1(\rho_0,q_0)}\Big).$$

We shall call such a solution the contact discontinuity combination (CDC) in the sequel.

If the condition (1.6) is not satisfied, then there exists a solution in the form of single delta shock wave as it was proved in [2]. The proof is obtained using a kind of measure spaces that does not suit our purposes. Therefore, we will proceed as follows. First, we repeat the results of [2] for shadow waves. One of the results is that there exists an entropic single shadow wave solution if and only if condition (1.6) does not hold. Let us remark that violation of (1.6) is equivalent to the fact that a shadow wave is overcompressive. Here we prove that that condition is equivalent to the entropy inequality for a convex entropy function. Moreover, it is enough to take a single entropy function that represents the energy. So, we have a unique solution just using the fact that energy can not rise in the system.

Note that the system (1.1) can not be solved using the canonical methods given in [16] due to the lack of linearity in both of the variables. Nevertheless the non-linearity in the flux can be resolved with shadow waves as we will present bellow. Some problems involving nonlinear operations with delta functions are successfully resolved in [11] and or [13] using approximate delta functions.

In this paper we analyze all possible cases of interactions. The incoming waves can be shadow waves and contact discontinuity combination so there are three possible cases: the interaction of two shadow waves, the shadow wave–contact discontinuity combination and the two contact discontinuity combinations. The last case is the simplest one - an outgoing wave after a double contact discontinuity interaction is either a contact discontinuity combination or a single shadow wave like in the case of the Riemann data. In the case of a double shadow wave interaction an outgoing wave is always a single weighted shadow wave (that can be interpreted as a delta shock with variable strength and variable speed). In the second case, when a shadow wave and contact discontinuity combination interacts, the possible solutions are two delta contact discontinuities (see [15] for the definition of delta contact discontinuity) or a single weighted shadow wave. In both cases these waves are followed by the second incoming contact discontinuity without an interaction. But there is also a possible case of special interest to us when neither elementary nor shadow wave solution exists. More precisely, if that is the case, each solution candidate we have used does not satisfies the entropy condition.[1]

Thus we shall present a new type of approximated solution consists of sequence of higher order shadow waves. Shadow waves of the second order are made by using new parameter $\varepsilon_1 \ll \varepsilon$ while the old one, ε, is treated as a constant. Each shadow wave of the higher order is made recursively by using a new negligible parameter $\varepsilon_{n+1} \ll \varepsilon_n$. There are two possibilities for a result of such a procedure. In the first one a sequence of higher order shadow waves has a distributional limit but only up to some finite time. That is, it can be used as an explanation of a blow-up mechanism for system admitting delta shock solutions in such case. The existence of entropic solution after the shadow wave blows-up is still open. We suspect that any kind of approximate solution does not have a distributional limit after the blow up time. In some way a similar situation is described in [14] for ionic gas model ([9]), where a wave that carries the delta function (a singular shock) disappear after interaction with another one (a rarefaction wave). But contrary to our case, there are a classical double shock wave solution after delta function annihilates. One can also look for some solutions of a different type as it was done in [18].

In the second one a sequence of the higher order waves turns to be a sequence of parametrized classical contact discontinuities and a distributional limit exists for all times.

2. Riemann problem

Let us start with a piecewise constant function of the following form called the simple shadow wave (SDW for short)

$$(\rho, q) = \begin{cases} (\rho_0, q_0), & x < (c-\varepsilon)t \\ (\rho_{0,\varepsilon}, q_{0,\varepsilon}), & (c-\varepsilon)t < x < ct \\ (\rho_{1,\varepsilon}, q_{1,\varepsilon}), & ct < x < (c+\varepsilon)t \\ (\rho_1, q_1), & x > (c+\varepsilon)t. \end{cases}$$

[1] Let us note that some author used relaxed entropy condition to get a solution [?]. We do not find any physical reason to support such solution and we omitted it in our analysis.

The SDW (p,q) solves (1.1,1.2) in the weak sense if

$$\lim_{\varepsilon\to 0}\langle \rho,\partial_t\phi\rangle+\langle q,\partial_x\phi\rangle=0$$

$$\lim_{\varepsilon\to 0}\langle q,\partial_t\phi\rangle+\Big\langle \frac{q^2-1}{\rho},\partial_x\phi\Big\rangle=0$$

for every test function $\phi\in C_0^\infty(\mathbb{R}_+^2)$. Using Lemma 1 from [16] one gets the following formulas for the derivatives

$$\begin{aligned}
\partial_t\rho &\approx \big(-c[\rho]+(\varepsilon\rho_{0,\varepsilon}+\varepsilon\rho_{1,\varepsilon})\big)\delta-c(\varepsilon\rho_{0,\varepsilon}+\varepsilon\rho_{1,\varepsilon})t\delta'\\
\partial_x q &\approx [q]\delta+(\varepsilon q_{0,\varepsilon}+\varepsilon q_{1,\varepsilon})t\delta'\\
\partial_t q &\approx \big(-c[q]+(\varepsilon q_{0,\varepsilon}+\varepsilon q_{1,\varepsilon})\big)\delta-c(\varepsilon q_{0,\varepsilon}+\varepsilon q_{1,\varepsilon})t\delta'\\
\partial_x\Big(\frac{q^2-1}{\rho}\Big) &\approx \Big[\frac{q^2-1}{\rho}\Big]\delta+\Big(\varepsilon\Big(\frac{q_{0,\varepsilon}^2-1}{\rho_{0,\varepsilon}}\Big)+\varepsilon\Big(\frac{q_{1,\varepsilon}^2-1}{\rho_{1,\varepsilon}}\Big)\Big)t\delta'.
\end{aligned}$$

Here and bellow $a_\varepsilon\approx b_\varepsilon$ means $\lim_{\varepsilon\to 0}a_\varepsilon-b_\varepsilon=0$ while $[y]:=y_1-y_0$ is the standard designation of a jump in the variable y across a shock front. The support of delta function and its derivative above is called a shock front. In the above formulas, the shock front is the line $x=ct$.

It is easy to see that the only possibility to avoid a trivial case when $\rho_{i,\varepsilon}$ and $q_{i,\varepsilon}$ disappear is that $\rho_{i,\varepsilon},q_{i,\varepsilon}\sim\varepsilon^{-1}$, $i=0,1$. Denoting $\xi_i:=\lim_{\varepsilon\to 0}\varepsilon\rho_i$, $\chi_i:=\lim_{\varepsilon\to 0}\varepsilon q_i$, $i=0,1$, we have $\frac{q_{0,\varepsilon}^2-1}{\rho_{0,\varepsilon}}\approx\frac{\chi_i^2}{\xi_i}$, $i=0,1$. So the Riemann problem (1.1,1.2) reduces to the following system of algebraic equations

$$\begin{aligned}
-c[\rho]+(\xi_0+\xi_1)+[q]&=0\\
c(\xi_0+\xi_1)&=\chi_0+\chi_1\\
-c[q]+(\chi_0+\chi_1)+\Big[\frac{q^2-1}{\rho}\Big]&=0\\
c(\chi_0+\chi_1)&=\frac{\chi_0^2}{\xi_0}+\frac{\chi_1^2}{\xi_1}.
\end{aligned}\tag{2.1}$$

Denote by $\kappa_1:=c[\rho]-[q]$ and $\kappa_2:=c[q]-\Big[\frac{q^2-1}{\rho}\Big]$ so-called the Rankine-Hugoniot deficits. One immediately gets $\kappa_2=c\kappa_1$ from the second equation. The third and fourth equation then imply

$$c=\frac{[q]\pm\sqrt{[q]^2-[\rho]\Big[\frac{q^2-1}{\rho}\Big]}}{[\rho]}=\frac{[q]+\kappa_1}{[\rho]}.\tag{2.2}$$

From the fourth equation one can see that the only possible relations between the unknowns ξ_i and χ_i, $i=0,1$, are

$$\xi_0=\frac{\chi_0}{c}\text{ and }\xi_1=\frac{\chi_1}{c}.$$

The first and the third equation in (2.1) uniquely determine strength of the SDW (ξ,χ) defined by

$$\xi:=\xi_0+\xi_1=\kappa_1,\ \chi:=\chi_0+\chi_1=\kappa_2=c\kappa_1.$$

The variable ρ denotes the density so $\kappa_1>0$ (the case $\kappa_1=0$ corresponds to the contact discontinuity solution). The positivity of κ_1 implies that one has to take the plus sign in expression (2.2) for the speed c. A simple computation gives

$$\begin{aligned}\kappa_1 &= \sqrt{\rho_0\rho_1\Big(\frac{q_0-1}{\rho_0}-\frac{q_1-1}{\rho_1}\Big)\Big(\frac{q_0+1}{\rho_0}-\frac{q_1+1}{\rho_1}\Big)}\\ &= \sqrt{\rho_0\rho_1(\lambda_1(\rho_0,q_0)-\lambda_1(\rho_1,q_1))(\lambda_2(\rho_0,q_0)-\lambda_2(\rho_1,q_1))}.\end{aligned}$$

It is obvious that the negation of condition (1.6) ensures the positivity of the term under the square root. Thus, we have well defined SDW solution whenever (1.6) is not satisfied i.e. when $\lambda_1(\rho_0,q_0)\geq\lambda_2(\rho_1,q_1)$. It remains to prove that the SDW is entropic. A SDW (ρ,q) is entropic (and thus admissible) if for every (semi)convex entropy function η and corresponding entropy flux function Q we have

$$\langle\partial_t\eta(\rho,q)+\partial_x Q(\rho,q),\phi\rangle\leq 0 \tag{2.3}$$

for every non-positive test function $\phi\in C_0^\infty$. According to formulas (4.2) and (4.3) from [16], a SDW solution (ρ,q) to (1.1) is entropic if and only if

$$\begin{aligned}\lim_{\varepsilon\to 0} -c(\varepsilon\eta(\rho_{0,\varepsilon},q_{0,\varepsilon})+\varepsilon\eta(\rho_{1,\varepsilon},q_{1,\varepsilon}))+\varepsilon Q(\rho_{0,\varepsilon},q_{0,\varepsilon})+\varepsilon Q(\rho_{1,\varepsilon},q_{1,\varepsilon}) &= 0\\ -c(\eta(\rho_1,q_1)-\eta(\rho_0,q_0))+Q(\rho_1,q_1)-Q(\rho_0,q_0)&\\ +\lim_{\varepsilon\to 0}(\varepsilon\eta(\rho_{0,\varepsilon},q_{0,\varepsilon})+\varepsilon\eta(\rho_{1,\varepsilon},q_{1,\varepsilon}))&\leq 0.\end{aligned} \tag{2.4}$$

Substituting (1.4) and (1.5) into the first relation above we obtain

$$-\frac{c}{2}\Big(\xi_0F(c)+\xi_1G(c)\Big)+\frac{1}{2}\Big(\chi_0F(c)+\chi_1G(c)\Big)=0$$

since $\chi_i=c\xi_i$, $i=0,1$. The second relation from (2.4) reduces to

$$\begin{aligned}-c\Bigg(\frac{\rho_1}{2}\Bigg(F\Big(\frac{q_1-1}{\rho_1}\Big)+G\Big(\frac{q_1+1}{\rho_1}\Big)\Bigg)-\frac{\rho_0}{2}\Bigg(F\Big(\frac{q_0-1}{\rho_0}\Big)+G\Big(\frac{q_0+1}{\rho_0}\Big)\Bigg)\Bigg)&\\ +\frac{1}{2}\Bigg((q_1+1)F\Big(\frac{q_1-1}{\rho_1}\Big)+(q_1-1)G\Big(\frac{q_1+1}{\rho_1}\Big)&\\ -(q_0+1)F\Big(\frac{q_0-1}{\rho_0}\Big)-(q_0-1)G\Big(\frac{q_0+1}{\rho_0}\Big)\Bigg)+\frac{\xi_0+\xi_1}{2}\Big(F(c)+G(c)\Big)&\leq 0.\end{aligned}$$

Here we give the proof for $G\equiv 0$ only. The proof for $F\equiv 0$ (and thus for a general case since one can prove the inequality for each addend separately) goes along the same lines so we omit it.

One has to prove

$$-\Bigg(\Big(q_0+1-c\rho_0\Big)F\Big(\frac{q_0-1}{\rho_0}\Big)+\Big(c\rho_1-(q_1+1)\Big)F\Big(\frac{q_1-1}{\rho_0}\Big)\Bigg)+\kappa_1F(c)\leq 0.$$

In order to do it, let us put

$$I:=-\Bigg(\frac{q_0+1-c\rho_0}{\kappa_1}F\Big(\frac{q_0-1}{\rho_0}\Big)+\frac{c\rho_1-(q_1+1)}{\kappa_1}F\Big(\frac{q_1-1}{\rho_0}\Big)\Bigg)+F(c).$$

Using

$$\frac{q_0+1-c\rho_0}{\kappa_1}+\frac{c\rho_1-(q_1+1)}{\kappa_1}=\frac{c(\rho_1-\rho_0)-(q_1-q_0)}{\kappa_1}$$

and the convexity of F one has

$$\begin{aligned} I \leq & -F\Big(\frac{q_0+1-c\rho_0}{\kappa_1}\frac{q_0-1}{\rho_0}+\frac{c\rho_1-(q_1+1)}{\kappa_1}\frac{q_1-1}{\rho_1}\Big)+F(c) \\ = & -F\left(\frac{1}{\kappa_1}\Big(\underbrace{c(\rho_1-\rho_0)-\Big(\frac{q_1^2-1}{\rho_1}-\frac{q_0^2-1}{\rho_0}\Big)}_{=\kappa_2=c\kappa_1}\Big)\right)+F(c)=-F(c)+F(c)=0. \end{aligned}$$

Note that it is necessary that both of $\dfrac{q_0+1-c\rho_0}{\kappa_1}$ and $\dfrac{c\rho_1-(q_1+1)}{\kappa_1}$ are non-negative. Thus the SDW has to satisfy

$$\lambda_2(\rho_0,q_0) \geq c \geq \lambda_2(\rho_1,q_1). \tag{2.5}$$

If one of the terms $q_0+1-c\rho_0$ or $c\rho_1-(q_1+1)$ is negative then it is easy to find an F such that $I>0$. Therefore $I \leq 0$ for every convex F if and only if (2.5) holds.

