INTRODUCTION TO
SYSTEMS OF CONSERVATION LAWS

LECTURE NOTES FOR DAAD SUMMER COURSE
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Part 1. Basic spaces

1. Classical function spaces

1.1. Space of differentiable functions. Denote by $\Omega \subset \mathbb{R}^n$ an open set, its closure $\overline{\Omega}$ and boundary $\partial\Omega$.

$\mathcal{C}^k(\Omega)$ is the set of all function $u : \Omega \to \mathbb{R}$ (or $\mathbb{C}$, but all functions in conservation laws are real-valued) with continuous derivatives of order k, $0 \leq k \leq \infty$.

$\mathcal{C}^k(\overline{\Omega})$ is the set of all functions $u \in \mathcal{C}^k(\Omega)$ such that there exist a function $\phi \in \mathcal{C}^k(\Omega')$, $u \equiv \phi$ on $\overline{\Omega} \subset \Omega'$, where Ω' is an open set.

$\mathcal{C}^k_b(\Omega)$ consists of functions from $\mathcal{C}^k(\Omega)$bounded together with all their derivatives.

It holds:

$$\mathcal{C}^k(\mathbb{R}^n)|_\Omega \subset \mathcal{C}^k(\overline{\Omega}) \subset \mathcal{C}^k(\Omega).$$

If Ω is bounded, then $\mathcal{C}^k(\overline{\Omega}) \subset \mathcal{C}^k_b(\Omega)$.

Denote by $\operatorname{supp} u$, $u : \Omega \to \mathbb{R}$, the complement of the largest open set Ω' such that $u|_{\Omega'} = 0$. The set $\operatorname{supp} u$ is called *support* of the function u. Since $\Omega \subset \mathbb{R}^n$,

$$\operatorname{supp} u = \overline{\{x \in \Omega :\ u(x) \neq 0\}}.$$

Notation $A \subset\subset B$ means that there exists a compact K such that $A \subset K \subset B$.

$\mathcal{C}^k_0(\Omega) = \{u \in \mathcal{C}^k(\Omega) :\ \operatorname{supp} u \subset\subset \Omega\}$.

Elements of $\mathcal{C}^\infty_0(\Omega)$ are called *test functions*.

1.2. L^p-Spaces. A set $A \subset \Omega \subset \mathbb{R}^n$ is of *Lebesgue measure zero*, $\mathcal{L}(A) = 0$, if for each $\varepsilon > 0$ there exists a numerable union $\bigcup_{i\in\mathbb{N}} C_i$ of parallelepipeds $C_i \subset \mathbb{R}^n$ such that

$$\operatorname{mes} \bigcup_{i=1}^{\infty} C_i < \varepsilon$$

(the measure of parallelopiped is the product of its edges lenghts).

In the set of all Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ (all elementary functions and their compositions are Lebesgue measurable, for example) we define the equivalence relation "equal almost everywhere in Ω", $f \sim g$, if

$$\mathcal{L}(\{x :\ f(x) \neq g(x)\}) = 0.$$

Let $1 \leq p < \infty$. From now on Ω will be an open connected set. "Measurable" stands for Lebesgue measurable.

$$L^p(\Omega) = \{f/\sim : \Omega \to \mathbb{R} : \ f \text{ is measurable }, \ \int_\Omega |f(x)|^p \,\mathrm{dx} < \infty\}.$$

It is Banach space with the norm

$$\|f\|_{L^p(\Omega)} = \Big(\int_\Omega |f(x)|^p \,\mathrm{dx}\Big)^{1/p}.$$

$L^2(\Omega)$ is Hilbert space, with the product $(f|g)$ defined by

$$(f|g) = \int_\Omega f(x)\overline{g(x)} \,\mathrm{dx},$$

where $\overline{g(x)}$ stands for complex conjugate of $g(x)$. If we are in the space of real-valued functions (which will usually be the case) then

$$(f|g) = \int_\Omega f(x)g(x) \,\mathrm{dx}\,.$$

For $p = \infty$ we have a different definition:

$$\begin{aligned} L^\infty(\Omega) = &\{f/\sim : \Omega \to \mathbb{R} : \ f \text{ is measurable and there exists} \\ &\text{real } M \text{ such that } |f(x)| \leq M, \text{ for every } x \in \Omega\}. \end{aligned} \tag{1}$$

$L^\infty(\Omega)$ is also Banach space with the norm

$$\|f\|_{L^\infty} = \inf M, \text{ where the constant } M \text{ is from (1).}$$

The most important spaces are L^2-spaces and L^1_{loc}-spaces which are defined by

$$\begin{aligned} L^p_{loc}(\Omega) = &\{f/\sim : \Omega \to \mathbb{R} : \ f \text{ is measurable and for every } K \subset\subset \Omega \\ &\int_K |f(x)|^p \,\mathrm{dx} \leq \infty\}. \end{aligned}$$

Functions from L^1_{loc} are called *locally integrable* ones.

Hölder inequality

(2)
$$\int_\Omega |u(x)v(x)| \,\mathrm{dx} \leq \|u\|_{L^p}\|v\|_{L^q}, \ u \in L^p(\Omega), \ v \in L^q(\Omega), \ \frac{1}{p} + \frac{1}{q} = 1.$$

will often be used. The special case $p = q = 2$ is called *Schwartz inequality.*

Corollaries of Hölder inequality:

1.
$$\mathrm{mes}(\Omega)^{-1/p}\|u\|_{L^p} \leq \mathrm{mes}(\Omega)^{-1/q}\|u\|_{L^q}, \ u \in L^q(\Omega), \ p \leq q.$$

2.
$$\|u\|_{L^q} \le \|u\|_{L^p}^{\lambda} \|u\|_{L^r}^{\lambda},\ u \in L^r(\Omega),\ p \le q \le r,\ \frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}.$$

3.
$$\int_\Omega u_1 ... u_m \,\mathrm{dx} \le \|u\|_{L^{p_1}} ... \|u\|_{L^{p_m}},$$
$$u_i \in L^{p_i}(\Omega),\ i = 1, ..., m,\ \frac{1}{p_1} + ... + \frac{1}{p_m} = 1.$$

2. Weak derivative and weak solution

2.1. **Weak derivative.** Denote by $|\alpha| = \alpha_1 + ... + \alpha_n$ multiindex $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ and

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}.$$

(If $\alpha_i = 0$ for some i, there is no derivative with respect to the variable x_i.)

Definition 1. A function $f \in L^1_{loc}(\Omega)$ has α-th weak derivative, $|\alpha| \le m$, denoted again by $\partial^\alpha f$, if there exist a function $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f(x) \partial^\alpha \phi(x) \,\mathrm{dx} = (-1)^{|\alpha|} \int_\Omega g(x) \phi(x) \,\mathrm{dx},$$

for every $\phi \in \mathcal{C}_0^\infty(\Omega)$. The function g will be called α-th weak derivative for f.

The following theorem is very useable and illustrative.

Theorem 1. *If there exists a weak derivative for a locally integrable function u, then u is almost everywhere differentiable and the weak derivative equals to a strong at the points where it exists.*

2.2. **Weak solution of partial differential equations.** Notion of a weak solution is not defined in a unique manner. It should be defined to fit a physical problem as much as it can.

First, we shall give the definition for first order systems. Later on, the definition will be easily adopted to an equation of higher order.

Definition 2. A system of first order partial differential equation is in *divergence form* if it can be written as

$$\partial_t a_0(t, x, u) + \partial_{x_1} a_1(t, x, u) + ... + \partial_{x_n} a_n(t, x, u) = b(t, x, u), \tag{3}$$

where $u = u(t, x_1, ..., x_n)$ is a vector valued function. Suppose that u satisfies initial condition $u(x, 0) = u_0(x)$. It is said that

$$u \in \left(L^1_{loc}([0,T) \times \Omega)\right)^n$$

is *weak solution* to system (3) with the above given initial data if

$$\begin{aligned}
&\int_0^t \int_\Omega \partial_t\phi(t,x)a_0(t,x,u) + \partial_{x_1}\phi(t,x)a_1(t,x,u) + ... \\
&+ \partial_{x_n}\phi(t,x)a_n(t,x,u)\,\mathrm{dx}\,\mathrm{dt} + \int_\Omega u_0(x)\,\mathrm{dx} \\
= &\int_0^t \int_\Omega b(t,x,u)\phi(t,x)\,\mathrm{dx}\,\mathrm{dt},
\end{aligned} \tag{4}$$

for every $\phi \in \mathcal{C}_0^\infty((-\infty,\infty)\times\Omega)$.

As one can see, vector valued function u is not necessary differentiable and the name "weak solution" comes from that fact.

Also, it is easy to check, using integration by parts, that every $\mathcal{C}^1$-solution of (3) also satisfies (4), i.e. it is weak solution, too.

For practical reasons we shall use the following simpler (and weaker) condition instead of (4):

$$\begin{aligned}
&\int_0^t \int_\Omega \partial_t\phi(t,x)a_0(t,x,u)\partial_{x_1}\phi(t,x)a_1(t,x,u) + ... \\
&+ \partial_{x_n}\phi(t,x)a_n(t,x,u)\,\mathrm{dx}\,\mathrm{dt} \\
= &\int_0^t \int_\Omega b(t,x,u)\phi(t,x)\,\mathrm{dx}\,\mathrm{dt} \\
&\lim_{t\to 0} u(t,x) = u_0 \text{ almost everywhere in } \Omega,
\end{aligned} \tag{5}$$

for every $\phi \in \mathcal{C}_0^\infty((0,\infty)\times\Omega)$. Note that now ϕ is defined on a smaller domain, i.e. equals zero on the x-axe ($t = 0$).

Remark 1. If a system is not given in the divergence form, then a definition of a weak solution is much more difficult to give and more specific. That was out of scope here.

Systems where t is distinguished variable are called *evolution systems* (or systems "written in evolution form").

3. Distribution spaces

In this section we shall present a simplified version of distribution theory. We shall use convergence in vector spaces and not topology.

Mapping from a vector space over some field into that field (usually $\mathbb{R}$ or $\mathbb{C}$) is called *functional*.

Let us introduce a convergence in the set $\mathcal{C}_0^\infty(\Omega)$.