Using the same procedure for G one can get that the wave is entropic if and only if

$$\lambda_1(\rho_0,q_0) \geq c \geq \lambda_1(\rho_1,q_1). \tag{2.6}$$

A wave satisfying (2.5) and (2.6) is said to be overcompressive.

So, we have proved the following theorem.

Theorem 2.1. *The Riemann problem (1.1,1.2) has a unique entropic solution which consists of two contact discontinuities if (1.6) holds. If (1.6) does not hold, the solution is a single SDW represented by*

$$(\rho,q)=\begin{cases}(\rho_0,q_0), & x<(c-\varepsilon)t\\ (\xi_0/\varepsilon,\chi_0/\varepsilon), & (c-\varepsilon)t<x<ct\\ (\xi_1/\varepsilon,\chi_1/\varepsilon), & ct<x<(c+\varepsilon)t\\ (\rho_1,q_1), & x>(c+\varepsilon)t,\end{cases}$$

with $c=\frac{q_1-q_0+\kappa_1}{\rho_1-\rho_0}$, $\xi_0+\xi_1=\kappa_1$, $\chi_0+\chi_1=c\kappa_1$, *where the Rankine-Hugoniot deficit is given by* $\kappa_1=\sqrt{\rho_0\rho_1(\lambda_1(\rho_0,q_0)-\lambda_1(\rho_1,q_1))(\lambda_2(\rho_0,q_0)-\lambda_2(\rho_1,q_1))}$.

Remark 2.1. The term “unique solution” in Theorem 2.1 should be understood as a weakly unique sense defined by Definition 4.1. in [16], that means that all SDW solutions have the same distributional limit. Next, all assertions are valid for any combination of ξ_0 and ξ_1 or χ_0 and χ_1 as long as ξ_0 and ξ_1 stay non-negative and have sums determined in the theorem. Particularly, it is safe to take $\rho_{0,\varepsilon}=\rho_{1,\varepsilon}=:\rho_\varepsilon$ (i.e. $\xi_0=\xi_1$) and $q_{0,\varepsilon}=q_{1,\varepsilon}=q_\varepsilon$ (i.e. $\chi_0=\chi_1$) in the sequel.

3. Weighted SDWs

In order to examine results of different wave interactions, let us start with the new initial data

$$(\rho,q)(x,0)=\begin{cases}(\rho_0,q_0), & x<-x_0\\ (\rho_1,q_1), & -x_0<x<0\\ (\rho_2,q_2), & -x>0.\end{cases} \tag{3.1}$$

where x_0 is a positive real. Assume that an SDW emerges from the point $(-x_0,0)$. The case when it comes from the right-hand side of the wave that it will interact with can be treated in the same way. The waves will always interact because any entropic SDW is overcompressive. Denote by T the time when the SDW reaches another wave and denote by ξ_{in} and χ_{in} the sum of the SDW strengths in ρ and q variable, respectively. One of solutions we are looking for when $t>T$ will be a weighted SDW (see [16]) of the general form

$$(\rho,q)=\begin{cases}(\rho_0,q_0), & x<c(t)-a_{0,\varepsilon}(t)\\ (\rho_{0,\varepsilon},q_{0,\varepsilon})(t), & c(t)-a_{0,\varepsilon}(t)<x<c(t)\\ (\rho_{1,\varepsilon},q_{1,\varepsilon})(t), & c(t)<x<c(t)+a_{1,\varepsilon}(t)\\ (\rho_2,q_2), & x>c(t)+a_{1,\varepsilon}(t).\end{cases} \tag{3.2}$$

Denote $\xi_i(t)=\lim_{\varepsilon\to0}\rho_{i,\varepsilon}(t)a_{i,\varepsilon}(t)$, $\chi_i(t)=\lim_{\varepsilon\to0}q_{i,\varepsilon}(t)a_{i,\varepsilon}(t)$, $i=0,1$, $\xi(t)=\xi_0(t)+\xi_1(t)$ and $\chi(t)=\chi_0(t)+\chi_1(t)$. It is enough to assume $\xi(T)=\xi_{in}$ and $\chi(T)=\chi_{in}$ and then the weighted SDW for $t>T$, if it exists, can be joined with the incoming waves (for $t<T$). That follows from Theorem 7.1 and the note after Lemma 10.1 in [16]. By Lemma 10.1 from [16], we have

$$\begin{aligned}\partial_t\rho=&-c'(t)[\rho]\delta+\lim_{\varepsilon\to0}\frac{d}{dt}\big(a_{0,\varepsilon}(t)\rho_{0,\varepsilon}(t)+a_{1,\varepsilon}(t)\rho_{1,\varepsilon}(t)\big)\delta\\ &-c'(t)\lim_{\varepsilon\to0}\big(a_{0,\varepsilon}(t)\rho_{0,\varepsilon}(t)+a_{1,\varepsilon}(t)\rho_{1,\varepsilon}(t)\big)\delta'\\ \approx&\big(-c'(t)[\rho]-(\xi_0'(t)+\xi_1'(t))\big)\delta-c'(t)(\xi_0(t)+\xi_1(t))\delta'\\ \partial_x q\approx&[q]\delta+(\chi_0'(t)+\chi_1'(t))\delta',\\ \partial_t q\approx&\big(-c'(t)[q]-(\chi_0'(t)+\chi_1'(t))\big)\delta-c'(t)(\chi_0(t)+\chi_1(t))\delta'\\ \partial_x\Big(\frac{q^2-1}{\rho}\Big)=&\Big[\frac{q^2-1}{\rho}\Big]\delta+\lim_{\varepsilon\to0}\Big(a_{0,\varepsilon}(t)\frac{q_{0,\varepsilon}^2(t)-1}{\rho_{0,\varepsilon}(t)}+a_{1,\varepsilon}(t)\frac{q_{1,\varepsilon}^2(t)-1}{\rho_{1,\varepsilon}(t)}\Big)\delta'\\ \approx&\Big[\frac{q^2-1}{\rho}\Big]\delta+\Big(\frac{\chi_0^2(t)}{\xi_0(t)}+\frac{\chi_1^2(t)}{\xi_1(t)}\Big)\delta'\end{aligned}$$

with the support of δ and δ' being the curve $x=c(t)$. Substitution of these expressions into the system yields the following system of differential equations

$$\begin{aligned}c'(t)[\rho]-[q]&=(\xi_0(t)+\xi_1(t))'\\ c'(t)(\xi_0(t)+\xi_1(t))&=\chi_0(t)+\chi_1(t)\\ c'(t)[q]-\Big[\frac{q^2-1}{\rho}\Big]&=(\chi_0(t)+\chi_1(t))'\\ c'(t)\big(\chi_0(t)+\chi_1(t)\big)&=\frac{\chi_0^2(t)}{\xi_0(t)}+\frac{\chi_1^2(t)}{\xi_1(t)}\end{aligned}$$

with the initial conditions $\xi_0(T)+\xi_1(T)=\xi_{in}$, $\chi_0(T)+\chi_1(T)=\chi_{in}$. We assume without a loss in generality (as in Remark 2.1) that $\xi_0=\xi_1$ and $\chi_0=\chi_1$. Then the above system reduces to

$$\begin{aligned}c'(t)[\rho]-[q]&=\xi'(t)\\ c'(t)\xi(t)&=\chi(t)\\ c'(t)[q]-\Big[\frac{q^2-1}{\rho}\Big]&=\chi'(t)\\ c'(t)\chi(t)&=\frac{\chi^2(t)}{\xi(t)},\ \xi(T)=\xi_{in},\ \chi(T)=\chi_{in}.\end{aligned} \tag{3.3}$$

Obviously, the fourth equation in (3.3) is satisfied if the second one is i.e. $\chi(t) = c'(t)\xi(t)$. More precisely, the first three equations determines all quantities $c(t)$, $\xi(t)$ and $\chi(t)$. After some straightforward calculations one finds that the SDW speed is

$$c'(t) = \frac{[q]}{[\rho]} + \frac{\left([q]^2 - [\rho]\left[\frac{q^2-1}{\rho}\right]\right)(t-T) + ([\rho]\chi_{in} - [q]\xi_{in})}{[\rho]\sqrt{\left([q]^2 - [\rho]\left[\frac{q^2-1}{\rho}\right]\right)(t-T)^2 + 2([\rho]\chi_{in} - [q]\xi_{in})(t-T) + \xi_{in}^2}}.$$

The choice of the proper sign in the above expression follows from the fact that Rankine-Hugoniot deficit in the first equation represent the density and therefore it should be nonnegative. Furthermore, one gets the following solution to (3.3)

$$\begin{aligned} c(t) =& c(0) + \frac{1}{[\rho]}\Bigg([q](t-T) - \xi_{in} \\ &+ \sqrt{\left([q]^2 - [\rho]\left[\frac{q^2-1}{\rho}\right]\right)(t-T)^2 + 2([\rho]\chi_{in} - [q]\xi_{in})(t-T) + \xi_{in}^2}\Bigg) \\ \xi(t) =& \sqrt{\left([q]^2 - [\rho]\left[\frac{q^2-1}{\rho}\right]\right)(t-T)^2 + 2([\rho]\chi_{in} - [q]\xi_{in})(t-T) + \xi_{in}^2}, \end{aligned} \tag{3.4}$$

and $\chi(t) = c'(t)\xi(t)$ as already known.

That is, a solution in the form of weighted SDW exists provided

$$0 < [q]^2 - [\rho]\left[\frac{q^2-1}{\rho}\right](t-T)^2 + 2([\rho]\chi_{in} - [q]\xi_{in})(t-T) + \xi_{in}^2.$$

But that is obviously true, at least in a neighborhood of T. Now, one has to check entropy conditions for such wave. The following lemma makes that task easier.

Lemma 3.1. *Weighted SDW given by (3.2) satisfy the overcompressibility condition $\lambda_1(\rho_0, q_0) \geq c'(t) \geq \lambda_2(\rho_2, q_2)$, if and only if satisfies the entropy condition for any convex entropy pair (1.4, 1.5).*

Proof. The entropy condition for the weighted SDW consists of the following two relations (see (10.6) and (10.7) in [16])

$$\begin{aligned} &\lim_{\varepsilon\to 0} c'(t)\big(a_{0,\varepsilon}(t)\eta(\rho_{0,\varepsilon}(t), q_{0,\varepsilon}(t)) + a_{1,\varepsilon}(t)\eta(\rho_{1,\varepsilon}(t), q_{1,\varepsilon}(t))\big) \\ &\quad - \big(a_{0,\varepsilon}(t)Q(\rho_{0,\varepsilon}(t), q_{0,\varepsilon}(t)) + a_{1,\varepsilon}(t)Q(\rho_{1,\varepsilon}(t), q_{1,\varepsilon}(t))\big) = 0 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &-c'(t)\big(\eta(\rho_2, q_2) - \eta(\rho_0, q_0)\big) + Q(\rho_2, q_2) - Q(\rho_0, q_0) \\ &+ \lim_{\varepsilon\to 0}\frac{d}{dt}\big(a_{0,\varepsilon}(t)\eta(\rho_{0,\varepsilon}(t), q_{0,\varepsilon}(t)) + a_{1,\varepsilon}(t)\eta(\rho_{1,\varepsilon}(t), q_{1,\varepsilon}(t))\big) \leq 0 \end{aligned} \tag{3.6}$$

Suppose that $\lim_{\varepsilon\to 0}\dfrac{q_{i,\varepsilon}(t)}{\rho_{i,\varepsilon}(t)} = \zeta_i$ exists for $i = 0, 1$. Then (3.5) is satisfied for the entropy pair (η, Q) given by (1.4, 1.5) if $\chi_i = c'(t)\xi_i(t)$, $i = 0, 1$:

$$\begin{aligned} c'(t)\Bigg(& a_{0,\varepsilon}(t)\frac{\rho_{0,\varepsilon}(t)}{2}\left(F\Big(\frac{q_{0,\varepsilon}(t)-1}{\rho_{0,\varepsilon}(t)}\Big) + G\Big(\frac{q_{0,\varepsilon}(t)+1}{\rho_{0,\varepsilon}(t)}\Big)\right) \\ &+ a_{1,\varepsilon}(t)\frac{\rho_{1,\varepsilon}(t)}{2}\left(F\Big(\frac{q_{1,\varepsilon}(t)-1}{\rho_{1,\varepsilon}(t)}\Big) + G\Big(\frac{q_{1,\varepsilon}(t)+1}{\rho_{1,\varepsilon}(t)}\Big)\right)\Bigg) \end{aligned}$$

$$
\begin{aligned}
&-\frac{a_{0,\varepsilon}(t)}{2}\Bigg((q_{0,\varepsilon}(t){+}1)F\Big(\frac{q_{0,\varepsilon}(t){-}1}{\rho_{0,\varepsilon}(t)}\Big){+}(q_{0,\varepsilon}(t){-}1)G\Big(\frac{q_{0,\varepsilon}(t){+}1}{\rho_{0,\varepsilon}(t)}\Big)\Bigg)\\
&-\frac{a_{1,\varepsilon}(t)}{2}\Bigg((q_{1,\varepsilon}(t){+}1)F\Big(\frac{q_{1,\varepsilon}(t){-}1}{\rho_{1,\varepsilon}(t)}\Big){+}(q_{1,\varepsilon}(t){-}1)G\Big(\frac{q_{1,\varepsilon}(t){+}1}{\rho_{1,\varepsilon}(t)}\Big)\Bigg)\\
&\approx\Big(\frac{1}{2}c'(t)\xi_0{-}\chi_0\Big)(F(\zeta_0(t)){+}G(\zeta_0(t))){+}\Big(\frac{1}{2}c'(t)\xi_1{-}\chi_1\Big)(F(\zeta_1(t)){+}G(\zeta_1(t)))=0,
\end{aligned}
$$

Note that the choice $\xi_0(t)=\xi_1(t)$, $\chi_0(t)=\chi_1(t)$ is safe again. It would be clear later that one can use that choice trough the paper without loss in generality in the distributional sense (needed for the weak uniqueness).

The second relation (3.6) is a bit more difficult to prove. We shall give a proof for $\eta=\rho F/2$, first.

Let us fix t for a moment and drop the dependence on t from the notation. The function F is a convex one, so the curve $(x,F(x))$ lies above each of its tangent lines $y=ax{+}b$. Chose $a=F'(c')$ and an appropriate constant b. Then

$$F(x)=ax{+}b{+}\overline{F}(x),\ \text{where } \overline{F}\geq 0,\ \overline{F}(c')=0 \text{ and } \overline{F}'(c')=0.$$

We have to find conditions that ensure $E(=E(t))\leq 0$, where

$$
\begin{aligned}
E=&-c'\Bigg(\frac{\rho_2}{2}\Big(a\frac{q_2{-}1}{\rho_2}{+}b{+}\overline{F}\Big(\frac{q_2{-}1}{\rho_2}\Big)\Big){-}\frac{\rho_0}{2}\Big(a\frac{q_0{-}1}{\rho_0}{+}b{+}\overline{F}\Big(\frac{q_0{-}1}{\rho_0}\Big)\Big)\Bigg)\\
&+\frac{q_2{+}1}{2}\Big(a\frac{q_2{-}1}{\rho_2}{+}b{+}\overline{F}\Big(\frac{q_2{-}1}{\rho_2}\Big)\Big){-}\frac{q_0{+}1}{2}\Big(a\frac{q_0{-}1}{\rho_0}{+}b{+}\overline{F}\Big(\frac{q_0{-}1}{\rho_0}\Big)\Big)\\
&+\frac{1}{2}\xi'(ac'{+}b{+}\overline{F}(c')){+}\frac{1}{2}\xi ac''=:I_a a{+}I_b b{+}I.
\end{aligned}
$$

We have

$$
\begin{aligned}
I_a&=\underbrace{-c'\Big(\frac{q_2}{2}{-}\frac{q_0}{2}\Big){+}\frac{q_2^2{-}1}{2\rho_2}{-}\frac{q_0^2{-}1}{2\rho_0}}_{=-\chi'/2}+\underbrace{\frac{1}{2}\xi'c'{+}\frac{1}{2}\xi c''}_{=(\xi c)'/2=\chi'/2}=0,\\
I_b&=\frac{1}{2}(\underbrace{-c'(\rho_2{-}\rho_0){+}q_2{-}q_0}_{=-\xi'}{+}\xi')=0.
\end{aligned}
$$

That was expected due to affinity. Also

$$
\begin{aligned}
I=&-c'\Big(\frac{\rho_2}{2}\overline{F}\Big(\frac{q_2{-}1}{\rho_2}\Big){-}\frac{\rho_0}{2}\overline{F}\Big(\frac{q_0{-}1}{\rho_0}\Big)\Big){+}\frac{q_2{+}1}{2}\overline{F}\Big(\frac{q_2{-}1}{\rho_2}\Big){-}\frac{q_0{+}1}{2}\overline{F}\Big(\frac{q_0{-}1}{\rho_0}\Big)\\
&+\underbrace{\frac{1}{2}\xi'\overline{F}(c')}_{=0}=-\Bigg((q_0{+}1{-}c'\rho_0)\overline{F}\Big(\frac{q_0{-}1}{\rho_0}\Big){+}(c'\rho_2{-}(q_2{+}1)\overline{F}\Big(\frac{q_2{-}1}{\rho_2}\Big)\Bigg).
\end{aligned}
$$

Due to the above construction $\overline{F}\geq 0$ and the sufficient condition for I to be non-positive is

$$q_0{+}1{-}c'\rho_0\geq 0 \text{ and } c'\rho_2{-}(q_2{+}1)\geq 0 \text{ i.e. } \lambda_2(\rho_2,q_2)\leq c'\geq \lambda_2(\rho_2,q_2).$$

Using exactly the same procedure for I where F is substituted by G, while the term $q{+}1$ is substituted by $q{-}1$ and oposite, one gets that I modified in such way is non-positive if and only if

$$q_0 - 1 - c'\rho_0 \geq 0 \text{ and } c'\rho_2 - (q_2 - 1) \geq 0 \ \Big(\text{i.e. } \lambda_1(\rho_0, q_0) \geq c' \geq \lambda_1(\rho_2, q_2)\Big).$$

Both conditions taken together are equivalent with the overcompressibility of the weighted SDW.

It is not so hard to see from the proof above that overcompressive SDW satisfies the entropy condition for each convex entropy–entropy flux pair. So we are allowed to use any of these two conditions in the rest of the paper. □

Note. One could see from the proof that the following three assertions are equivalent:

- An SDW is overcompressive.
- An SDW satisfies the entropy inequality for every convex entropy–entropy flux pair (1.4, 1.5).
- An SDW satisfies the entropy inequality for a single convex entropy–entropy flux pair (energy conservation law, for example)

4. SDW shock interactions

First, let us look at the simpler case, a double SDW interaction.

Theorem 4.1. *Suppose that two SDWs emerge from the points* $(-x_0, 0)$ *and* $(0, 0)$. *The solution of the interaction problem is a single weighted SDW emerging from the interaction point.*

Proof. In that case relation (1.6) is not satisfied for any of the Riemann problems, i.e. $\lambda_1(\rho_0, q_0) \geq \lambda_2(\rho_1, q_1)$ and $\lambda_1(\rho_1, q_1) \geq \lambda_2(\rho_2, q_2)$ imply

$$\lambda_1(\rho_0, q_0) \geq \lambda_2(\rho_2, q_2). \tag{4.1}$$

Thus one can try to find an overcompressive weighted SDW solution after the interaction. The following notation will be used:

$$\begin{aligned} A &:= [q]^2 - [\rho]\Big[\frac{q^2-1}{\rho}\Big] = \rho_0\rho_2(\lambda_1(\rho_0, q_0) - \lambda_1(\rho_2, q_2))(\lambda_2(\rho_0, q_0) - \lambda_2(\rho_2, q_2)), \\ B_2 &:= [q] - \frac{\chi_{in}}{\xi_{in}}[\rho],\ [x] := x_2 - x_0, \text{ here.} \end{aligned}$$

With these variables we have

$$\begin{aligned} c'(t) &= \frac{[q]}{[\rho]} + \frac{A(t-T) + B\chi_{in}}{[\rho]\sqrt{A^2(t-T)^2 - 2B\chi_{in} + \chi_{in}^2}}, \\ \xi(t) &= \sqrt{A^2(t-T)^2 - 2B\chi_{in} + \chi_{in}^2}, \text{ and } \chi(t) = c'(t)\xi(t). \end{aligned}$$

Denote

$$D(t) := A^2(t-T)^2 - 2B\chi_{in} + \chi_{in}^2$$

and use $\bar{\cdot}$ for data of the SDW on left-, and y $\tilde{\cdot}$ for data of the one on right-hand side respectively (the speed of the left(right)-hand SDW is denoted $\bar{c}$ ($\tilde{c}$), etc). In order to verify (3.4), we have to prove that $D(t) \geq 0$ for $t > T$. We already know that the first variable represents the density so all ξ's should be nonnegative and $A > 0$ due to (4.1). Next, assume $\rho_2 > \rho_0$. Then

$$\frac{\chi_{in}}{\xi_{in}} = \frac{\bar{\chi}T + \tilde{\chi}T}{\bar{\xi}T + \tilde{\xi}T} = \frac{\bar{c}\bar{\xi}T + \tilde{c}\tilde{\xi}T}{\bar{\xi}T + \tilde{\xi}T} = \frac{(\bar{c} - \tilde{c})\bar{\xi}T}{\bar{\xi}T + \tilde{\xi}T} + \tilde{c} > \tilde{c}$$

and

$$\begin{aligned}B_2 =&q_2-q_0-(\rho_2-\rho_0)\frac{\chi_{in}}{\xi_{in}} \leq q_2-q_0-(\rho_2-\rho_0)\tilde{c} \leq q_2-q_0-(\rho_2-\rho_0)\frac{q_2+1}{\rho_2}\\ =&\frac{1}{\rho_2}\underbrace{\left(-\rho_2(q_0+1)+\rho_0(q_2+1)\right)}_{<0 \text{ due to (4.1)}} < 0.\end{aligned}$$

If $\rho_0 > \rho_2$, then

$$\frac{\chi_{in}}{\xi_{in}} = \bar{c}+\frac{(\tilde{c}-\bar{c})\tilde{\xi}T}{\bar{\xi}T+\tilde{\xi}T} < \bar{c}$$

and

$$\begin{aligned}B_2 \leq& q_2-q_0-(\rho_2-\rho_0)\bar{c} \leq q_2-q_0-(\rho_2-\rho_0)\frac{q_0-1}{\rho_0}\\ =&\frac{1}{\rho_0}\underbrace{\left(-\rho_2(q_0-1)+\rho_0(q_2-1)\right)}_{<0 \text{ due to (4.1)}} < 0.\end{aligned}$$

In both cases we have $D(T) \geq \xi_{in}^2 > 0$, $D'(t) = 2(A(t-T)-B_2\xi_{in}) > 0$, and consequently, $D(t) > 0$ for all $t \geq T$. If $\rho_2 = \rho_0$ then $A = B_2^2$ and $D(t) = (B_2(t-T)-\xi_{in})^2 \geq 0$. That is, all expressions in (3.4) are well defined and we can check the SDW admissibility using Lemma 3.1. The speed of the outgoing wave equals

$$c'(t) = \frac{[q]}{[\rho]}+\frac{A(t-T)-B_2\xi_{in}}{[\rho]\sqrt{A(t-T)^2-2B_2\xi_{in}+\xi_{in}^2}}, \text{ with } c'(T) = \frac{[q]}{[\rho]}+\frac{[\rho]\frac{\chi_{in}}{\xi_{in}}-[q]}{[\rho]}.$$

We already saw that $\dfrac{\chi_{in}}{\xi_{in}} = \dfrac{\bar{c}\bar{\xi}T+\tilde{c}\tilde{\xi}T}{\bar{\xi}T+\tilde{\xi}T} \in (\tilde{c},\bar{c})$ because $\bar{\xi}$ and $\tilde{\xi}$ are positive. The incoming SDWs are both overcompressive, so $\lambda_i(\rho_0,q_0) \geq \bar{c} > \dfrac{\chi_{in}}{\xi_{in}} > \tilde{c} \geq \lambda_i(\rho_2,q_2)$, $i = 1,2$, and the resulting wave is also overcompressive in a neighborhood of $t = T$. Differentiating $c'(t)$ once again

$$c''(t) = \frac{A\xi(t)^2-\left(A(t-T)-B_2\xi_{in}\right)^2}{[\rho]\xi(t)^3} = \frac{\left(A-B_2^2\right)\xi_{in}^2}{[\rho]\xi(t)^3} \tag{4.2}$$

since $\xi(t) > 0$ is given by (3.4). It is now easy to see that the sign of $c''(t)$ does not depend on t at all: $\mathrm{sign}(c''(t)) = \mathrm{sign}(A-B_2^2)\,\mathrm{sign}(\rho_2-\rho_0)$. So, $c'(\infty) = \lim_{t\to\infty} c'(t) = \dfrac{[q]}{[\rho]}+\dfrac{\sqrt{A}}{[\rho]}$. Therefore, if the wave is overcompressive in $t = T$ and at infinity, then it is overcompressive for all $t \in [T,\infty)$ since $c'(t)$ is monotone. Let us assume $\rho_2 > \rho_0$ ($\rho_0 > \rho_2$, resp). Then

$$\begin{aligned}\frac{q_0-1}{\rho_0} \geq& \frac{q_2-q_0}{\rho_2-\rho_0}+\frac{\sqrt{A}}{\rho_2-\rho_0}\\ \rho_2\Big(\frac{q_0-1}{\rho_0}-\frac{q_2-1}{\rho_2}\Big) \geq& \ (\leq)\sqrt{A} \text{ (after multiplication by } (\rho_2-\rho_0))\\ \rho_2^2\Big(\frac{q_0-1}{\rho_0}-\frac{q_2-1}{\rho_2}\Big)^2 \geq& \ (\leq)\rho_0\rho_2\Big(\frac{q_0-1}{\rho_0}-\frac{q_2-1}{\rho_2}\Big)\Big(\frac{q_0+1}{\rho_0}-\frac{q_2+1}{\rho_2}\Big)\\ \rho_2\Big(\frac{q_0-1}{\rho_0}-\frac{q_2-1}{\rho_2}\Big) \geq& \ (\leq)\rho_0\Big(\frac{q_0+1}{\rho_0}-\frac{q_2+1}{\rho_2}\Big)\end{aligned}$$

$$=\rho_0\Big(\frac{q_0-1}{\rho_0}-\frac{q_2-1}{\rho_2}+\underbrace{\frac{2}{\rho_0}-\frac{2}{\rho_2}}_{=\frac{2}{\rho_0\rho_2}(\rho_2-\rho_0)}\Big)$$

$$\frac{q_0-1}{\rho_0}-\frac{q_2-1}{\rho_2}\geq\frac{2}{\rho_2}\text{ (after another multiplication by }(\rho_2-\rho_0)^{-1}).$$

But the last inequality is equivalent to $\lambda_1(\rho_0,q_0)\geq\lambda_2(\rho_2,q_2)$ which follows from (4.1). In almost the same manner one can prove the inequality

$$\frac{q_2-q_0}{\rho_2-\rho_0}+\frac{\sqrt{A}}{\rho_2-\rho_0}\geq\frac{q_2+1}{\rho_2}.$$

So we have proved that the resulting SDW is overcompressive and thus entropic. □

We deal with the interaction of a SDW with a wave consisting of two contact discontinuities in the following assertions. Suppose that (ρ_0,q_0) is connected to (ρ_1,q_1) by a SDW, (ρ_1,q_1) is connected to (ρ_s,q_s) by a CD_1 that is connected to (ρ_2,q_2) by a CD_2.