{testkon}

Definition 3. A sequence $\{\phi_j\} \subset \mathcal{C}_0^\infty(\Omega)$ converge to zero as $j \to \infty$ if

- There exists a compact $K \subset\subset \Omega$ such that $\operatorname{supp} \phi_j \subset K$, for every $j \in \mathbb{N}$.
- For each $\alpha \in \mathbb{N}_0^n$, $\|\partial^\alpha \phi\|_{L^\infty(\Omega)} \to 0$, as $j \to \infty$.

This convergence is denoted by $\xrightarrow{\mathcal{D}}$.

The set $\mathcal{C}_0^\infty(\Omega)$ with the convergence defined in this way will be denoted by $\mathcal{D}(\Omega)$. Elements of this space will be called test functions.

{distrib}

Definition 4. Linear continuous functional S with the domain $\mathcal{D}(\Omega)$ is called *distribution*. Its acting on the test function ϕ is denoted by $\langle S, \phi \rangle$.

Continuity is understood in the means of convergence: S is continuous if for each sequence of test functions $\{\phi_j\}_j$ converging to zero as $j \to \infty$ it holds $\langle S, \phi_j \rangle \to 0$, as $j \to \infty$.

Vector space of distributions is denoted by $\mathcal{D}'(\Omega)$.

Now, we shall give some important examples of distributions. The first one shows how locally integrable function can be treated as distributions and the second one is an example of distribution which can not be treated as a usual function.

Example 1. Let $f \in L^1_{loc}(\Omega)$ and ϕ be a test function. Then mapping from $\mathcal{D}$ into $\mathbb{R}$ defined by

$$S_f : \langle S_f, \phi \rangle = \int_\Omega f(x)\phi(x)\,\mathrm{dx}$$

define a distribution: Functional S_f is obviously linear and

$$|\langle S_f, \phi \rangle| \le \|\phi\|_{L^\infty(\Omega)} \int_{\operatorname{supp}\phi} |f(x)|\,\mathrm{dx}\,.$$

That means that if a sequence $\{\phi_j\}$ converges to zero in $\mathcal{D}$, then $\langle S_f, \phi_j \rangle \to 0$ as $j \to \infty$, i.e. S_f is a distribution.

Example 2. Let $a \in \Omega$. Relation

$$\langle \delta_a, \phi \rangle = \phi(a)$$

defines *Dirac delta distribution* at the point a. If $a = 0$, then we write just δ instead δ_0.

4. Properties and operations with distributions

(1) For a sequence of distributions $\{S_j\} \subset \mathcal{D}'(\Omega)$ is said to *converge to zero* if
$$\langle S_j, \phi\rangle \to 0, \text{ as } j \to \infty,$$
for every $\phi \in \mathcal{D}(\Omega)$. Convergence in the distribution space is denoted by $\xrightarrow{\mathcal{D}'}$. (In distribution theory this convergence is called "weak"). Convergence to zero is enough since the distribution space is a vector one: $S_j \to T$, $T \in \mathcal{D}'(\Omega)$ if and only if
$$\langle S_j - T, \phi\rangle \to 0, \text{ as } j \to \infty,$$
for every test function ϕ.

(2) $S \in \mathcal{D}'(\Omega)$ is zero on $\omega \subset \Omega$ if
$$\langle S, \phi\rangle = 0,$$
for every test function ϕ with a support in ω.

Definition 5. *Support of a distribution* $S \in \mathcal{D}'(\Omega)$, $\operatorname{supp} S$, *is a complement of the maximum open set where* $S = 0$ (i.e. set of point in Ω which do not have a neighborhoods ω where $S = 0$).

Definition 6. $\mathcal{E}'(\Omega)$ is the *space of distributions with compact support.*

Example 3. $\operatorname{supp}\delta = \{0\}$, because for each $x \in \Omega$, $x \neq 0$, there exists its neighborhoods ω not containing zero and there exist a test function ϕ with a support in ω. That means
$$\langle \delta, \phi\rangle = \phi(0) = 0.$$

Definition 7. *Distributional derivative* S of order $\alpha \in \mathbb{N}_0^n$ is defined by
$$\langle \partial^\alpha S, \phi\rangle := (-1)^{|\alpha|}\langle S, \partial^\alpha \phi\rangle, \text{ for every } \phi \in \mathcal{D}(\Omega).$$

Since $\partial^\alpha \phi$ is also in $\mathcal{D}(\Omega)$, one can see that the definition makes sense, i.e. each distribution has a derivative of every order. That fact is the main reason why distributions are so important.

{nep}

Lemma 1. *Differentiation is a continuous operation in the distribution space.*

Example 4. We can easily calculate each derivative of the delta distribution,
$$\langle \partial^\alpha \delta, \phi\rangle = (-1)^{|\alpha|}\langle \delta, \partial^\alpha \phi\rangle = (-1)^{|\alpha|}\partial^\alpha \phi(0).$$

One can easily verify the following. If $g \in L^1_{loc}(\Omega)$ is α-th weak derivative of $f \in L^1_{loc}(\Omega)$, then $S_g = \partial^\alpha S_f$, where S_f (or S_g) is the distributional image of f (or g).

Example 5. Define Heaviside function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases}$$

Since H is locally integrable function we can identify it with a distribution defined on $\mathbb{R}$. We will show that its derivative is the delta distribution. Let ϕ be an arbitrary test function on $\mathbb{R}$. Then

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int_0^\infty \phi'(x)\,\mathrm{dx} = \phi(0) = \langle \delta, \phi \rangle.$$

If $W^k(\Omega)$ stands for the space of locally integrable functions on Ω having all derivatives of order less or equal to k, then

$$\mathcal{C}^k(\Omega) \subset W^k(\Omega) \subset \mathcal{D}'(\Omega).$$

(Here, function is identified with its image in the space of distributions.) If $f \in \mathcal{C}^\infty(\Omega)$, then we can define its product with a distribution S, $T = Sf$, in the following way

$$\langle T, \phi \rangle := \langle S, f\phi \rangle, \ \phi \in \mathcal{D}(\Omega).$$

But, there is no general definition of the product if f is not smooth. This is the main disadvantage of distributions.

At the end of the paper, we shall give a possibility to overcome that fact by introducing Colombeau-type generalized function spaces.

5. Sobolev spaces

5.1. **Definitions.** Let $m \in \mathbb{N}_0$, $p \geq 1$ and Ω be an open subset of $\mathbb{R}^n$. Denote by $W^k(\Omega)$ the vector space of all locally integrable functions on Ω which has all weak derivatives of order less or equal to k. We are in position to define its subspaces which will have the advantage to be normed (space $W^k(\Omega)$ is only locally convex, with topology defined by a sequence of seminorms).

Definition 8. Sobolev space $H^{m,p}(\Omega)$ is the set of functions $u \in W^m(\Omega)$, such that

$$\partial^\alpha u \in L^p(\Omega),$$

for every $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m$.

Norma is given by

$$\|u\|_{H^{m,p}(\Omega)} = \|u\|_{m,p,\Omega} := \Big(\int_\Omega \sum_{|\alpha| \leq m} |\partial^\alpha u(x)|^p \,\mathrm{dx} \Big)^{1/p}.$$

An equivalent norm to the above one is given by

$$\|u\|'_{H^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{L^p(\Omega)}.$$

In the sequel we shall not distinguish them by notation, i.e. each of these two norms will be denoted by $\|u\|_{H^{m,p}(\Omega)}$.

If $p=2$, then we shall omit that number in the superscript.

Theorem 2. *For each $m \in \mathbb{N}_0$, $H^m(\Omega)$ is Hilbert space with the product*

$$(u|v) = \int_\Omega u(x)v(x)\,\mathrm{dx} + \sum_{i=1}^m \int_\Omega \nabla^m u(x)\nabla^m v(x)\,\mathrm{dx}\,. \tag{6}$$

If $p \geq 1$, then $H^{m,p}(\Omega)$ is only Banach space.

Denote by $\langle\xi\rangle = \sqrt{1+\xi^2}$. Then, the usual norm of u in $H^m(\mathbb{R}^n)$ is equivalent with the following one

$$\|u\|''_{H^m} := \sup_{\xi\in\mathbb{R}^n} \sum_{j=0}^m \|\langle\xi\rangle\hat{u}\|_{L^2(\Omega)}.$$

We are in position to define $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ now: $u \in H^s(\mathbb{R}^n)$ if and only if $\|u\|''_{H^m} < 0$.

The following spaces are important for defining value of an element in Sobolev space on the boundary in a simple way.

Definition 9. $H_0^{m,p}(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ in a norm $H^{m,p}(\Omega)$. $v \in H_0^{m,p}(\Omega)$ means that there exists a sequence $\{\phi_j\} \subset \mathcal{C}_0^\infty(\Omega)$ such that

$$v_j \stackrel{H^{m,p}}{\to} v,\ j \to \infty.$$

For $u \in H^{m,p}(\Omega)$, boundary condition $u|_{\partial\Omega} = 0$ in a weak sense means that $u \in H_0^{m,p}(\Omega)$. If $v \in H^{m,p}(\Omega)$, then $u = v$ on $\partial\Omega$ if and only if $u - v \in H_0^{m,p}(\Omega)$.

5.2. **Imbedding theorems.** We shall give only few of the large number of these important theorems.

Definition 10. Banach space B_1 is *continuously imbedded* in a Banach space B_2, $B_1 \to B_2$, if there exists bounded linear injection from B_1 into B_2.

Theorem 3. *For an open set $\Omega \subset \mathbb{R}^n$ it holds*

$$H^{m,p}(\Omega) \to L^q(\Omega),\ mp > n,\ p \leq q \leq \infty$$
$$H^{m,p}(\Omega) \to L^q(\Omega),\ mp = n,\ p \leq q < \infty$$
$$H^{m,p}(\Omega) \to \mathcal{C}_b^0(\Omega),\ mp > n$$

Theorem 4. *Let Ω be bounded and posses* conus property*: For each $x \in \Omega$ there exists a conus of the height h with the edge at x completely lying in Ω. Then*

$$H^{m,p}(\Omega) \to L^q(\Omega),\ p \le q \le np/(n-mp)$$
$$H^{m+j,1}(\Omega) \to \mathcal{C}_b^j(\Omega),\ mp > n$$

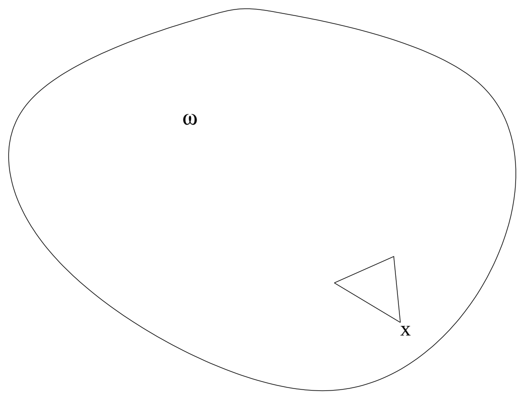


FIGURE 1. A set with a conus property {sl_con}

Part 2. **Waves**

6. Introduction

There is no precise definition of wave, but one can describe it as a signal traveling from one place to another one with clearly visible speed.

The signal can be any disturbance, like some kind of maxima or change of some quantity.

We shall define two kinds of waves.

{d_waves}

Definition 11. (i) Hyperbolic waves. They are solution to hyperbolic equations.
(ii) Dispersive waves. They are solutions to some equation(s) of the form

(7) {dis1} $$\varphi = a\psi(kx - \omega t),$$

where its *frequency* ω is some real function of *wave number* k, and $\omega(k)$ is defined via a system of partial or ordinary differential or integral equations.

Phase velocity is $\dfrac{\omega(k)}{k}$. The wave is *dispersive* if $\omega'(k)$ is not a constant, i.e. $\omega''(k) \neq 0$.

Group velocity, defined by

$$c(k) = \frac{\mathrm{d}\,\omega}{\mathrm{d}\,k},$$

is especially important for observation of wave propagation.

There exist partial differential equations belonging to both of the groups. One of these is Klein-Gordon equation

$$u_{xx} - u_x + u = 0$$

. It is hyperbolic equation with solutions of form (7), with $\omega^2 = k^2 + 1$. But that group is relatively small.

7. Kinetic waves

In a lot of physical problems a disturbance in a material, or in a state of a medium can arise. So we shall describe the basic building blocks, density $\rho(x,t)$, flux $q(x,t)$ and the flow velocity:

$$v(x,t) = \frac{q(x,t)}{\rho(x,t)} = \frac{\text{flux}}{\text{density}}.$$

as well as relations between them.

Homogeneous relation between ρ and q is the simplest one: $q = Q(\rho)$. Denote $c(\rho) = Q'(\rho)$. The above equation now reads

$$\rho_t + q_x = 0 \tag{8}$$

i.e.

$$\rho_t + c(\rho)\rho_x = 0 \tag{9}$$

if ρ and q are regular enough.

First, let us note that the characteristics for (9) are given by the following ordinary differential equations

$$\gamma : \frac{\mathrm{dx}}{\mathrm{dt}} = c(\rho).$$

Since we are dealing with a conservation law (right-hand side of (8) equals zero), the curves given by γ are straight lines, i.e. speed of a wave, $c(\rho)$, is constant.

8. Traffic flow

Obviously, the flow velocity

$$v(\rho) = \frac{q(\rho)}{\rho}$$

is obviously decreasing function with respect to ρ which take values from a maximum one, at $\rho = 0$, to zero, as $\rho \to \rho_y$. Here ρ_y is maximal car density at a road (cars touches one another). The flux q is therefore a convex function (see Fig. 2) and has a maximal value q_m for some density ρ_m, while $q(0) = q(\rho_y) = 0$.

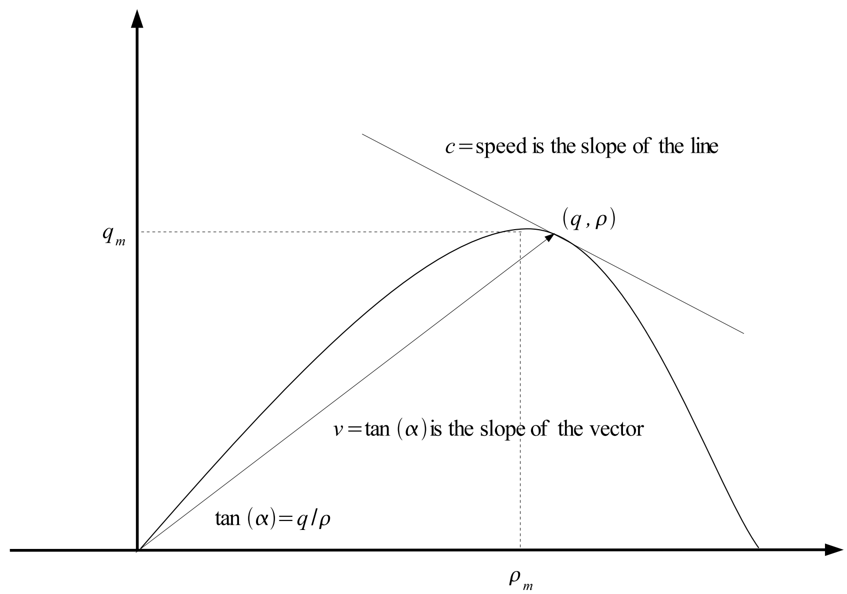


Figure 2. Flux function for traffic flow model {sl1}

After observations made in Lincoln tunnel, New York, experimental data for one right of way are: $\rho_y \approx 225\frac{\text{verichles}}{\text{mile}}$, $\rho_m \approx 80\frac{\text{verichles}}{\text{hour}}$. (The maximum flow for the above data could be obtained for car speeds $q_m \approx 20\frac{\text{milja}}{\text{sat}}$).

A rough model for more than one right of way can be obtained by multiplying the above values with their multiplicity.

Suppose that q depends only on ρ, $q := Q(\rho)$. Speed of waves is given by

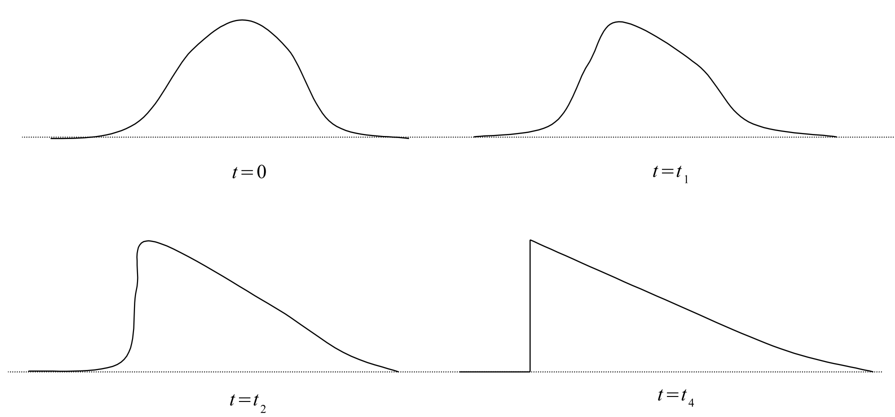
$$c(\rho) = Q'(\rho) = v(\rho) + \rho v'(\rho).$$

Since $v'(\rho) < 0$, it is less than flow velocity. It means that drivers can see a disturbance ahead.

In this particular case, speed of waves c is speed of cars, and a flow velocity is an average velocity of motion of a road relative to all of cars. Let us note that $c > 0$ for $\rho < \rho_m$ (cars are moving faster than average if density is small) and $c < 0$ for $\rho > \rho_m$ (opposite case: high density of the cars has lower speed than average).

Greenberg's model for the above tunnel are calculated in the following way: $Q(\rho) = a\rho \log \frac{\rho_j}{\rho}$, $a = 17.2\frac{m}{h}$, $\rho_j = 228\frac{v}{m}$. $\rho_m = 83\frac{v}{m}$, $\rho_m = 1430\frac{v}{h}$. Logarithmic function definitely does not approximate states in a neighborhoods of the point $\rho = 0$ in a good way, but this is practically not interesting case, anyway.

A solution to the above problem is just illustrated in the Fig. 3.



{sl2} FIGURE 3. Density of cars

9. Sedimentation in a river, chemical reactions

This model describes exchange processes between river-bed and fluid in the river, i.e. sedimentation transport, more precisely.

Denote

ρ_1 . . . . density of a fluid

ρ_0 . . . . density of a solid material.

Then, the density is given by

$$\rho = \rho_0 + \rho_1$$

and flux by $q = u\rho_1$, where u is a fluid speed. Conservation of mass law is given by

$$(\rho_1 + \rho_0)_t + u\rho_{1x} = 0,$$

with the supposition that fluid speed is a constant.

Reaction between these two materials are given by

$$\frac{\partial \rho_0}{\partial t} = k_1(A_1 - \rho_0)\rho_1 - k_2\rho_0(A_2 - \rho_1),$$

where k_1 and k_2 are coefficients depending on a reaction speed, and A_1, A_2 are constants depending on material specifications (both solid and fluid ones).

Let us take a special case, so called quasi-equilibrium, when changes of solid material density due chemical reactions are neglected, i.e.

$$\frac{\partial \rho_0}{\partial t} = 0.$$

We shall also suppose that space-time position is negligible, i.e.

$$\rho_0 = r(\rho_1).$$

Then we have the following system

$$\begin{aligned} &\rho_{1t}(1 + r'(\rho_1)) + u\rho_{1x} = 0 \\ &\text{i.e.} \\ &\rho_{1t} + \frac{u}{1 + r'(\rho_1)}\rho_{1x} = 0. \end{aligned}$$

In some models, one can take

$$r(\rho_1) = \frac{k_1 A_1 \rho_1}{k_2 B + (k_1 - k_2)\rho_1}.$$

Equation which describes waves in this case follows from law of mass conservation. In general, flux is given by $q = \rho u$, where $u \neq \text{const}$, so we need one more equation (for speed u).

10. Shallow water equations

Let us fix some notation:

ρ . . . . height of water level – its depth ($\approx$ density)
u . . . . speed of water flow

This model is used for description of river flow when depth is not so big (in the later case one can safely take that the depth equals infinity). It can be also used for flood, sea near beach, channel flow, avalanche,...