There are different results depending on the relation between the constants in (3.1). Each of the following assertions is devoted to one possibility. We start with the case when system (1.1) with the Riemann data

$$(\rho,q)(x,0)=\begin{cases}(\rho_0,q_0), & x<0\\(\rho_2,q_2), & x>0\end{cases}$$

has a solutions consisting of two contact discontinuities. Now, there is a delta function in the initial data so an elementary CDC solution is impossible to find. But one can try with the so called delta contact discontinuities (defined in [15]). These waves are carrying delta measure and acting in a linear way – their strength do not change with time and they are spreading along the characteristics. A typical representative of the i-th delta contact discontinuity connecting the states (ρ_l,q_l) and (ρ_r,q_r) is given by

$$(\rho,q)=\begin{cases}(\rho_l,q_l), & x<c_it-\varepsilon\\(\rho_{0,\varepsilon},q_{0,\varepsilon}), & c_it-\varepsilon<x<c_it\\(\rho_{1,\varepsilon},q_{1,\varepsilon}), & c_it<x<c_it+\varepsilon\\(\rho_r,q_r), & x>c_it+\varepsilon,\end{cases}\tag{4.3}$$

where $c_i=\lambda_i(\rho_l,q_l)=\lambda_i(\rho_r,q_r)$.

Theorem 4.2. *Suppose that a solution to problem (1.1,3.1) consists of a SDW emanating from $(-x_0,0)$ followed by two centered contact discontinuities. Denote by (ρ_s,q_s) an intermediate state between these contact discontinuities. If $\lambda_1(\rho_0,q_0)<\lambda_2(\rho_s,q_s)$, i.e. if*

$$\frac{2}{\rho_s}>\frac{q_0-1}{\rho_0}-\frac{q_1+1}{\rho_1}+\frac{2}{\rho_1}=\lambda_1(\rho_0,q_0)-\lambda_1(\rho_1,q_1)\tag{4.4}$$

then there exists a unique solution to the interaction problem in the form of two delta contact discontinuities emanating from the interaction point. It does not interact with the second incoming contact discontinuity.

Proof. Our first task is to find a solution to (1.1) with the initial data

$$(\rho,q)|_{t=T}=\begin{cases}(\rho_0,q_0), & x<x_T\\ (\rho_s,q_s), & x>x_T\end{cases}+(\xi_{in},\chi_{in})\delta_{(x_T,T)} \tag{4.5}$$

at time T of the interaction of the SDW and the first CD. Here, x_T is a space component of the interaction point while ξ_{in} and χ_{in} are incoming SDW strengths for ρ and q variable, respectively. Note that an overcompressive SDW can not follow or be followed by any other wave. Substitution of a weighted SDW into the equation with the new initial data (4.5) leads again to ODEs system (3.3). A necessary condition for existence of a non-constant solution to the system is $\rho_s(q_0-1)-\rho_0(q_s+1)>0$. But that clearly contradicts assumption (4.4). Thus, only possible choice for a SDW solution is a non-overcompressive wave $\xi(t)=\text{const}$ and $\chi(t)=\text{const}$. Then $c(t)=\text{const}\cdot t$ and the weighted SDW reduces to a wave of the form (4.3). The first and third equation in the system are just the Rankine–Hugoniot conditions. The other itwo equations in (3.3) are satisfied if χ/ξ equals the speed of wave. The Rankine–Hugoniot conditions further implies that (ρ_r,q_r) lies on a $\mathrm{CD}_1(\rho_l,q_l)$ or $\mathrm{CD}_2(\rho_l,q_l)$. Let us observe that relation (4.4) is equivalent to (1.6) with (ρ_1,q_1) substituted by (ρ_s,q_s). Thus there exist a state (ρ_{ss},q_{ss}) such that it lies on $\mathrm{CD}_1(\rho_0,q_0)$ and $(\rho_s,q_s)\in\mathrm{CD}_2(\rho_{ss},q_{ss})$. That enables us to construct an SDW solution to (1.1,4.5) consisting of two delta contact discontinuities

$$(\rho,q)=\begin{cases}(\rho_0,q_0), & x-x_T<c_1(t-T)-\varepsilon\\ (\rho_{0,\varepsilon},q_{0,\varepsilon}), & c_1(t-T)-\varepsilon<x-x_T<c_1(t-T)\\ (\rho_{1,\varepsilon},q_{1,\varepsilon}), & c_1(t-T)<x-x_T<c_1(t-T)+\varepsilon\\ (\rho_{ss},q_{ss}), & c_1(t-T)+\varepsilon<x-x_T<c_2(t-T)-\varepsilon\\ (\rho_{2,\varepsilon},q_{2,\varepsilon}), & c_2(t-T)-\varepsilon<x-x_T<c_2(t-T)\\ (\rho_{3,\varepsilon},q_{3,\varepsilon}), & c_2(t-T)<x-x_T<c_2(t-T)+\varepsilon\\ (\rho_s,q_s), & c_2(t-T)+\varepsilon<x-x_T,\end{cases} \tag{4.6}$$

where $c_1=\lambda_1(\rho_0,q_0)=\lambda_1(\rho_{ss},q_{ss})$, $c_2=\lambda_2(\rho_s,q_s)=\lambda_2(\rho_{ss},q_{ss})$, $\rho_{0,\varepsilon}=\rho_{1,\varepsilon}=\dfrac{\xi^1}{2\varepsilon}$, $q_{0,\varepsilon}=q_{1,\varepsilon}=\dfrac{\chi^1}{2\varepsilon}$, $\rho_{2,\varepsilon}=\rho_{3,\varepsilon}=\dfrac{\xi^2}{2\varepsilon}$, $q_{2,\varepsilon}=q_{3,\varepsilon}=\dfrac{\chi^2}{2\varepsilon}$. The positive reals ξ^1,ξ^2 are solutions to the system

$$\begin{aligned}\xi^1+\xi^2&=\xi_{in},\\ c_1\xi^1+c_2\xi^2&=\chi_{in},\end{aligned}$$

and $\chi^i=c_i\xi^i$, $i=1,2$. The above system has a unique solution because $c_1<c_2$.

The above solution (4.6) is already unique in the weak sense but the entropy condition has to be proved. The first entropy relation (3.5) is satisfied since $c_i\xi^i=\chi^i$ (see the proof of Lemma 3.1). For the first wave in (4.6), the second entropy condition (3.6) reduces to

$$-c_1\big(\eta(\rho_{ss},q_{ss})-\eta(\rho_0,q_0)\big)+Q(\rho_{ss},q_{ss})-Q(\rho_0,q_0)\le 0.$$

The left-hand side of the above inequality equals

$$-\underbrace{\frac{q_0-1}{\rho_0}}_{=\frac{q_{ss}-1}{\rho_{ss}}}\left(\frac{\rho_{ss}}{2}\left(F\Big(\frac{q_{ss}-1}{\rho_{ss}}\Big)+G\Big(\frac{q_{ss}+1}{\rho_{ss}}\Big)\right)-\frac{\rho_0}{2}\left(F\Big(\frac{q_0-1}{\rho_0}\Big)+G\Big(\frac{q_0+1}{\rho_0}\Big)\right)\right)$$
$$+\frac{q_{ss}+1}{2}F\Big(\frac{q_{ss}-1}{\rho_{ss}}\Big)+\frac{q_{ss}-1}{2}G\Big(\frac{q_{ss}+1}{\rho_{ss}}\Big)$$
$$-\frac{q_0+1}{2}F\Big(\frac{q_0-1}{\rho_0}\Big)-\frac{q_0-1}{2}G\Big(\frac{q_0+1}{\rho_0}\Big)=0.$$

The same holds for the second wave in (4.6) so the solution is admissible.

Finally, one can see that the speed of the second delta contact discontinuity equals $c_2=\lambda_2(\rho_{ss},q_{ss})=\lambda_2(\rho_s,q_s)$ which is the same as the speed of the incoming $\mathrm{CD}_2=\lambda_2(\rho_s,q_s)=\lambda_2(\rho_2,q_2)$. That means that these waves will never interact, and the proof is completed. □

The condition equivalent to (4.4) written without involving the state (ρ_s,q_s) is

$$q_2<-1+\frac{q_0-1}{\rho_0}\rho_2.$$

Suppose now that it is not satisfied. Then we have the following assertion as the first possibility.

Theorem 4.3. *Suppose that the solution to problem (1.1,3.1) consists of a SDW emanating from $(-x_0,0)$ followed by two centered contact discontinuities as in the previous theorem. If $\bar c\geq\lambda_2(\rho_s,q_s)$, i.e. if*

$$\frac{2}{\rho_s}\leq\bar c-\frac{q_1-1}{\rho_1}=\bar c-\lambda_1(\rho_1,q_1)=\bar c-\lambda_1(\rho_s,q_s) \tag{4.7}$$

then there exists a weakly unique weighted SDW solution to the above interaction problem. The symbol $\bar c$ denotes the speed of the incoming SDW,

$$\bar c=\frac{[q]_1+\sqrt{[q]_1^2-[\rho]_1\left[\frac{q^2-1}{\rho}\right]_1}}{[\rho]_1},\ \textit{with } [y]_1:=y_1-y_0.$$

Proof. Note that (4.7) imply $\frac{2}{\rho_s}\leq\lambda_1(\rho_0,q_0)-\lambda_1(\rho_s,q_s)$ since the incoming SDW is overcompressive. That excludes (4.4), so the solution is weakly unique. As before, denote by T the interaction time of the initial SDW and CD_1. A weighted SDW solution (3.2) to (1.1,4.5), with (ρ_2,q_2) substituted by (ρ_s,q_s), exists since $\lambda_1(\rho_0,q_0)>\lambda_2(\rho_s,q_s)$. One just have to prove that the wave is overcompressive due to Lemma 3.1. The proof is similar to the one of Theorem 4.1 and we give just the main points and differences. Note that the initial speed of the outgoing SDW is $c'(T)=\dfrac{\chi_{in}}{\xi_{in}}=\bar c$. The incoming SDW is overcompressive and assume a strict inequality $\lambda_1(\rho_0,q_0)>\bar c=c'(T)$, $\bar c>\lambda_1(\rho_s,q_s)+\dfrac{2}{\rho_s}=\lambda_2(\rho_s,q_s)$ for a moment. The cases of equality will be considered later. Thus, the outgoing SDW is entropic in a neighborhood of $t=T$. Denote $[y]=y_s-y_0$. The speed $c'(t)$ has the same formula as in the proof of Theorem 4.1,

$$c'(t)=\frac{[q]}{[\rho]}+\frac{A(t-T)-B_2\xi_{in}}{[\rho]\sqrt{A(t-T)^2-2B_2\xi_{in}+\xi_{in}^2}}$$

with

$$A=\rho_0\rho_s(\lambda_1(\rho_0,q_0)-\lambda_1(\rho_s,q_s))(\lambda_2(\rho_0,q_0)-\lambda_2(\rho_s,q_s)),\ B_2=[q]-\frac{\chi_{in}}{\xi_{in}}[\rho].$$

Also $c''(t)$ has a constant sign for $t>T$ due to (4.2), so $c'(t)$ is monotone. At infinity $c'(t)$ takes the value $c'(\infty)=\dfrac{q_s-q_0+\sqrt{A}}{\rho_s-\rho_0}$, so we have $\lambda_1(\rho_0,q_0)\geq c'(\infty)\geq \lambda_2(\rho_s,q_s)$ obtained in the same way as in the proof of Theorem 4.1 just substituting ρ_2 by ρ_s. Thus the outgoing SDW is overcompressive for all $t\geq T$. On the other hand, $c'(t)>\lambda_2(\rho_s,q_s)$ implies $c'(t)>\lambda_2(\rho_2,q_2)$ since (ρ_s,q_s) and (ρ_2,q_2) are connected by CD_2 and $\lambda_2(\rho_s,q_s)=\lambda_2(\rho_2,q_2)$. Thus the outgoing SDW is faster than incoming CD_2. Denote by (T_1,X_1) the interaction point of the outgoing SDW and the incoming CD_2. Then we have to solve the problem (1.1) with the initial data

$$(\rho,q)|_{t=T_1}=\begin{cases}(\rho_0,q_0), & x<X_1\\ (\rho_2,q_2), & x>X_1\end{cases}+(\xi(T_1),\chi(T_1))\delta_{(T_1,X_1)}.$$

But $\lambda_1(\rho_0,q_0)>c'(T_1)\geq\lambda_2(\rho_s,q_s)=\lambda_2(\rho_2,q_2)$ means that (4.1) is satisfied so the complete proof of Theorem 4.1 can be applied in the present situation: There exists only a single weighted SDW as a solution after $t>T_1$. Weak uniqueness easily follows as above.