Basic assumption in this model is that a fluid is incompressible and homogenous (forming of "waves", moving of a water visible on its surface, is possible). Bottom of a river is not necessary flat, but for a flat one equations are homogenous – flux is independent of space-time coordinates. That eases finding global solutions to a system.

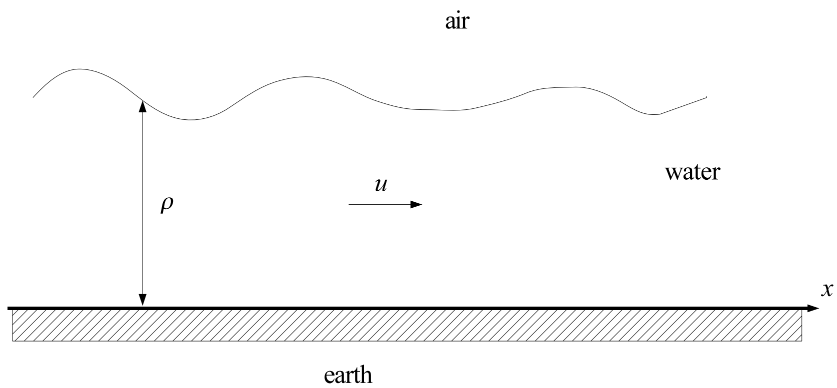


FIGURE 4. Shallow water {sl3}

Mass conservation law gives

$$\rho_t + (\rho u)_x = 0. \quad (10) \quad \{\texttt{shw01}\}$$

In order to solve the above equation we shall introduce new partial differential equation involving the speed u and Newton's second law:

$$(mu)^{\cdot} = f \text{ (“force = impuls change per time”)}.$$

Take a space interval $[x_1, x_2]$ during a time interval $[t_1, t_2]$. Then

$$\int_{x_1}^{x_2} \rho(x,t_2)u(x,t_2)\,\mathrm{dx} - \int_{x_1}^{x_2} \rho(x,t_1)u(x,t_1)\,\mathrm{dx}$$
$$= \int_{t_1}^{t_2} \big(\rho(x_1,t)u^2(x_1,t) - \rho(x_2,t)u^2(x_2,t)\big)\,\mathrm{dt}$$
$$+ \int_{t_1}^{t_2} \big(p(x_1,t) - p(x_2,t)\big)\,\mathrm{dt}$$

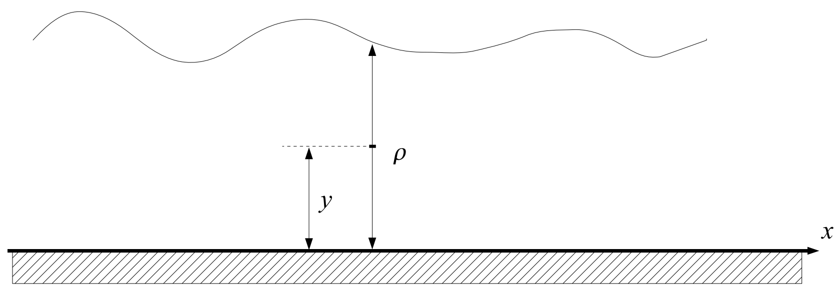
“impuls change per time = kinetic energy + force due to preassure”

Contraction of a time-space interval: $t_1, t_2 \longrightarrow t$ and $x_1, x_2 \longrightarrow x$ for some pint (x,t), gives the following PDE

$$(\rho u)_t + (\rho u^2)_x + p_x = 0. \quad (11)$$

The pressure in the above equation is the hydraulic pressure. One gets (we shall assume that density of water equals 1)

$$\pi(y) = g(\rho - y) \;.\;.\;.\;.\;.\;.\;.\;. \text{ hydraulic preassure,}$$



{sl4} FIGURE 5. Hydraulic pressure

where g is the universal gravitational constant (see Fig. 5), and

$$p = \int_0^\rho \pi(y)\,\mathrm{dy} = \int_0^\rho g(\rho - y)\,\mathrm{dy} = g\frac{\rho^2}{2}.$$

Substituting this relation into (10) and (11) gives

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + \big(\rho u^2 + g\frac{\rho^2}{2}\big)_x = 0. \end{aligned} \tag{12}$$ {shw1}

Let us differentiate the second equation in the above system assuming enough regularity of solutions:

$$\rho_t u + \rho u_t + 2\rho u u_x + \rho_x u^2 + g\rho\rho_x = 0.$$

Then substitute ρ_t from the first equation in the modified second equation. After that procedure we get

$$u_t + uu_x + g\rho\rho_x = 0,$$

and finally the system becomes

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &u_t + \big(\frac{u^2}{2} + g\rho\big)_x = 0. \end{aligned} \tag{13}$$ {shw2}

If solutions are not necessarily differentiable, one substitute $\omega = \rho u$ (ω is a flux) into system (12) so we get a different one

$$\begin{aligned} &\rho_t + \omega_x = 0 \\ &\omega_t + \big(\frac{\omega^2}{\rho} + g\frac{\rho^2}{2}\big)_x = 0. \end{aligned} \tag{14}$$ {shw3}

In subsequent sections one will see that systems (13) and (14) are not equivalent in practice (concerning weak solutions) due to the use of differentiation.

11. Gas dynamics (viscous)

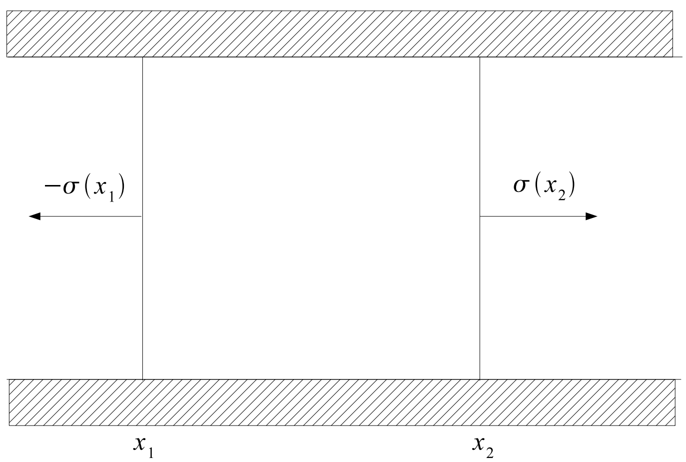
We shall use the following notation:
ρ . . . . gas density
u . . . . gas velocity (gas molecule speed)
σ . . . . pressure (force/area)
As before, we have the following system of conservation laws

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2)_x - \sigma_x = 0.$$

The following relation holds in general:

$$\sigma = -p + \nu u_x,$$

where p is a pressure of a gas without moving, and ν is a viscosity ($\ll 1$) (see Fig. 6).



{s15} Figure 6. Pressure in share gas

Thus,

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2)_x + p_x = \nu u_{xx}$$

holds for viscous fluids. For gases it holds $\nu \to 0$, so one can often take $\nu \equiv 0$.

11.1. **Thermodynamically effects with gases.** We shall continue to use notation from the previous section. Let $p = p(\rho, S)$, where new independent variable S stands for entropy.

In order to close the system we need an extra equation. For adiabatic case one can take

$$S_t + uS_x = 0,$$

for example.

For an isotropic, ideal gas, one takes

$$S \equiv \text{const},\ \nu \equiv 0.$$

Now, the viscous case is modeled by

$$\begin{aligned}
&\rho_t + (\rho u)_x = 0\\
&(\rho u)_t + (\rho u^2)_x + (p(\rho))_x = 0\\
&p(\rho) = \kappa\rho^\gamma,\ 1 < \gamma < 3,\ \gamma = 1 + 2/n,
\end{aligned}$$

where κ stands for universal gas constant, and n is a number of atoms in gas molecule.

Let us note that for constant density, $\rho = \rho_0 \in \mathbb{R}$, there is no changes in pressure and speed of the gas – no gas movements.

In more than one space dimensions we have well known Navier-Stokes equation

$$\begin{aligned}
&\rho_t + \operatorname{div}(\rho\vec{u}) = 0\\
&(\rho u)_t + \vec{u}\cdot\operatorname{grad}(\rho\vec{u}) + (\rho\vec{u})\cdot\operatorname{div}\vec{u} + \operatorname{grad} p = 0\\
&(\text{or } = \nu\triangle\vec{u} \text{ for viscous fluids}).
\end{aligned}$$

Part 3. **Weak solutions and elementary waves**

12. Rankin-Hugoniot conditions

Let $u \in C^1(\mathbb{R}\times[0,\infty))$ be a solution to the following partial differential equation

$$\begin{aligned}
&u_t + (f(u))_x = 0\\
&u(x,0) = u_0(x).
\end{aligned} \tag{15} \quad \{\texttt{sr1}\}$$

Take $\varphi \in C_0^1(\mathbb{R}\times[0,\infty))$, i.e. smooth function such that its support intersected by $\mathbb{R}\times[0,\infty)$ is compact.

Then

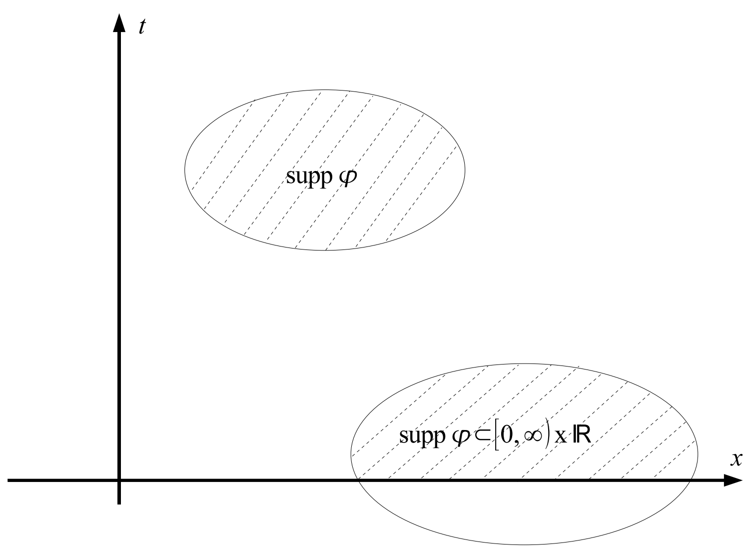
$$
\begin{aligned}
0 &= \int_0^\infty \int_{-\infty}^\infty (u_t(x,t) + (f(u))_x \varphi(x,t)\,\mathrm{dt}\,\mathrm{dx} \\
&= -\int_0^\infty \int_{-\infty}^\infty f(u)\varphi_x\,\mathrm{dt}\,\mathrm{dx} + \int_{-\infty}^\infty u(x,t)\varphi(x,t)\,\mathrm{dx}\,\Big|_{t=0}^{t=\infty} \\
&\quad - \int_0^\infty \int_{-\infty}^\infty u\varphi_t\,\mathrm{dx}\,\mathrm{dt} \\
&= -\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\,\mathrm{dx}\,\mathrm{dt} - \int_{-\infty}^\infty u_0(x)\varphi(x,0)\,\mathrm{dx}\,.
\end{aligned}
$$

The above calculation inspired the following definition of weak solution for (15).

Definition 12. $u \in L^\infty(\mathbb{R} \times (0,\infty))$ (u is bounded function up to a set of Lebesgue measure zero) is called *weak solution* of (15) if

$$
\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\,\mathrm{dx}\,\mathrm{dt} + \int_{-\infty}^\infty u_0(x)\varphi(x,0)\,\mathrm{dx} = 0,
$$

for every $\varphi \in C_0^1(\mathbb{R} \times [0,\infty))$



{s16} FIGURE 7. Supports of test functions in halfplane

Remark 2. (1) All classical solutions are also weak.
(2) If u is a weak solution, then u is also a distributive solution.

(3) If $u \in C^1(\mathbb{R} \times [0,\infty))$ is a weak solution, then it is a classical, too.

If we do not say differently, "solution" will mean weak solution from now on.

In a few steps we shall find necessary conditions for existence of piecewise differentiable weak solution to some conservation law.

{trh}

Theorem 5. *Necessary and sufficient condition that*

$$u(x,t) = \begin{cases} u_l(x,t), & x < \gamma(t),\ t \geq 0 \\ u_d(x,t), & x > \gamma(t),\ t \geq 0, \end{cases}$$

where u_l and u_d are C^1 solutions on their domains, be a weak solution to (15) is

$$\dot{\gamma} = \frac{f(u_d) - f(u_l)}{u_d - u_l} =: \frac{[f(u)]_\gamma}{[u]_\gamma}. \tag{16} \quad \{\texttt{rh}\}$$

Proof. The proof will be given in few steps.
1. Let

$$u(x,t) = \begin{cases} u_l(x,t), & x < \gamma(t),\ t \geq 0 \\ u_d(x,t), & x > \gamma(t),\ t \geq 0, \end{cases}$$

where u_l and u_d are defined above, be a weak solution to (15). Then

$$\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\,\mathrm{dx}\,\mathrm{dt} + \int_{-\infty}^\infty u(x,0)\varphi(x,0)\,\mathrm{dx} = 0,$$

for every $\varphi \in (\mathbb{R} \times [0,\infty))$.

Also $(u_l)_t + f(u_l)_x = 0$ for $x < \gamma(t)$ and $t > 0$ as well as $(u_d)_t + f(u_d)_x = 0$ for $x > \gamma(t)$ and $t > 0$.

That is consequence of the fact that

$$\begin{aligned} 0 &= \int\int u_l\varphi_t + f(u_l)\varphi_x\,\mathrm{dx}\,\mathrm{dt} \\ &= -\int\int (u_l)_t\varphi + (f(u_l))_x\varphi\,\mathrm{dx}\,\mathrm{dt}, \end{aligned}$$

for every φ, $\operatorname{supp}\varphi \subset \{(x,t) :\ x < \gamma(t),\ t > 0\}$ and C^1-function u_l. And since φ is arbitrary, we have

$$(u_l)_t + (f(u_l))_x = 0.$$

The same arguments hold for u_d, too.

2.

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\,\mathrm{dx}\,\mathrm{dt} + \int_{-\infty}^\infty u_0(x)\varphi(x,0)\,\mathrm{dx}\\
=&\int_0^\infty\int_{-\infty}^{\gamma(t)} (u_l\varphi_t + f(u_l)\varphi_x)\,\mathrm{dx}\,\mathrm{dt} + \int_0^\infty\int_{\gamma(t)}^\infty (u_d\varphi_t + f(u_d)\varphi_x)\,\mathrm{dx}\,\mathrm{dt}\\
&+\int_{-\infty}^\infty u_0(x)\varphi(x,0)\,\mathrm{dx}\,.\end{aligned}$$

3. Let us calculate the first integral from above. It holds

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{-\infty}^{\gamma(t)} u_l\varphi\,\mathrm{dx}\\
=&\dot\gamma(t)u_l(\gamma(t),t)\varphi(\gamma(t),t) + \int_{-\infty}^{\gamma(t)} ((u_l)_t\varphi + u_l\varphi_t)\,\mathrm{dx}\,.\end{aligned}$$

That implies

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^{\gamma(t)} u_l\varphi_t\,\mathrm{dx}\,\mathrm{dt} = -\int_0^\infty\int_{-\infty}^{\gamma(t)} (u_l)_t\varphi\,\mathrm{dx}\,\mathrm{dt}\\
&-\int_0^\infty \dot\gamma(t)u_l(\gamma(t),t)\varphi(\gamma(t),t)\,\mathrm{dt} + \int_0^\infty\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{-\infty}^{\gamma(t)} u_l\varphi\,\mathrm{dx}\,\mathrm{dt}\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^{\gamma(t)} f(u_l)\varphi_x\,\mathrm{dx}\,\mathrm{dt} = -\int_0^\infty\int_{-\infty}^{\gamma(t)} f(u_l)_x\varphi\,\mathrm{dx}\,\mathrm{dt}\\
&+\int_0^\infty f(u_l(\gamma(t),t))\varphi(\gamma(t),t))\,\mathrm{dt}\end{aligned}$$

Adding these terms and using the fact that u_l is a solution of PDE on the left-hand side of the curve $(\gamma(t),t)$, one gets the following

$$\int_0^\infty (f(u_l)-\dot\gamma u_l)\varphi\,\mathrm{dt} + \int_0^\infty \frac{d}{\mathrm{dt}}\int_{-\infty}^{\gamma(t)} u_l\varphi\,\mathrm{dx}\,\mathrm{dt}$$

as a value of that integral.

4. Analogously, concerning the right-hand side, one can see that the second integral equals

$$-\int_0^\infty (f(u_d)-\dot\gamma u_d)\varphi\,\mathrm{dt} + \int_0^\infty \frac{d}{\mathrm{dt}}\int_{\gamma(t)}^\infty u_d\varphi\,\mathrm{dx}\,\mathrm{dt}\,.$$

5. After adding all the above integrals one gets

$$
\begin{aligned}
0 = &\int_0^\infty (f(u_l) - f(u_d) - (u_l - u_d)\dot{\gamma})\varphi \,\mathrm{dt} \\
&+ \int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}} \int_{-\infty}^\infty u\varphi \,\mathrm{dx}\,\mathrm{dt} + \int_{-\infty}^\infty u_0(x)\varphi(x,0)\,\mathrm{dx},
\end{aligned}
$$

and

$$
\int_{-\infty}^\infty u(x,t)\varphi(x,t)\,\mathrm{dx}\,\Big|_{t=0}^{t=\infty} = -\int_{-\infty}^\infty u_0(x)\varphi(x,0)\,\mathrm{dx}\,.
$$

That is true if

$$
\dot{\gamma} = \frac{f(u_d) - f(u_l)}{u_d - u_l} =: \frac{[f(u)]_\gamma}{[u]_\gamma}.
$$

Obviously the above condition is sufficient. The proof is complete.

Condition (16) is called *Rankine-Hugoniot* (RH) condition.

Example 6. Consider the following Riemann problem

$$
\begin{aligned}
&u_t + \left(\frac{u^2}{2}\right)_x = 0 \\
&u_0 = \begin{cases} u_l \in \mathbb{R}, & x < 0 \\ u_d \in \mathbb{R}, & x > 0. \end{cases}
\end{aligned}
\tag{17}
$$

{pr1}

Since u_l and u_d are constants, there exist two trivial solutions of (17) out of the discontinuity curve, and RH-condition gives

$$
\dot{\gamma}(t) = \frac{u_d^2 - u_l^2}{2(u_d - u_l)} = \frac{u_d + u_l}{2},
$$

i.e. $\dot{\gamma}(t) = ct$, $c = \frac{u_l + u_d}{2}$ and (see Fig. 8)

$$
u(x,t) = \begin{cases} u_l, & x < ct \\ u_d, & x > ct, \end{cases}
\tag{18}
$$

{pr2}

If $u_l < u_d$, then except the above solution there exist also the following solutions (Fig. 9):

$$
u(x,t) = \begin{cases} u_l, & x < u_l t \\ \frac{x}{t}, & u_l t \le x \le u_d t \\ u_d, & x > u_d t \end{cases}
\tag{19}
$$

{pr3}

or, (Fig. 10))