Let us now check cases $\lambda_1(\rho_0,q_0)=\bar{c}$ and $\bar{c}=\lambda_2(\rho_s,q_s)$. In both of them we have to check overcompressibility condition in a neighborhood of $t=T$ by more precise control of $c'(t)$ there.

If $\bar{c}=\lambda_2(\rho_s,q_s)$ then

$$A-B_2^2=\rho_0(\rho_s-\rho_0)\Big(\lambda_2(\rho_0,q_0)-\lambda_2(\rho_s,q_s)\Big)\underbrace{\Big(\lambda_1(\rho_0,q_0)-\lambda_1(\rho_s,q_s)-\frac{2}{\rho_s}\Big)}_{=\lambda_1(\rho_0,q_0)-\lambda_2(\rho_s,q_s)}.$$

Therefore, $\dfrac{A-B_2^2}{\rho_s-\rho_0}>0$ and $c''(t)>0$. Thus $c'(t)$ is increasing and $c'(t)>\lambda_2(\rho_s,q_s)$ for $t>T$.

If $\bar{c}=\lambda_1(\rho_0,q_0)$ then

$$A-B_2^2=\rho_s(\rho_0-\rho_s)(\lambda_1(\rho_0,q_0)-\lambda_1(\rho_s,q_s))\Big(\lambda_1(\rho_0,q_0)-\lambda_1(\rho_s,q_s)-\frac{2}{\rho_s}\Big),$$

$$\frac{A-B_2^2}{\rho_s-\rho_0}<0 \text{ and } c'(t)<\lambda_1(\rho_0,q_0) \text{ for } t>T.$$

That completes the proof. □

5. Higher order SDWs

The remaining case is completely different from the others and we shall examine it in this section. Suppose that all assumptions of Theorem 4.3 are valid but the inequality (4.7) do not hold, i.e. $\bar{c}<\lambda_2(\rho_s,q_s)$. One could see that there are no difficulties to construct the same SDW solution as in that theorem, but the initial speed of the resulting SDW would satisfy $c'(0)=\bar{c}<\lambda_2(\rho_s,q_s)=\lambda_2(\rho_2,q_2)$ and the outgoing wave can not be overcompressive (entropic).

Therefore we have to use different approach to find out what is going on in this case. Assume that a SDW starting at the point $(-x_0,0)$ interacts with a centered

CDC wave. The interaction happens at the time $T = \dfrac{x_0}{\bar{c}+\varepsilon-\lambda_1(\rho_s, q_s)}$ when the SDW front from the right-hand side $x = -x_0+(\bar{c}+\varepsilon)t$ meets the centered CD_1 supported by $x = \lambda_1(\rho_1, q_1)$ at the point $Y_1 = \dfrac{\lambda_1(\rho_s, q_s)}{\bar{c}+\varepsilon-\lambda_1(\rho_s, q_s)}x_0$. Now, we do not let $\varepsilon \to 0$ as we done it above, but we use a more precise procedure. We stop the initial solution at $t = T$ so there are three new Riemann problems.

- The first Riemann problem connects (ρ_0, q_0) with $(\rho_\varepsilon, q_\varepsilon)$ at the point $X_1 = \dfrac{\lambda_1(\rho_s, q_s)-2\varepsilon}{\bar{c}+\varepsilon-\lambda_1(\rho_s, q_s)}x_0$.
- The second Riemann problem connects the states $(\rho_\varepsilon, q_\varepsilon)$ and (ρ_s, q_s) at the point Y_1.
- For the third Riemann problem at the point $(\lambda_2(\rho_2), q_2)t$ there is no need for anything to do because (ρ_s, q_s) and (ρ_2, q_2) are still connected by the same incoming CD_2. The same will be at any other interaction of that CD_2-wave.

While the incoming speed of SDW at the interaction point violates the overcompressibility condition on the right-hand side, there are two main possibilities concerning the left hand-side: $\lambda_1(\rho_0, q_0) > \bar{c}$ or $\lambda_1(\rho_0, q_0) = \bar{c}$. The first one is called the compressive while the second one is called the marginal interaction.

5.1. **The compressive interaction.** Suppose $\lambda_1(\rho_0, q_0) > \bar{c}$. We will use a sequence of small parameters $\{\varepsilon_n\}_{n\in\mathbb{N}}$, ... $\ll \varepsilon_2 \ll \varepsilon_1 \ll \varepsilon$ to describe new waves. The general idea goes as follows. After an interaction occurs we fix the parameter from the previous interaction, ε_n, and introduce a new one $\varepsilon_{n+1} \ll \varepsilon_n$ to describe a new approximate solution. Now we present the details.

The first interaction. Let ε be fixed. For the first Riemann problem in X_1, the condition (1.6) is violated since $\lambda_2(\rho_\varepsilon, q_\varepsilon) = \dfrac{q_\varepsilon+1}{\rho_\varepsilon} = \bar{c}+\xi_0\varepsilon$ and $\lambda_1(\rho_0, q_0) > \bar{c}$, too. Put $\xi_0 := \rho_\varepsilon\varepsilon = \frac{1}{2}\kappa_1$ (due to Theorem 2.1) and $\chi_0 := q_\varepsilon\varepsilon = \frac{1}{2}\kappa_2 = \bar{c}\xi_0$. We are in position to use Theorem 2.1 with ε substituted by ε_1. The speed of the new SDW (i.e. SDW of the second order) is $c_1 = \dfrac{q_\varepsilon-q_0+\kappa_{1,1}}{\rho_\varepsilon-\rho_0}$ with

$$\begin{aligned}
\kappa_{1,1} &= \sqrt{\rho_0\rho_\varepsilon(\lambda_1(\rho_0, q_0)-\lambda_1(\rho_\varepsilon, q_\varepsilon))(\lambda_2(\rho_0, q_0)-\lambda_2(\rho_\varepsilon, q_\varepsilon))} \\
&= \sqrt{\rho_0\frac{\xi_0}{\varepsilon}\Big(\frac{q_0-1}{\rho_0}-\bar{c}+\frac{\xi_0}{\varepsilon}\Big)\Big(\frac{q_0+1}{\rho_0}-\bar{c}-\frac{\xi_0}{\varepsilon}\Big)} \\
&\approx \sqrt{\rho_0\xi_0\Big(\frac{q_0-1}{\rho_0}-\bar{c}\Big)\Big(\frac{q_0+1}{\rho_0}-\bar{c}\Big)}\varepsilon^{-1/2}
\end{aligned}$$

being the Rankine–Hugoniot deficit in the first equation. Thus

$$c_1 \approx \frac{\bar{c}\xi_0-\varepsilon q_0+\sqrt{C_\xi\varepsilon}}{\xi_0-\varepsilon\rho_0} \approx \bar{c}+\frac{2\sqrt{C_\xi}}{\xi_0}\varepsilon^{1/2},\ C_\xi = \frac{1}{4}\rho_0\Big(\frac{q_0-1}{\rho_0}-\bar{c}\Big)\Big(\frac{q_0+1}{\rho_0}-\bar{c}\Big).$$

Denote the inner value in the outgoing SDW by $(\rho_{\varepsilon_1}, q_{\varepsilon_1})$ (again identifying both sides of SDW). The fronts of the SDW have speeds $c_{1,-} = c_1-\varepsilon_1$ and $c_{1,+} = c_1+\varepsilon_1$. Denote $\xi_1 := \varepsilon_1\rho_{\varepsilon_1} = \frac{1}{2}\kappa_{1,1}$ and $\chi_1 := \varepsilon_1 q_{\varepsilon_1}$. As before, $\chi_1 = c_1\xi_1$, but ξ_1 and χ_1 are not constants now and they depend on ε.

For the second problem at Y_1, the condition (1.6) is satisfied because of the assumption $\bar{c} < \lambda_2(\rho_s, q_s)$, so the solution consists of two contact discontinuities. The speed of the first one is $\lambda_1(\rho_\varepsilon, q_\varepsilon) = \dfrac{q_\varepsilon - 1}{\rho_\varepsilon} = \bar{c} - \xi_0 \varepsilon$ while the speed of the second one is $\lambda_2(\rho_s, q_s)$. The components of the intermediate state are $\rho_{s_1} = \dfrac{2}{\lambda_2(\rho_s, q_s) - \lambda_1(\rho_\varepsilon, q_\varepsilon)} \approx \dfrac{2}{\lambda_2(\rho_2, q_2) - \bar{c}}$ and $q_{s_1} = \dfrac{\lambda_2(\rho_s, q_s) + \lambda_1(\rho_\varepsilon, q_\varepsilon)}{\lambda_2(\rho_s, q_s) - \lambda_1(\rho_\varepsilon, q_\varepsilon)} \approx \dfrac{\lambda_2(\rho_2, q_2) + \bar{c}}{\lambda_2(\rho_2, q_2) - \bar{c}}$.

The second interaction. Take $\xi_1 = \varepsilon_1 \rho_{\varepsilon_1}$ The distance between X_1 and Y_1 is $2T\varepsilon$ while the difference between the speeds of the interacting waves starting at these point equals

$$c_{1,+} - \lambda_1(\rho_\varepsilon, q_\varepsilon) = c_1 + \varepsilon_1 - \left(\bar{c} - \frac{\varepsilon}{\xi_0}\right) \approx \frac{2\sqrt{C_\xi}}{\xi_0} \varepsilon^{1/2}.$$

The next interaction time of the waves after $t > T$ will be denoted by T_1 and

$$T_1 = \frac{2\varepsilon T}{c_1 + \varepsilon_1 - \lambda_1(\rho_\varepsilon, q_\varepsilon)} \approx \frac{\xi_0 T}{C_\xi} \varepsilon^{1/2}.$$

At $t = T_1$ with ε and ε_1 fixed we have again three Riemann problems at the points (X_2, T_1), (Y_2, T_1) and $(\lambda_2(\rho_2, q_2)T_1, T_1)$, where $X_2 = X_1 + (c_1 - \varepsilon_1(T_1 - T))$ and $Y_2 = X_1 + (c_1 + \varepsilon_1(T_1 - T))$. Let us and introduce a new parameter $\varepsilon_2 \ll \varepsilon_1$ that will be used for construction of an approximate solution when $t > T_1$. Riemann problem solutions at these points are similar to the ones for $t = T$: at the point X_2 we have an SDW with the inner value $(\rho_{\varepsilon_2}, q_{\varepsilon_2})$, while there is an CDC solution with the intermediate state $(\rho_{s_2, q_{s_2}})$ at Y_2. As we already noted, the third solution is just the incoming wave. Using notation analogous to the previous interaction analysis, put $\xi_2 = \varepsilon_2 \rho_{\varepsilon_2} = \frac{1}{2}\kappa_{2,1}$ and $\chi_2 = \varepsilon_2 q_{\varepsilon_2} = c_2 \xi_2$.

Due to overcompressibility, the SDW starting from X_2 will interact with CD_1 starting from Y_2 and the procedure repeats recursively. Let us show how. (One can see the illustration at Figure 5.1.)