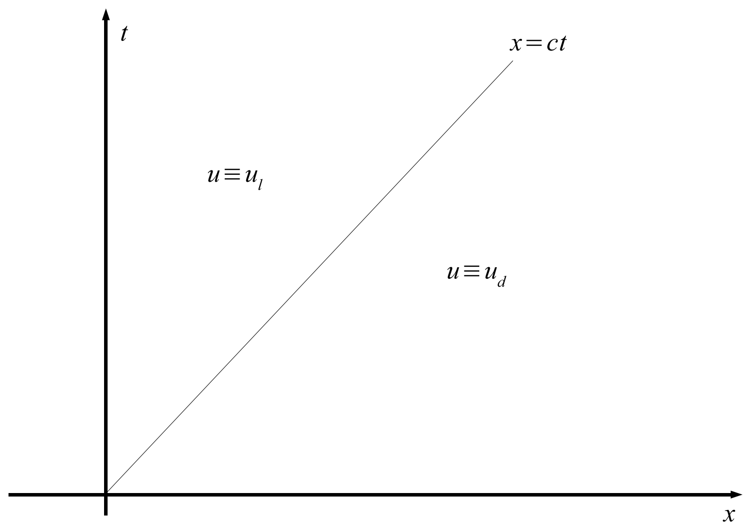


FIGURE 8. Shock wave {sl7}

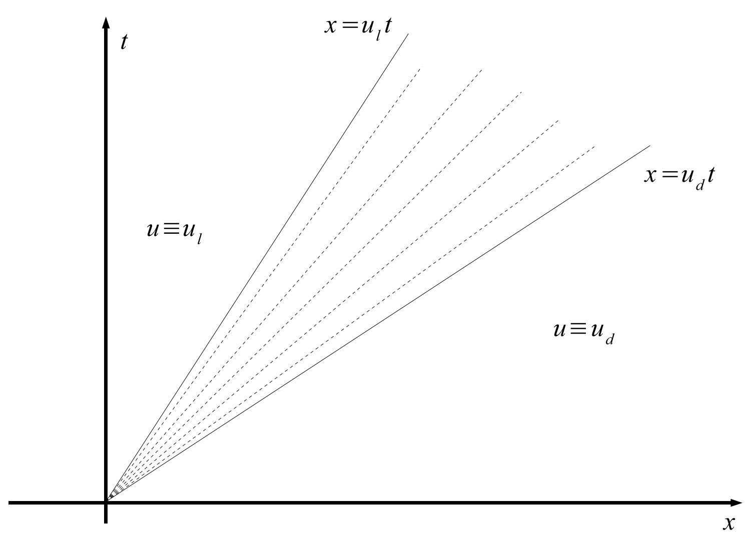


FIGURE 9. Rarefaction wave {sl8}

$$u(x,t) = \begin{cases} u_l, & x < u_l t \\ \frac{x}{f}, & u_l t \le x \le at \\ a, & at \le x \le \frac{a+u_d}{2} t \\ u_d, & x \ge \frac{a+u_d}{2} t, \end{cases} \tag{20}$$ {pr4}

for some $a \in (u_l, u_d)$.

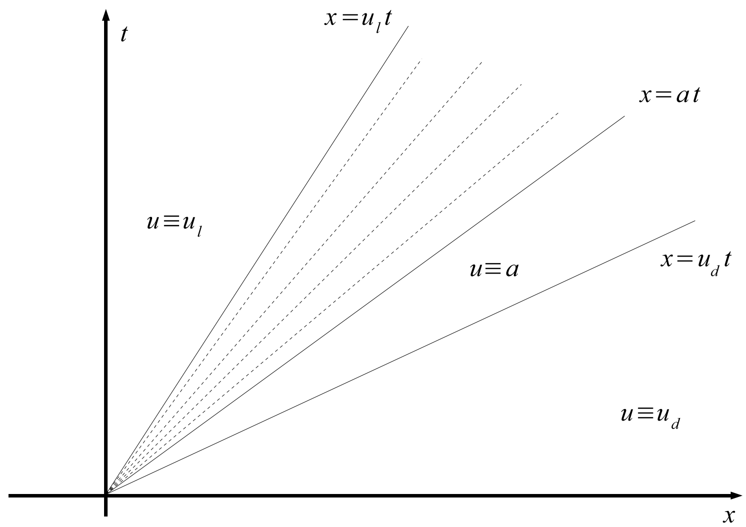


FIGURE 10. Non-entropic weak solution {s19}

One can see that there is no uniqueness of solution in the case $u_l < u_d$. That problem (finding admissible or so called "entropy" solutions) will be approached later on.

Example 7. Let us multiply partial differential equation(17) by u and transfer it into divergence form

$$\begin{aligned}
&u_t + uu_x = 0 \quad / \cdot u \\
&uu_t + u^2 u_x = 0 \\
&\left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3\right)_x = 0.
\end{aligned}$$

After nonlinear change of variables $\frac{1}{2}u^2 \mapsto v$, one gets the following conservation law

$$\begin{aligned}
&v_t + (\frac{2\sqrt{2}}{3}v^{3/2})_x = 0 \\
&v\Big|_{t=0} = \begin{cases} v_l = \frac{1}{2}u_l^2, & x < 0 \\ v_d = \frac{1}{2}u_d^2, & x > 0. \end{cases}
\end{aligned}$$

RH-conditions give the following speed of shock wave c and the discontinuity line is $\gamma = ct$:

$$\dot\gamma(t) = \frac{[\frac{3}{2}v^{3/2}]}{[v]} = \frac{\frac{2\sqrt{2}}{3}\frac{1}{2}(u_d^2)^{3/2} - \frac{2\sqrt{2}}{3}\frac{1}{2}(u_l^2)^{3/2}}{\frac{1}{2}(u_d^2 - u_l^2)}$$
$$= \frac{\frac{1}{3}(u_d^3 - u_l^3)}{\frac{1}{2}(u_d^2 - u_l^2)} \neq \frac{u_l + u_d}{2} \text{ in general.}$$

(For example, for $u_l = 1$, $u_d = 0$ one has $\frac{\frac{1}{3}}{\frac{1}{2}} \neq \frac{1}{2}$.)

This was an "unpleasant" example, because simple but nonlinear transformations of variables do not preserve solutions.

Because of that a precise interpretation of a physical model is of the crucial importance.

13. Rarefaction waves for single conservation law

Solution of equation (15) of the form $u(x,t) = \tilde u(\frac{x}{t})$ is called *self-similar solution*. Now we shall try to find such a solution of (15) in a simple way, just by substituting a function of this form into the equation. After the differentiation we have

$$-\frac{x}{t^2}\tilde u'\Big(\frac{x}{t}\Big) + f'\Big(\tilde u\Big(\frac{x}{t}\Big)\Big)\frac{1}{t}\tilde u'\Big(\frac{x}{t}\Big) = 0$$

after multiplication of the equation with

t and the substitution $\dfrac{x}{t} \mapsto y$ one gets ODE

$\tilde u'(y)(f'(\tilde u(y)) - y) = 0$

After neglecting constant, so called trivial solutions ($\tilde u' \neq 0$), one can see that solution is given by the implicit relation

$$f'(\tilde u) = y, \text{ ie. } \tilde u(y) = f'^{-1}(y),$$

if f' is bijection (locally).

One can interpret the initial data in the following way:

$$u(x,0) = \begin{cases} u_l, & x<0 \\ u_d, & x>0 \end{cases} \Longrightarrow \tilde u(+\infty) = u_d,\ \tilde u(-\infty) = u_l. \tag{21}$$

If $f'' > 0$ (f is convex), then f' is an increasing function and solution $\tilde u$ to the equation satisfying (21) exists if $u_l < u_d$. Such solution is called *centered rarefaction wave* (the initial data has a singularity at zero).

14. Linear hyperbolic systems

Homogeneous linear scalar Cauchy problem with constant coefficients

$$\begin{aligned} &u_t + \lambda u_x = 0 \\ &u(x,0) = \bar{u}(x),\ \lambda \in C(\mathbb{R}),\ \bar{u} \in C^1([0,\infty) \times \mathbb{R}) \end{aligned} \tag{22}$$

has a simple solution in a traveling wave form

$$u(x,t) = \bar{u}(x - \lambda t). \tag{23}$$

{ls2}

If $\bar{u} \in L^1_{loc}$, then the above function (23) is a weak solution to (22), what one can show easily.

Let a homogeneous system with constant coefficients

$$\begin{aligned} &u_t + Au_x = 0 \\ &u(x,0) = \bar{u}(x) \end{aligned} \tag{24}$$

{ls3}

be given, where A is $n \times n$ hyperbolic matrix with real characteristic values $\lambda_1 < \ldots < \lambda_n$ and left-hand sided l_i (resp. right-hand sided r_i), $i = 1, \ldots, n$, eigenvectors. They are chosen in a way that $l_i r_i = \delta_{ij}$, $i, j = 1, \ldots, n$. Denote by $u_i := l_i * u$ coordinates of the vector $u \in \mathbb{R}^n$ with respect to the base $\{r_1, \ldots, r_n\}$. Multiplying (24) from the left-hand side with l_i one gets

$$\begin{aligned} &(u_i)_t + \lambda_i (u_i)_x = (l_i u)_t + \lambda_i (l_i u)_x = l_i u_t + l_i A u_x = 0 \\ &u_i(x,0) = l_i \bar{u}(x) =: \bar{u}_1(x). \end{aligned}$$

So, (24) decouples into n scalar Cauchy problems, which can be solved like (22), one by one. Using (23) one can see that

$$u(x,t) = \sum_{i=1}^{n} \bar{u}_i(x - \lambda_i t) r_i \tag{25}$$

{ls4}

solution to (24) because

$$u_t(x,t) = \sum_{i=1}^{n} -\lambda_i (l_i \bar{u}_x(x - \lambda_i t)) r_i = -Au_x(x,t).$$

Thus, initial profile decouples into a sum of n waves with speeds $\lambda_1, \ldots, \lambda_n$.

As a spacial case, take Riemann problem

$$\bar{u}(x) = \begin{cases} u_l, & x < 0 \\ u_d, & x > 0. \end{cases}$$

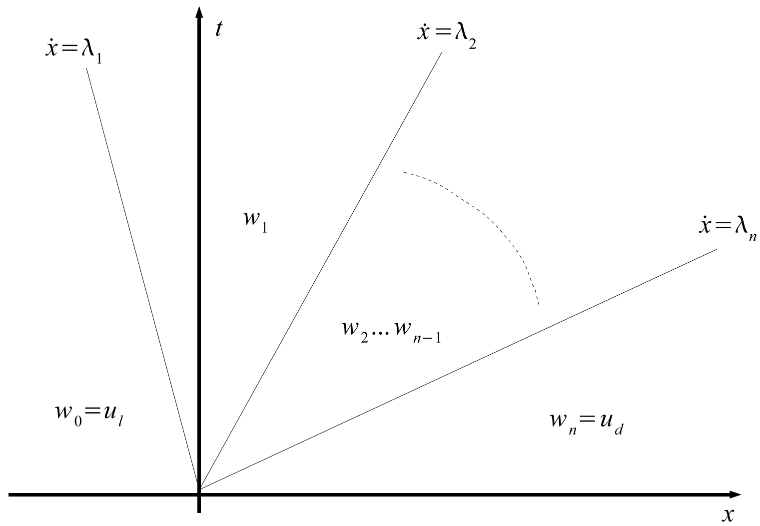


FIGURE 11. Waves and linear system {sl10}

Let us write down a solution to (25) using

$$u_d - u_l = \sum_{j=1}^{n} c_j r_j.$$

Let interstates be given by

$$w_i := u^l + \sum_{j \le i} c_j r_j, \; i = 0, \ldots, n,$$

such that $w_i - w_{i-1}$ is $(i-n)$-th characteristic vector of A. Solution is of the form (Fig. 11)

$$u(x,t) = \begin{cases} w_0 = u_l, & \frac{x}{t} < \lambda_1 \\ \ldots, & \\ w_i, & \lambda_i < \frac{x}{t} < \lambda_{i+1} \\ \ldots, & \\ w_n = u_d, & \frac{x}{t} > \lambda_n. \end{cases} \tag{26}$$

Part 4. Elementary waves for conservation laws

15. Basic definitions

One can find very usefull the class of functions with finite total variation, where

{totvar}

Definition 13. *Total variation* of a function v is defined by

(27) {totalna varijacija}
$$\mathrm{TV}(v) = \sup \sum_{j=1}^{N} |v(\xi_j) - v(\xi_{j-1})|,$$

where the supremum is taken by all partitions of the real line

$$-\infty = \xi_0 < \xi_1 < ... < \xi_N = \infty.$$

One can write 27 in the form

$$\mathrm{TV}(v) = \limsup_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |v(x) - v(x - \varepsilon)|.$$

Let

(28) {et1}
$$\begin{gathered} \frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x} f_1(u_1, \dots, u_n) = 0 \\ \vdots \\ \frac{\partial}{\partial t} u_n + \frac{\partial}{\partial x} f_n(u_1, \dots, u_n) = 0 \end{gathered}$$

be $n \times n$ one-dimensional conservation laws system, where

$$u = (u_1, \dots, u_n) \in \mathbb{R}^n, \ f = (f_1, \dots, f_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Denote by $A(u) := Df(u)$ Jacobi matrix of f at a point u. The above system reads (using vector notation)

(29) {et2}
$$u_t + f(u)_x = 0.$$

If a solution is smooth enough (C^1), then quasilinear form

(30) {et3}
$$u_t + A(u)u_x = 0$$

defines the equivalent system.

System is called *strictly hyperbolic* if all characteristic values of $A(u)$ are real and distinct. They are ordered in the following way

$$\lambda_1(u) < \lambda_2(u) < .... < \lambda_n(u).$$

If there exist n linearly independent characteristic vectors, the system is called hyperbolic.

Left-hand sided $l_1(u), ..., l_n(u)$ and right-hand sided $r_1(u), \dots, r_n(u)$ characteristics vectors are determined in a way that it holds

$$l_i(u) r_j(u) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

16. Elementary waves – an introduction

16.1. **Shock waves.** Like in the case of $n = 1$ we shall suppose that $x = \gamma(t)$ defines a discontinuity curve of piecewise smooth solutions $u_l(x,t)$ and $u_d(x,t)$, i.e.

$$u(x,t) = \begin{cases} u_l(x,t), & x < \gamma(t) \\ u_d(x,t). & x > \gamma(t) \end{cases}$$

In order that u defines a weak solution one has to find γ from Rankin-Hugoniot conditions for system

$$\dot{\gamma} \cdot (u_d - u_l) = f(u_d) - f(u_l). \tag{31}$$

Now, u_d, u_l, $f(u_d)$ and $f(u_l)$ are n-dim vectors. That means that a discontinuity curve $x = \gamma(t)$ can not be found in a direct way like in the case of a single equation. That is, it is not true that for each pair of constant initial vectors u_l, u_d there exists a shock wave solution (like in the case of a single equation).

Denote by

$$A(u,v) := \int A(\theta u + (1-\theta)v) d\theta$$

averaged matrix, where $\lambda_i(u,v)$, $i = 1, \ldots, n$, are its characteristic values. Then (31) can be written in the equivalent form

$$\dot{\gamma} \cdot (u_d - u_l) = f(u_d) - f(u_l) = A(u_d, u_l)(u_d - u_l). \tag{32}$$

In the other word, RH conditions hold if (u_d, u_l) is a characteristic vector of the averaged matrix $A(u_d, u_l)$, and speed $\dot{\gamma}$ equals its characteristic value.

16.2. **Rarefaction waves.** Let us find solutions of the form $u = u\left(\frac{x}{t}\right)$ (selfsimilar solutions) for system (30):

$$u_t + A(u)u_x = -\frac{x}{t^2} u'(y) + \frac{1}{t} A(u(y))u'(y) = 0,$$

where $y = \frac{x}{t}$. From the last equation it follows

$$A(u)u' = yu',$$

what means that u' is equal to the right-hand sided characteristic vector r_i and $y = \lambda_i$, for $i = 1, \ldots, n$.

17. Entropy conditions

As one could see, even for the case $n = 1$ there is a problem of uniqueness for weak solutions. In order to chose physically relevant solution we will use so called *entropy conditions*. The solution which satisfies it is called *admissible*.

17.1. **Entropy conditions 1 – vanishing viscosity.** A weak solution u to (28) is admissible if there exists a sequence of smooth solutions u_ε to

$$u_{\varepsilon t} + A(u_\varepsilon)u_{\varepsilon x} = \varepsilon u_{\varepsilon xx}$$

which converges to u in L^1 as $\varepsilon \to 0$.

17.2. **Entropy conditions 2 – entropy inequality.** C^1–function $\eta : \mathbb{R}^n \to \mathbb{R}$ is called *entropy* for system (28) with appropriate *entropy flux* $q : \mathbb{R}^n \to \mathbb{R}$ if

$$D\eta(u)Df(u) = Dq(u),\ u : \mathbb{R}^n \to \mathbb{R}^n. \tag{33}$$ {en1}

Note that (33) implies

$$(\eta(u))_t + (q(u))_x = 0,$$

for $u \in C^1$ as a solution to (28). When one substitutes $u_t = -Df(u)u_x$ into the above equation,

$$D\eta(u)u_t + Dq(u)u_x = D\eta(u)(-Df(u)u_x) + Dq(u)u_x = 0.$$

A weak solution u to (28) is admissible if

$$(\eta(u))_t + (q(u))_x \leq 0$$

in a distributional sense, i.e.

$$-\int \eta(u)\varphi_t + q(u)\varphi_x \geq 0,$$

for every $\varphi \geq 0$, $\varphi \in C_0^\infty(\mathbb{R} \times [0, \infty))$.

Thus,

$$D\eta(u)u_t + Dq(u)u_x = 0$$

outside a discontinuity, and

$$\dot{x}_\alpha(\eta(u(x_\alpha+)) - \eta(u(x_\alpha-))) \geq q(u(x_\lambda+)) - q(u(x_\alpha-))$$

on the discontinuity curve $x = \dot{x}_\alpha(t)$.

17.3. **Entropy condition 3 – Lax condition.** Shock wave connecting states u_l i u_d and has a speed $\dot{\gamma} = \lambda_i(u_l, u_d)$ is admissible if

$$\lambda_i(u_l) \geq \lambda_i(u_l, u_d) = \dot{\gamma} \geq \lambda_i(u_d). \tag{34}$$

(34) {en2}

Because of the ordering of characteristic values

$$\lambda_j(u_l) > \dot{\gamma},\ j > i$$
$$\lambda_j(u_d) < \dot{\gamma},\ j < i.$$

Such a wave is called i-th shock wave.

18. Rarefaction (RW) and shock wave (SW) curves

Fix $u_0 \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$. Integral curve for vector field r_i through u_0 is called *i-th rarefaction curve* (RW_i). One can get it explicitly by solving the Cauchy problem

$$\frac{du}{d\sigma} = r_i(u),\ u(0) = u_0. \tag{35}$$

That curve will be denoted by

$$\sigma \mapsto R_i(\sigma)(u_0).$$

Due to the above definition, u_0 can be joined with $u \in RW_i(u_0)$ by a single rarefaction wave.

Note that a curve parameterization depends on a choice of r_i. If $|r_i| \equiv 1$ then the curve is parametrizated by its length.

Fix $u_0 \in \mathbb{R}^n$ again. Let u be a right-hand state which can be joined to u_0 with i-th shock wave. (One uses RH conditions and Lax condition (34)). So, vector $u - u_0$ is a right-hand sided i-th characteristic vector for $A(u, u_0)$. By basic theorem of linear algebra this is true if and only if $u - u_0$ is orthogonal to l_j, $j \neq i$ (j-th left-hand sided characteristic vector for $A(u, u_0)$) i.e.

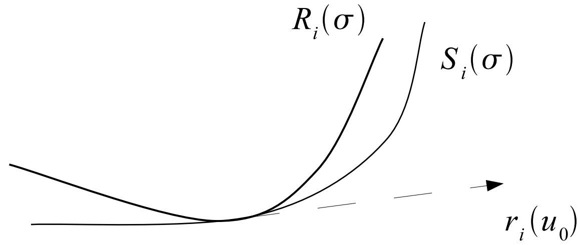
$$l_j(u, u_0)(u - u_0) = 0,\ \forall j \neq i,\ \dot{\gamma} = \lambda_i(u, u_0). \tag{36}$$

One can see that (36) is the system of $n - 1$ scalar equation with n unknowns (components of $u \in \mathbb{R}^n$). Linearizing (36) in a neighborhood of u_0 one gets linear system

$$l_j(u_0)(w - u_0) = 0,\ j \neq i.$$

It has a solution $w = u_0 + Cr_i(u_0)$, $C \in \mathbb{R}$. By Implicit Function Theorem, a set of solutions forms a regular curve (C^1–class) in a neighborhood of u_0 with a tangent vector r_i in the point u_0. That curve is called the *curve of i-th shock wave* and denoted by

$$\sigma \mapsto S_i(\sigma)(u_0).$$



{sl11} FIGURE 12. Shock wave and rarefaction curves

Both of the above curves exist in a neighborhood of u_0 (if f is smooth enough), and it can be proved that they have the same tangent in the point u_0 parallel to $r_i(u_0)$.