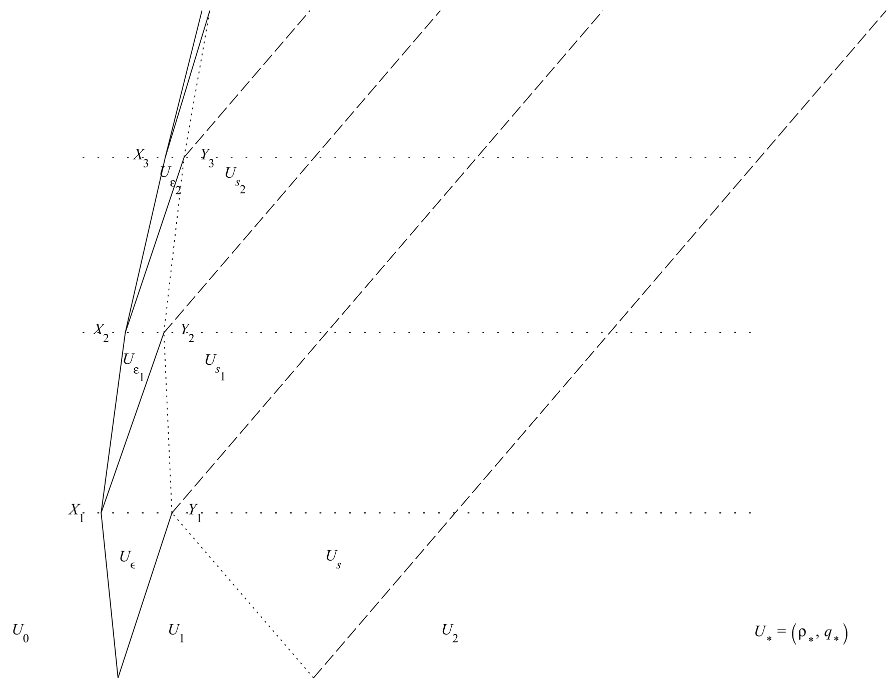

FIGURE 1. The compressive case

Further interactions. Suppose that an interaction occurs at a time $t = T_n$ and a point Y_n. Values X_n, ε_n, ρ_n, q_n, ξ_n, χ_n are defined as above. Let us fix ε_n and introduce $\varepsilon_{n+1} \ll \varepsilon_n$. As before, at the point X_n, there is a SDW solution of order higher by one with speed c_{n+1} and inner value $(\rho_{\varepsilon_{n+1}}, q_{\varepsilon_{n+1}})$. In the point Y_n there is an CDC solution with the intermediate state $(\rho_{s_{n+1}}, q_{s_{n+1}})$. The initial CD_2 continues to run on the right-hand side of Y_n as we already noticed. Denote $\xi_{n+1} = \varepsilon_{n+1}\rho_{n+1}$ and T_{n+1} the time of next interaction is determined by the intersection of the lines $x = X_n + (c_{n+1} + \varepsilon_{n+1})(t - T_n)$ and $x = Y_n + \lambda_1(\rho_{\varepsilon_n}, q_{\varepsilon_n})(t - T_n)$.

Let us now estimate some of the important quantities.

First, note that $\xi_{n+1} > \xi_n$ and $c_n \approx \bar{c}$. For every n, $\varepsilon_n \to 0$, but one has to keep in mind the relation $\ldots \varepsilon_n \ll \cdots \ll \varepsilon_1 \ll \varepsilon$. We have

$$\begin{aligned}
\xi_{n+1} &= \frac{1}{2}\sqrt{\rho_0\rho_{\varepsilon_n}\Big(\frac{q_0-1}{\rho_0} - \frac{q_{\varepsilon_n}-1}{\rho_{\varepsilon_n}}\Big)\Big(\frac{q_0+1}{\rho_0} - \frac{q_{\varepsilon_n}+1}{\rho_{\varepsilon_n}}\Big)} \\
&\approx \frac{1}{2}\sqrt{\rho_0\frac{\xi_n}{\varepsilon_n}\Big(\frac{q_0-1}{\rho_0} - c_n + \frac{\varepsilon_n}{\xi_n}\Big)\Big(\frac{q_0+1}{\rho_0} - c_n - \frac{\varepsilon_n}{\xi_n}\Big)} \approx C_\xi^{1/2}\varepsilon_n^{-1/2}\xi_n^{1/2}.
\end{aligned}$$

Using the above recursive relation one gets

$$\begin{aligned}
\xi_{n+1} &\approx C_\xi^{\sum_{i=1}^{n+1} 2^{-i}} \varepsilon^{-2^{-(n+1)}} \prod_{i=1}^{n} \varepsilon_i^{-2^{-(n+1-i)}} < C_\xi \varepsilon^{2^{-(n+1)}} \prod_{i=1}^{n} \varepsilon_i^{-2^{-(n+1-i)}}, \\
&\text{as } \cdots \ll \varepsilon_1 \ll \varepsilon \to 0, \text{ for some } C_\xi \text{ independent of } \varepsilon\text{'s.}
\end{aligned} \tag{5.1}$$

We have used that $\xi_1 \approx C_\xi^{1/2}\xi_0^{1/2}\varepsilon^{-1/2}$.

Then

$$c_{n+1}-c_n = \frac{q_{\varepsilon_n}-q_0+2\xi_{n+1}}{\rho_{\varepsilon_n}-\rho_0}-c_n = \frac{\varepsilon_n(c_n\rho_0-q_0)+2\varepsilon_n\xi_{n+1}}{\xi_n-\varepsilon_n\rho_0} \approx 2\varepsilon_n\frac{\xi_{n+1}}{\xi_n}$$
$$\sim \varepsilon^{2^{n+1}}\varepsilon_n\prod_{i=1}^{n}\varepsilon_i^{-2^{-(n+1-i)}}\prod_{i=1}^{n-1}\varepsilon_i^{2^{(n-i)}} = \prod_{i=1}^{n}\varepsilon_i^{2^{-(n+1-i)}}.$$

The last important quantity is the distance between two successive interactions $\Delta T_{n+1} = T_{n+1}-T_n$. We have

$$\Delta T_{n+1} = \frac{2\varepsilon_n\Delta T_n}{c_{n+1}+\varepsilon_{n+1}-\lambda_1(\rho_{\varepsilon_n}, q_{\varepsilon_n})} = \frac{2\varepsilon_n\Delta T_n}{c_{n+1}+\varepsilon_{n+1}-c_n+\frac{\varepsilon_n}{\xi_n}}$$
$$\approx 2\frac{\xi_n}{\xi_{n+1}}\Delta T_n \sim \varepsilon^{-2^{-(n+1)}}\varepsilon_n^{1/2}\prod_{i=1}^{n-1}\varepsilon_i^{-2^{-(n+1-i)}}\Delta T_n.$$

Using that recursive formula we have

$$\Delta T_{n+1} \sim \varepsilon^{-1/2+2^{-(n+1)}}\prod_{i=1}^{n}\varepsilon_i^{2^{-(n+1-i)}}\underbrace{\Delta T_1}_{\approx\frac{\xi_0}{C_\xi^{1/2}}T\varepsilon^{1/2}} \sim \varepsilon^{2^{-(n+1)}}\prod_{i=1}^{n}\varepsilon_i^{2^{-(n+1-i)}}T. \quad (5.2)$$

So, each ΔT_{n+1} is less than ΔT_n so the first question one should ask is whether the sum $\sum_{n=1}^{\infty} T_n$ is convergent or not. That is, whether the interaction procedure has a time limit or not. The answer depends on relations between ε, ε_1, ..., ε_n,.... In order to find them, we will check when the piecewise constant function constructed by the above procedure

$$(\rho, q)|_{t\in[T_n,T_{n+1})} = \begin{cases} (\rho_0, q_0), & x < X_n+(c_{n+1}-\varepsilon_{n+1})(t-T_n) \\ (\rho_{\varepsilon_{n+1}}, q_{\varepsilon_{n+1}}), & X_n+(c_{n+1}-\varepsilon_{n+1})(t-T_n) < x \\ & < X_n+(c_{n+1}+\varepsilon_{n+1})(t-T_n) \\ (\rho_{\varepsilon_n}, q_{\varepsilon_n}), & X_n+(c_{n+1}+\varepsilon_{n+1})(t-T_n) < x \\ & < Y_n+\lambda_1(\rho_{s_n}, q_{s_n})(t-T_n) \\ (\rho_{s_n}, q_{s_n}), & Y_n+\lambda_1(\rho_{s_n}, q_{s_n})(t-T_n) < x \\ & < Y_n+\lambda_2(\rho_{s_n}, q_{s_n})(t-T_n) \\ \vdots & \\ (\rho_s, q_s), & Y_1+\lambda_2(\rho_s, q_s)(t-T) < x < \lambda_2(\rho_s, q_s)t \\ (\rho_2, q_2), & x > \lambda_2(\rho_s, q_s)t \end{cases} \quad (5.3)$$

$n = 1, 2, \ldots$ satisfies system (1.1) in the approximated sense and when it satisfies the entropy inequality (2.3). Here, T is identified with T_0.

We will use the same method as in [16] for the normal (i.e. first order) SDWs. Let us consider the interval $[T_n, T_{n+1}]$. Put $f(\rho, q) = \Big(q, \dfrac{q^2-1}{\rho}\Big)$ and $U = (p, q)$. Then

$$(I_1, I_2) := \langle\partial_t U_\varepsilon+\partial_x f(U_\varepsilon), \phi\rangle = \sum_{n=1}^{\infty} J_n,$$

where

$$
\begin{aligned}
J_n =& -\int\limits_{T_n}^{T_{n+1}} (c_n-\varepsilon_n)(U_{\varepsilon_n}-U_0)\phi((c_n-\varepsilon_n)(t-T_n)+X_n,t)dt \\
& -\int\limits_{T_n}^{T_{n+1}} (c_n+\varepsilon_n)(U_{\varepsilon_{n-1}}-U_{\varepsilon_n})\phi((c_n+\varepsilon_n)(t-T_n)+X_n,t)dt \\
& -\int\limits_{T_n}^{T_{n+1}} (U_{s_n}-U_{\varepsilon_n})\lambda_1(U_{\varepsilon_{n-1}})\phi(\lambda_1(U_{\varepsilon_n})(t-T_n)+Y_n,t)dt \\
& (\text{here } \lambda_1(U_{\varepsilon_{n-1}}) = \lambda_1(U_{s_n})) \\
& +\int\limits_{T_n}^{T_{n+1}} (f(U_{\varepsilon_n})-f(U_0))\phi((c_n-\varepsilon_n)(t-T_n)+X_n,t)dt \\
& +\int\limits_{T_n}^{T_{n+1}} (f(U_{\varepsilon_{n-1}})-f(U_{\varepsilon_n}))\phi((c_n+\varepsilon_n)(t-T_n)+X_n,t)dt \\
& +\int\limits_{T_n}^{T_{n+1}} \lambda_1(U_{\varepsilon_{n-1}})(f(U_{s_n})-f(U_{\varepsilon_{n-1}}))\phi(\lambda_1(U_{\varepsilon_{n-1}})(t-T_n)+Y_n,t)dt.
\end{aligned}
$$

As in [16] the next step is using the Taylor expansions of the test function around $\Gamma_n := \{(x,t) : \ (c_n(t-T_n)+X_n,t)\}$ after the time $t = T_n$. First, note that all integrals supported by the characteristics cancels due to the contact discontinuity construction. We expand the other integrals in neighborhood of the line Γ_n. Then

$$
\begin{aligned}
J_n \approx& -\int\limits_{T_n}^{T_{n+1}} (c_n-\varepsilon_n)(U_{\varepsilon_n}-U_0)\big(\phi|_{\Gamma_n}-\partial_x\phi|_{\Gamma_n}\cdot\varepsilon_n(t-T_n)\big)dt \\
& -\int\limits_{T_n}^{T_{n+1}} (c_n+\varepsilon_n)(U_{\varepsilon_{n-1}}-U_{\varepsilon_n})\big(\phi|_{\Gamma_n}+\partial_x\phi|_{\Gamma_n}\cdot\varepsilon_n(t-T_n)\big)dt \\
& +\int\limits_{T_n}^{T_{n+1}} (f(U_{\varepsilon_n})-f(U_0))\big(\phi|_{\Gamma_n}-\partial_x\phi|_{\Gamma_n}\cdot\varepsilon_n(t-T_n)\big)dt \\
& +\int\limits_{T_n}^{T_{n+1}} (f(U_{\varepsilon_{n-1}})-f(U_{\varepsilon_n}))\big(\phi|_{\Gamma_n}+\partial_x\phi|_{\Gamma_n}\cdot\varepsilon_n(t-T_n)\big)dt.
\end{aligned}
$$

Neglecting terms converging to zero with the rate at least as ε and using $c_n \approx \bar{c}$ we have

$$
J_n \approx \int_{T_n}^{T_{n+1}} \big(2\varepsilon_n U_{\varepsilon_n}+c_nU_0-c_nU_{\varepsilon_{n-1}}-\underbrace{\varepsilon_n U_{n-1}}_{\to 0}-f(U_0)+f(U_{\varepsilon_{n-1}})\big)\phi\_\Gamma_n dt
$$

$$+\int_{T_n}^{T_{n+1}}\big(c_n\varepsilon_n U_{\varepsilon_n}-\underbrace{c_n\varepsilon_n U_{n-1}}_{\to 0}c_n+\varepsilon_n U_{\varepsilon_n}-2\varepsilon_n f(U_{\varepsilon_n})\big)(t-T_n)\partial_x\phi\_\Gamma_n dt$$

Using the fact that and $\varepsilon_n^{-1}\varepsilon_{n+1}\approx 0$ as we have used const$\cdot\varepsilon\approx 0$ in the case of normal SDWs, one get the convergence to zero for two terms above for each n, i.e. both components of $U_{\varepsilon_n}=(\rho_{\varepsilon_n},q_{\varepsilon_n})$ go to infinity slower than ε_{n+1}^{-1}. But there are infinitely many such terms in the sum $\sum_{n=1}^{\infty}J_n$, so one has to control their behavior. That can be done by supposing that $\dfrac{\varepsilon_{n+1}}{\varepsilon_n}\le d_\varepsilon\to 0$, uniformly with respect to the first perturbation parameter ε). The most natural choice is $\varepsilon_n=\varepsilon^{n+1}$ ($d_\varepsilon=\varepsilon$), as it was ussual in weak asymptotic methods (see [12] or [7] for example).

With such assumption, the following relations should be fulfilled

$$\begin{aligned}
c_n(\rho_{\varepsilon_{n-1}}-\rho_0)-(q_{\varepsilon_{n-1}}-q_0)+2\varepsilon_n\rho_{\varepsilon_n}&\approx 0\\
c_n\varepsilon_n\rho_{\varepsilon_n}-\varepsilon_n q_{\varepsilon_n}&\approx 0\\
c_n(q_{\varepsilon_{n-1}}-q_0)-\Big(\frac{q_{\varepsilon_{n-1}-1}}{\rho_{\varepsilon_{n-1}}}-\frac{q_0-1}{\rho_0}\Big)+2\varepsilon_n q_{\varepsilon_n}&\approx 0\\
c_n\varepsilon_n\rho_{\varepsilon_n}-\varepsilon_n q_{\varepsilon_n}&\approx 0\\
c_n\varepsilon_n q_{\varepsilon_n}-\frac{\varepsilon_n q_{\varepsilon_n}-1}{\rho_{\varepsilon_n}}&\approx 0.
\end{aligned}$$

But all these relations are already satisfied during the construction of $(n+1)$-th order SDWs (after fixing ε_{n-1} and introducing $\varepsilon_n\ll\varepsilon_{n-1}$) by setting

$$\lim_{\varepsilon_n\to 0}\varepsilon_n\rho_{\varepsilon_n}=\xi_n=\frac{1}{2}\kappa_{n,1},\ \lim_{\varepsilon_n\to 0}\varepsilon_n q_{\varepsilon_n}=\chi_n,\ c_n=\frac{\chi_n}{\xi_n}.$$

We have used the notation $\varepsilon_0=\varepsilon$ while $\kappa_{n,1}$ is the Rankine-Hugoniot deficit for the first equation in the strip $[T_n,T_{n+1}]$. Note that the relations for $n=0$ have been already established when the Riemann problem was solved.

Thus we have proved the existence of a sequence of higher order SDWs that solves our problem in approximate sence. The next step is to prove that the solution is admissible, i.e.

$$\partial_t\eta(\rho,q)+\partial_x Q(\rho,q)\le 0 \text{ for } (\rho,q) \text{ given by (5.3)}$$

in distributional sense for each convex entropy pair (η,Q) given by (1.4,1.5). Like in the previous cases, we shall prove the above inequality for $\eta=\frac{\rho}{2}F\Big(\frac{q-1}{\rho}\Big)$, F is convex, while the appropriate entropy flux is then $Q=\frac{1}{2}(q+1)F\Big(\frac{q-1}{\rho}\Big)$.

The case $\eta=\frac{\rho}{2}G\Big(\frac{q+1}{\rho}\Big)$ can be treated in the same way. Admissibility of the waves before $t=T$ has been already proved. Using the above calculations with $U=\eta(\rho,q)$ and $f(U)=Q(\rho,q)$ one gets that the following relations have to be satisfied in each interval $[T_n,T_{n+1}]$:

$$\begin{aligned}
-c_n(\eta(\rho_{\varepsilon_{n-1}},q_{\varepsilon_{n-1}})-\eta(\rho_0,q_0))+Q(\rho_{\varepsilon_{n-1}},q_{\varepsilon_{n-1}})-Q(\rho_0,q_0)&\\
+2\varepsilon_n\eta(\rho_{\varepsilon_n},q_{\varepsilon_n})\le 0,\ \varepsilon_n\ll 1&\\
\text{and } c_n\varepsilon_n\eta(\rho_{\varepsilon_n},q_{\varepsilon_n})-\varepsilon_n Q(\rho_{\varepsilon_n},q_{\varepsilon_n})\approx 0.&
\end{aligned}$$

The second relation is true since $q_{\varepsilon_n} = c_n\rho_{\varepsilon_n}$ so the difference between $\varepsilon_n \frac{1}{2}(q_{\varepsilon_n} + 1)F\Big(\frac{q_{\varepsilon_n}-1}{\rho_{\varepsilon_n}}\Big)$ and $\varepsilon_n \frac{1}{2}\rho_{\varepsilon_n} F\Big(\frac{q_{\varepsilon_n}-1}{\rho_{\varepsilon_n}}\Big)$ equals $\frac{1}{2}\varepsilon_n F(\bar{c}_n) \approx 0$. Letting all ε_n to zero one gets

$$2\varepsilon_n \eta(\rho_{\varepsilon_n}, q_{\varepsilon_n}) \approx \xi_n F(c_n).$$

The first relation is satisfied if for every n,

$$E_n := c_n(\eta(\rho_{\varepsilon_{n-1}}, q_{\varepsilon_{n-1}}) - \eta(\rho_0, q_0)) - (Q(\rho_{\varepsilon_{n-1}}, q_{\varepsilon_{n-1}}) - Q(\rho_0, q_0)) \geq \xi_n F(c_n).$$

But

$$\begin{aligned}
E_n =& \frac{c_n\rho_{\varepsilon_{n-1}} - q_{\varepsilon_{n-1}} - 1}{2} F\Big(\frac{q_{\varepsilon_{n-1}}-1}{\rho_{\varepsilon_{n-1}}}\Big) + \frac{q_0+1-c_1\rho_0}{2} F\Big(\frac{q_0-1}{\rho_0}\Big) \\
=& \underbrace{\frac{c_n(\rho_{\varepsilon_{n-1}}-\rho_0)-(q_{\varepsilon_{n-1}}-q_0)}{2}}_{=\kappa_{n,1}>0} \Big(\frac{c_n\rho_{\varepsilon_{n-1}} - q_{\varepsilon_{n-1}} - 1}{c_n(\rho_{\varepsilon_{n-1}} - rho_0) - (q_{\varepsilon_{n-1}} - q_0)} F\Big(\frac{q_{\varepsilon_{n-1}}-1}{\rho_{\varepsilon_{n-1}}}\Big) \\
&+ \frac{q_0+1-c_1\rho_0}{c_n(\rho_{\varepsilon_{n-1}} - rho_0) - (q_{\varepsilon_{n-1}} - q_0)} F\Big(\frac{q_0-1}{\rho_0}\Big)\Big) \\
\geq& \frac{\kappa_{n,1}}{2} F\Bigg(\frac{c_n(q_{\varepsilon_{n-1}}-q_0) - \Big(\frac{q_{\varepsilon_{n-1}}-1}{\rho_{\varepsilon_{n-1}}} - \frac{q_0-1}{\rho_0}\Big)}{\kappa_{n,1}}\Bigg) = \frac{\kappa_{n,1}}{2} F\Big(\frac{\kappa_{n,2}}{\kappa_{1,1}}\Big).
\end{aligned}$$

As $\frac{\kappa_{n,2}}{\kappa_{n,1}} = \frac{\chi_{\varepsilon_{n-1}}}{\xi_{\varepsilon_{n-1}}} = c_n$, we have $E_2 \geq \frac{1}{2}\kappa_{n,1}F(c_n) = \xi_n F(c_n)$. That proves the admissibility of the approximate solution in $[0, T] \cup \bigcup_{n=1}^{\infty}[T_{n-1}, T_n]$.

Taking into account the relation $\frac{\varepsilon_{n+1}}{\varepsilon_n} \leq d_\varepsilon$, $d_\varepsilon \to 0$ one can find more precise estimates for the parameters in (5.1) and (5.2):

$$\begin{aligned}
\xi_{n+1} \sim& d_\varepsilon^{-n+1-2^{-n}} \varepsilon^{1-2^{-(n+1)}} \\
\sum_{n=1}^{\infty} \Delta T_n \sim& \sum_{n=1}^{\infty} d_\varepsilon^{n-2+2^{-(n-1)}} \varepsilon^{1-2^{-n}} \leq \varepsilon^{1/2}(1 + d_\varepsilon^{1/2} + d_\varepsilon^{1+1/4} + \ldots\ldots \\
&\leq \varepsilon^{1/2} \underbrace{\sum (d_\varepsilon^{1/2})^n}_{\text{convergent for } \varepsilon \ll 1} \to 0, \text{ as } \varepsilon \to 0.
\end{aligned}$$

That is, there is a limit $\lim_{\varepsilon\to 0} T_n = \bar{T}$ that can be considered as a lifespan of SDW solution - note that the first interaction point T tends to $\bar{T} = \frac{x_0}{\bar{c} - \lambda_1(\rho_s, q_s)}$. Put $\bar{X} = \lim_{\varepsilon\to 0} X_n$ ($= \lim_{\varepsilon\to 0} Y_n$).

The total mass after T is now $\sum_{n=1}^{\infty} P_n$, where

$$\begin{aligned}
P_n =& \underbrace{\frac{1}{2}\rho_{\varepsilon_{n+1}}\varepsilon_{n+1}(T_{n+1}-T_n)^2}_{\text{mass in } \triangle X_n Y_{n+1} X_{n+1}} + \underbrace{\frac{1}{2}\rho_{\varepsilon_n}\varepsilon_n(T_{n+1}-T_n)(T_n-T_{n-1})}_{\text{mass in } \triangle X_n Y_n Y_{n+1}} \\
&+ \underbrace{\frac{1}{2}\rho_{s_{n+1}}(\lambda_2(\rho_{s_n}, q_{s_n}) - \lambda_1(\rho_{s_n}, q_{s_n}))(T_{n+1}-T_n)}_{\text{mass in CDC part}} \\
\sim& \frac{1}{2}\xi_{n+1}(\Delta T_{n+1})^2 + \frac{1}{2}\xi_n \Delta T_n \Delta T_{n+1} + \frac{\rho_{s_{n+1}}}{\rho_{s_n}} \Delta T_{n+1}
\end{aligned}$$

$$\sim \varepsilon^{2^{-(n+1)}} \prod_{i=1}^{n} \varepsilon_i^{2^{-(n+1-i)}} T^2 \sim \Delta T_{n+1}.$$

Therefore, $\sum_{n=1}^{\infty} P_n \sim \varepsilon^{1/2} \to 0$ as $\varepsilon \to 0$.

Due to the above analysis, the entropic approximate higher order SDW solution depending on the parameters $\varepsilon \gg \varepsilon_1 \gg$ .... exists only below $\bar{T} < \infty$. Its limit is the Riemann data $(\rho, q)|_{t=\bar{T}} = \begin{cases} (\rho_0, q_0), & x < \bar{X} \\ (\rho_s, q_s), & x > \bar{X} \end{cases}$. One can construct an entropic weak solution in $t > \bar{T}$ for such initial data, but that solution can not be a continuation of the initial solution for $t < \bar{T}$ since a distributional limit for $t < \bar{T}$ contains a delta function while the one for $t > \bar{T}$ does not.

One could say that SDW solution "blows up" at time $\bar{T}$. So the search for entropic weak solution after that time is left as an open problem.

5.2. **The marginal case.** Now, consider the case when $\bar{c} = \lambda_1(\rho_0, q_0)$. A solution to the Riemann problem at the point (T, X_1) is now CDC since $\lambda_1(\rho_0, q_0) = \bar{c} < \bar{c} + \frac{\varepsilon}{\xi_0} = \lambda_2(\rho_\varepsilon, q_\varepsilon)$. The values between CDs are $\rho_{r_1} = \dfrac{2}{\lambda_2(\rho_\varepsilon, q_\varepsilon) - \lambda_1(\rho_0, q_0)} = \dfrac{2\xi_0}{\varepsilon}$ and $q_{r_1} = \dfrac{\lambda_2(\rho_\varepsilon, q_\varepsilon) + \lambda_1(\rho_0, q_0)}{\lambda_2(\rho_\varepsilon, q_\varepsilon) - \lambda_1(\rho_0, q_0)} = \dfrac{2\xi_0 \bar{c} + \varepsilon}{\varepsilon}$. The solution to the Riemann problem at (T, Y_1) is the same as in the previous case consisting of a CDC with the intermediate state (ρ_{s_1}, q_{s_1}). The difference between slopes of the interacting waves started at the points X_1 and Y_1 is $\lambda_2(\rho_\varepsilon, q_\varepsilon) - \lambda_1(\rho_\varepsilon, q_\varepsilon) = \dfrac{2}{\rho_\varepsilon} = \dfrac{2\varepsilon}{\xi_0}$. Thus the next interaction time T_1 equals

$$T_1 = T + \frac{2\varepsilon T}{\lambda_2(\rho_\varepsilon, q_\varepsilon) - \lambda_1(\rho_\varepsilon, q_\varepsilon)} = T + \xi_0 T.$$

At that moment there are two unsolved Riemann problems at the points $X_2 = X_1 + \lambda_1(\rho_0, q_0)(T_1 - T)$ and $Y_2 = X_1 + \lambda_2(\rho_{r_1}, q_{r_1})(T_1 - T) Y_1$. At the point X_2, $\lambda_1(\rho_0, q_0) = \bar{c} < \lambda_2(\rho_{r_1}, q_{r_1}) = \lambda_2(\rho_\varepsilon, q_\varepsilon)$ and the solution to the Riemann problem in that point is a CDC with the intermediate state (ρ_{r_2}, q_{r_2}), $\rho_{r_2} = \dfrac{2}{\lambda_2(\rho_{r_1}, q_{r_1}) - \lambda_1(\rho_0, q_0)}$ and $q_{r_2} = \dfrac{\lambda_2(\rho_{r_1}, q_{r_1}) + \lambda_1(\rho_0, q_0)}{\lambda_2(\rho_{r_1}, q_{r_1}) - \lambda_1(\rho_0, q_0)}$. It is exactly the same as the one at the point X_1 at $t = T$ since $\lambda_2(\rho_{r_1}, q_{r_1}) = \lambda_2(\rho_\varepsilon, q_\varepsilon)$. At X_2, $\lambda_1(\rho_{r_1}, q_{r_1}) = \lambda_1(\rho_0, q_0) = \bar{c} < \lambda_2(\rho_{s_1}, q_{s_1}) = \lambda_2(\rho_s, q_s)$ and the solution is also a CDC with $\rho_{s_2} = \dfrac{2}{\lambda_2(\rho_s, q_s) - \lambda_1(\rho_{r_1}, q_{r_1})}$, $q_{s_2} = \dfrac{\lambda_2(\rho_s, q_s) - \lambda_1(\rho_{r_1}, q_{r_1})}{\lambda_2(\rho_s, q_s) - \lambda_1(\rho_{r_1}, q_{r_1})}$. Note that the later one is bounded with respect to ε. The difference between slopes of interacting waves started at the points X_2 and Y_2 is $\lambda_2(\rho_\varepsilon, q_\varepsilon) - \lambda_1(\rho_\varepsilon, q_\varepsilon) = \frac{1}{\rho_\varepsilon} = \frac{\varepsilon}{\xi_0}$, while the distance between X_2 and Y_2 is ε. So the next interaction time is $T_2 = T_1 + \xi_0$. Again, we have to solve two Riemann problems, but the solutions are the same as the ones at points X_2 and Y_2. And the procedure repeats again with the same solutions at new interaction points. Thus, we have the following approximate solution. For $T < t < T_1$,

$$(\rho, q) = \begin{cases} (\rho_0, q_0) & x < X_1 + \bar{c}(t-T) \\ (\rho_{r_1}, q_{r_1}), & X_1 + \bar{c}(t-T) < x < X_1 + \lambda_2(\rho_\varepsilon, q_\varepsilon)(t-T) \\ (\rho_\varepsilon, q_\varepsilon), & X_1 + \lambda_2(\rho_\varepsilon, q_\varepsilon)(t-T) < x < Y_1 + \lambda_1(\rho_\varepsilon, q_\varepsilon)(t-T) \\ (\rho_{s_1}, q_{s_1}), & Y_1 + \lambda_1(\rho_\varepsilon, q_\varepsilon)(t-T) < x < Y_1 + \lambda_2(\rho_s, q_s)(t-T) \\ (\rho_s, q_s), & Y_1 + \lambda_2(\rho_s, q_s)(t-T) < x < \lambda_2(\rho_2, q_2)t \\ (\rho_2, q_2), & x > \lambda_2(\rho_2, q_2)t \end{cases}.$$

And for $t > T_1$,

$$(\rho, q) = \begin{cases} (\rho_0, q_0) & x < X_2 + \bar{c}(t-T_1) \\ (\rho_{r_1}, q_{r_1}), & X_2 + \bar{c}(t-T_1) < x < Y_2 + \bar{c}(t-T_1) \\ (\rho_{s_2}, q_{s_2}), & Y_2 + \bar{c}(t-T_1) < x < Y_2 + \lambda_2(\rho_s, q_s)(t-T_1) \\ (\rho_s, q_s), & Y_2 + \lambda_2(\rho_s, q_s)(t-T_1) < x < \lambda_2(\rho_2, q_2)t \\ (\rho_2, q_2), & x > \lambda_2(\rho_2, q_2)t \end{cases}.$$

Contrary to the case $\lambda_1(\rho_0, q_0) > \bar{c}$ when the distribution limit of the approximate solution does not exists after a finite time, now there exists a distributional limit for $T > 0$. Denote $\lim_{\varepsilon\to 0} T = \bar{T}$, $\lim_{\varepsilon\to 0} X_1 = \bar{X}$. Then (ρ, q) tents to

$$(\bar{\rho}, \bar{q}) = \begin{cases} (\rho_0, q_0), & x < X_T + \bar{c}(t-\bar{T}) \\ \left(\frac{2}{\lambda_2(\rho_s, q_s) - \bar{c}}, \frac{\lambda_2(\rho_s, q_s) + \bar{c}}{\lambda_2(\rho_s, q_s) - \bar{c}}\right), & \bar{X} + \bar{c}(t-\bar{T}) < x < \bar{X} + \lambda_2(\rho_s, q_s)(t-\bar{T}) \\ (\rho_s, q_s), & x > \bar{X} + \lambda_2(\rho_s, q_s)(t-\bar{T}) \end{cases}$$
$$+\sigma(t)(1, \bar{c})\delta_{x=\bar{X}+\bar{c}(t-\bar{T})},$$

for $t > \bar{T}$. Here $\sigma(t) = \xi_0^2(t-\bar{T})$. The above analysis is illustrated in Figure 5.2.

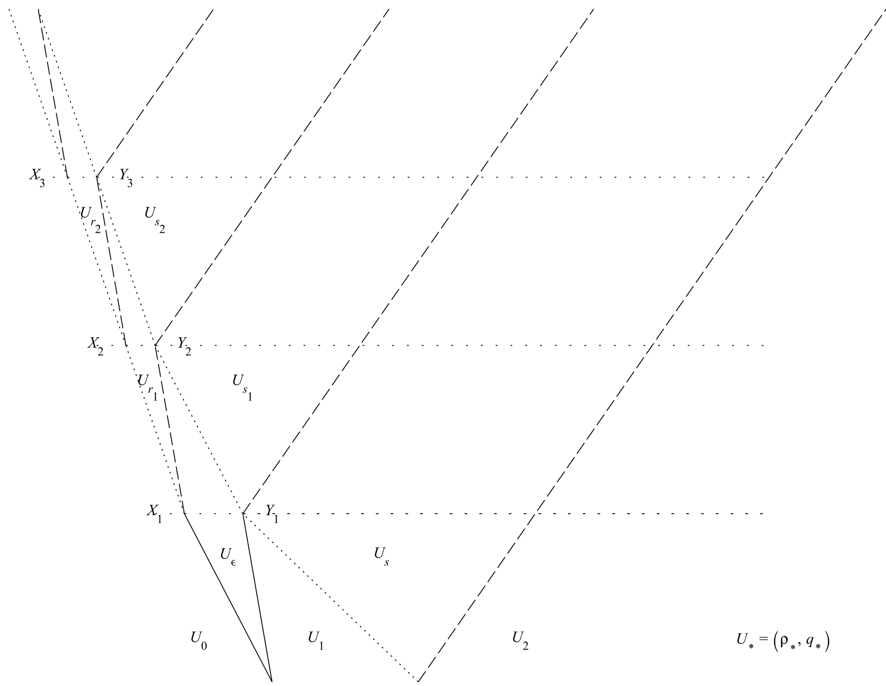

FIGURE 2. The marginal case

6. CONCLUSION

Let us conclude with an overview of the different cases of the initial data (3.1) that yield the wave interaction with an SDW on the left-hand side. The right-hand side can be done in an analogous way and we omit it here.

The necessary condition for existence of an incoming SDW on the left-hand side is $\lambda_1(\rho_0, q_0) \geq \lambda_2(\rho_1, q_1)$. Let us fix the states (ρ_0, q_0) and (ρ_1, q_1) satisfying that relation and denote by $\bar{c}$ the speed of the incoming SDW.

(a) If $\lambda_1(\rho_1, q_1) \geq \lambda_2(\rho_2, q_2)$ i.e. $q_2 \leq -1 + \dfrac{q_1 - 1}{\rho_1}\rho_2$, then the second incoming wave is also an SDW, and the outgoing wave is a single weighted SDW.

(b) If $q_2 > -1 + \dfrac{q_1 - 1}{\rho_1}\rho_2$ the second incoming wave is an CDC. Let (ρ_s, q_s) denotes the state between these contact discontinuities. Then $\lambda_1(\rho_1, q_1) = \lambda_1(\rho_s, q_s)$ and $\lambda_2(\rho_s, q_s) = \lambda_2(\rho_2, q_2)$ imply $\dfrac{2}{\rho_s} = \lambda_2(\rho_2, q_2) - \lambda_1(\rho_1, q_1) = \dfrac{q_2 + 1}{\rho_2} - \dfrac{q_1 - 1}{\rho_1}$.

   (1) If $\dfrac{2}{\rho_s} > \lambda_1(\rho_0, q_0) - \lambda_1(\rho_1, q_1)$ i.e. $q_2 > -1 + \dfrac{q_0 - 1}{\rho_0}\rho_2$ the outgoing wave consist of two delta contact discontinuities. The case is possible since we have $\dfrac{q_0 - 1}{\rho_0} > \dfrac{q_1 - 1}{\rho_1}$ already.

(2) If $\bar{c} \geq \dfrac{q_2{+}1}{\rho_2}$, i.e. $q_2 \leq -1{+}\bar{c}\rho_2$ then the solution is a single weighted SDW. Again, $\bar{c} > \dfrac{q_2{+}1}{\rho_2} {-} \dfrac{2}{\rho_1}$ so the above situation can happen.

(3) If $\bar{c} < \dfrac{q_2{+}1}{\rho_2}$ and $\bar{c} < \dfrac{q_0{-}1}{\rho_0}$ there exists an approximated piecewise constant solution in the form of sum of higher order shadow waves until the critical time $\bar{T} = \dfrac{x_0}{\bar{c}{-}\lambda_2(\rho_2, q_2)}$. The question of existence of entropic weak (approximate) solution after that time is still open.

(4) If $q_2 > -1{+}\bar{c}\rho_2$ and $\bar{c} = \dfrac{q_0{-}1}{\rho_0}$ we apply the same procedure as above, but now we get a sequence of CDCs as an approximate solution. That solution is a global one unlike the previous one and also has a distributional limit for each $T > 0$.

(c) The case of a double CDC interaction can be solved by a straightforward use of Riemann problem solutions as follows. Suppose that an CD_2 connecting states (ρ_0, q_0) and (ρ_1, q_1) interact with an CD_1 connecting states (ρ_1, q_1) and (ρ_2, q_2). Then the result of the interaction depends on the relation between (ρ_0, q_0) and (ρ_2, q_2) in the following way.

(1) If $\dfrac{q_0{-}1}{\rho_1} < \dfrac{q_1{+}1}{\rho_0}$, then the outgoing wave consists of the CDs with the state

$$(\rho_s, q_s) = \Big(\frac{2\rho_0\rho_1}{\rho_0(\rho_1{+}1){-}\rho_1(q_0{-}1)}, \frac{\rho_0(\rho_1{+}1){+}\rho_1(q_0{-}1)}{\rho_0(\rho_1{+}1){-}\rho_1(q_0{-}1)}\Big).$$

(2) If $\dfrac{q_0{-}1}{\rho_1} \geq \dfrac{q_1{+}1}{\rho_0}$, then the outgoing wave is a single simple SDW as in the cases where Theorem 2.1 is applied.

References

[1] M. C. Bento, O. Bertolami, A. A. Sen, Generalized Chaplygin gas, accelerated Expansion and dark energy-matter unification, Phys.Rev. **D66** 043507 (2002).

[2] Y. Brenier, Solutions with concentration to the Riemann problem for the one-dimensional Chaplygin gas equations. Journal of Mathematical Fluid Mechanics **7**(3) (2005), 326–331.

[3] Y. Brenier, E. Grenier, Sticky particles and scalar conservation laws, SIAM J. Numer. Anal. **35** (1998), no. 6, 2317–2328.

[4] A. Bressan, Hyperbolic Systems of Conservation Laws, Oxford University Press, New York, 2000.

[5] G-Q. Chen, H. Liu, Formation of δ-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids, SIAM J. Math. Anal. **34** (2003), no. 4, 925–938.

[6] C. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Springer-Verlag, Heidelberg, 2000.

[7] V. G. Danilov, V. M. Shelkovich, Dynamics of propagation and interaction of shock waves in conservation law systems, J. Differ. Equations, **211**, 333-381 (2005).

[8] W. E, Y. G. Rykov, Ya. G. Sinai, Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics. Comm. Math. Phys. **177** (1996), no. 2, 349–380.

[9] B. L. Keyfitz, H. C. Kranzer, Spaces of weighted measures for conservation laws with singular shock solutions, J. Differ. Equations **118**,2, 420-451 (1995).

[10] A. Kamenshchik, U. Moschella, V. Pasquier, An alternative to quintessence, Phys. Lett. **511** (2001) 265.

[11] N. Krejić, T. Krunić, M. Nedeljkov, Numerical verification of delta shock waves for pressureless gas dynamics, Journal of Mathematical Analysis and Applications, **345**(2000), 1, 243-257.
[12] V.P. Maslov, G.A. Omel'anov, Asymptotic soliton-like solutions of equations with small dispersion (Russian), Uspekhi Mat. Nauk **36** (1981), no. 3(219), 63-126, 255.
[13] M. Mazzotti, Nonclassical Composition Fronts in Nonlinear Chromatography: Delta-Shock, Industrial & Engineering Chemistry Research **48** (2009), 16, 7733-7752.
[14] M. Nedeljkov, Singular shock waves in interactions. Quart. Appl. Math. **66** (2008), no. 2, 281–302.
[15] M. Nedeljkov, M. Oberguggenberger, Interactions of delta shock waves in a strictly hyperbolic system of conservation laws. J. Math. Anal. Appl. **344** (2008), no. 2, 1143–1157. J. Math. Anal. Appl. (2008).
[16] M. Nedeljkov, Shadow waves, entropies and interactions for delta and singular shocks, Arch. Ration. Mech. Anal. **197**,2 (2010), 489–537.
[17] M. Nedeljkov, Generalized Chaplygin gas dynamics and nonclassical solutions, Preprint, 2012.
[18] E.Yu. Panov, V.M. Shelkovich, δ'-Shock waves as a new type of solutions to systems of conservation laws, J. Differ. Equations **228** (2006), 49–86.

Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia
E-mail address: `marko@dmi.uns.ac.rs`