19. RIEMANN PROBLEM

{drip}

Definition 14. We say that i-th characteristic field is *genuinely nonlinear* if

$$D\lambda_i(u)r_i(u) \neq 0.$$

If

$$D\lambda_i(u)r_i(u) \equiv 0,$$

then i-th field is said to be *linearly degenerate*

Note that in the case when i-th field is genuinely nonlinear one can chose the orientation of r_i (by changing its sign, eventually) such that

$$D\lambda_i(u)r_i(u) > 0.$$

Let us suppose that the above relation holds in the sequel for genuinely nonlinear fields.

In the rest of the paper we shall use the following general assumption:

System (28) is strictly hyperbolic with smooth coefficients. For each $i \in \{1, \dots, n\}$, i-th characteristic field is either genuinely nonlinear or linearly degenerate.

19.1. **Centered rarefaction wave.** Let i-th field be genuinely nonlinear and suppose that u_d lies on a positive part of RW curve starting from u_l, i.e. $u_d = R(\sigma)(u_l)$ for some $\sigma > 0$.

Theorem 6. *Let us define*

$$\lambda_i(s) = \lambda_i(R_i(s)(u_l))$$

for every $s \in [0, \sigma]$.

Because of genuine nonlinearity, mapping $s \mapsto \lambda_i(s)$ is strictly increasing. Let $t \geq 0$. Function

$$u(x,t) = \begin{cases} u_l, & x < t\lambda_i(u_l) \\ R_i(s)(u_l), & x = t\lambda_i(s) \\ u_d = R_i(\sigma)(u_l), & x > t\lambda_i(u_d), \end{cases} \tag{37}$$

{crw1}

where $\frac{x}{t} = y = \lambda_i(s)$, $s \in [0, \sigma]$, is piecewise smooth solution to Riemann problem

$$u_t + f(u)_x = 0$$

$$u\Big|_{t=0} := u_0 = \begin{cases} u_l, & x < 0 \\ u_d, & x > 0. \end{cases}$$

Proof. One can easily see that

$$\lim_{t \to 0} \|u(x,t) - u_0\|_{L^1} = 0.$$

Besides that, (28) trivially holds true for $x < t\lambda_i(u_l)$ and $x > t\lambda_i(u_d)$, because $u_t = u_x = 0$. Suppose $x = t\lambda_i(s)$, for some $s \in (0, \sigma)$. Since $u \equiv \text{const}$ along each halfline $\{(x,t): \ x = t\lambda_i(s)\}$, there holds

$$u_t(x,t) + \lambda_i(s) u_x(x,t) = 0. \tag{38}$$

Since

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x} = \frac{dR_i(s)(u_l)}{ds} \\ &= \Big(\frac{d\lambda_i(s)}{ds}\Big)^{-1} \frac{d\lambda_i}{\mathrm{dx}} = r_i(u) \Big(\frac{d_i(s)}{ds}\Big)^{-1} \frac{1}{t}, \end{aligned}$$

u_x is a characteristic vector for $A(u)$ when $\lambda_i(s) = \lambda_i(u(t,x))$, i.e.

$$A(u) u_x = \lambda_i u_x.$$

Note that assumption $\sigma > 0$ is crucial for the above construction of a solution. If $\sigma < 0$, (37) would define a triple valued function in the area $\frac{x}{t} \in [\lambda_i(u_d), \lambda_i(u_l)]$.

19.2. **Shock waves.** Let i-th characteristic field be genuinely nonlinear and let u_d be connected with u_l by i-shock wave, $u_d = S_i(\sigma)(u_l)$. Then $\lambda := \lambda_i(u_d, u_l)$ is the speed of that wave and

$$u(x,t) = \begin{cases} u_l, & x < \lambda t \\ u_d, & x > \lambda t \end{cases} \tag{39}$$

is piecewise constant solution to the above Riemann problem.

Note that in the case $\sigma < 0$ that solution is admissible in the Lax-sense, because

$$\lambda_i(u_d) < \lambda_i(u_l, u_d) < \lambda_i(u_l).$$

For $\sigma > 0$, one would have

$$\lambda_i(u_l) < \lambda_i(u_d)$$

and Lax condition could not be satisfied.

19.3. **Contact discontinuities.** Suppose that i-th characteristic field is linearly degenerate and $u_d = R_i(\sigma)(u_l)$ for some σ. By the assumption, λ_i is constant along that curve. Putting $\lambda := \lambda_i(u_l)$, one can see that piecewise constant function given by (39) solves the above Cauchy problem, because RH condition is satisfied at discontinuity curve.

$$\begin{aligned} f(u_d) - f(u_l) &= \int_0^\sigma Df(R_i(s)(u_l))r_i(R_i(s)(u_l))\,\mathrm{ds} \\ &= \int_0^\sigma \lambda_i(R_i(s)(u_l))r_i(R_i(s)(u_l)\,\mathrm{ds} \\ &= \lambda_i(u_l)\int_0^\sigma \frac{\mathrm{d}\,R_i(s)(u_l)}{\mathrm{d}\,s}\,\mathrm{ds} = \lambda_i(u_l)(R_i(\sigma)(u_l) - u_l). \end{aligned}$$

We have used here that

$$\begin{aligned} \frac{d}{ds}\lambda_i(R_i(s)(u_l)) &= D\lambda_i(R_i(s)(u_l))\frac{\mathrm{d}\,R_i(s)(u_l))}{\mathrm{d}\,s} \\ &= (D\lambda_i r_i)\Big(R_i(s)(u_l)\Big) = 0, \end{aligned}$$

as well as the definition of linear degeneracy.

In that case Lax conditions hold thus regardless to the sign of σ, because

$$\lambda_i(u_d) = \lambda_i(u_l, u_d) = \lambda_i(u_l).$$

From the above calculations one can deduce that

$$R_i(\sigma)(u_0) = S_i(\sigma)(u_0),$$

for every σ.

19.4. **General solutions.** As we have seen before, the set of points $\{u_d : u \in \mathbb{R}^n\}$ which could be connected with a left-hand side state of Riemann problem is just a curve. In order to connect two arbitrary points u_l, $u_d \in \mathbb{R}^n$ with an entropic solution of Riemann problem one can insert at most $n-1$ vectors

$$u_l, u_1, u_2, \ldots, u_{n-1}, u_d$$

such that between each pair (u_l, u_1), (u_1, u_2), ..., (u_{n-1}, u_d) there is one of the previously described elementary waves.

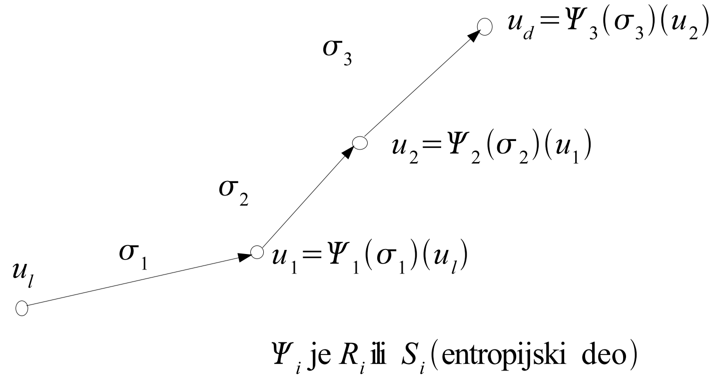


FIGURE 13. Sketch of a solution to Riemann problem {sl12}

If the initial condition belongs to L^∞, then we shell approximate it by piecewise constant function. So there are a lot of Riemann problems which have to be simultaneously solved. One by one solution in the form of elementary waves can be easily find, but the main problem is how to deal with a huge number of mutual wave interactions.

We shall describe two procedures for that purpose.

(1) **Glimm scheme**. Before the first interaction of the initial elementary waves, one approximates a solution with new piecewise constant function again. That function becomes a new initial data and procedure is repeated as many times as needed. Rarefaction wave is approximated by a fan of non-admissible shock waves in this procedure.

The procedure will converge for small enough variation of initial states, i.e. total variation of the initial data is small enough.

There are a lot of technical problems concerning the above scheme, so a lot of effort was given to find a new procedure, the following one.

(2) **Front-tracking method)**. Again, rarefaction wave is approximated with a fan of non-entropic shock waves. But now waves are permitted to interact. In a point of interaction there is a new Riemann problem. One can solve it accurately or approximatively. In the later case, one constructs non-physical shock wave with small amplitude, but with the larger speed of all possible waves in order to prevent blow-up effect.

After that one can again use the same method for later interactions.

Again, this procedure will converge when total variation of the initial data is small enough.

Part 5. Unbounded generalized solutions

Lately, a number of papers appeared dealing with non-classical weak solutions to conservation law systems. The main reason for that is to overcome common deficiencies in both of the ideas presented at the end of the previous part:

Total variation of initial data has to be sufficiently small.

All fields are either genuinely nonlinear or linearly degenerate.

We are dealing mostly with strictly hyperbolic systems.

Between different approaches, we choose the one using Colombeau generalized functions.

20. Generalized function spaces

We shall briefly repeat some definitions of Colombeau algebra given in [17]. Denote $\mathbb{R}^2_+ := \mathbb{R}\times(0,\infty)$, $\overline{\mathbb{R}^2_+} := \mathbb{R}\times[0,\infty)$ and let $C^\infty_b(\Omega)$ be the algebra of smooth functions on Ω bounded together with all their derivatives. Let $C^\infty_{\overline{b}}(\mathbb{R}^2_+)$ be a set of all functions $u\in C^\infty(\mathbb{R}^2_+)$ satisfying $u|_{\mathbb{R}\times(0,T)}\in C^\infty_b(\mathbb{R}\times(0,T))$ for every $T>0$. Let us remark that every element of $C^\infty_b(\mathbb{R}^2_+)$ has a smooth extension up to the line $\{t=0\}$, i.e. $C^\infty_b(\mathbb{R}^2_+) = C^\infty_b(\overline{\mathbb{R}^2_+})$. This is also true for $C^\infty_{\overline{b}}(\mathbb{R}^2_+)$.

Definition 15. $\mathcal{E}_{M,g}(\mathbb{R}^2_+)$ is the set of all maps $G:(0,1)\times\mathbb{R}^2_+\to\mathbb{R}$, $(\varepsilon,x,t)\mapsto G_\varepsilon(x,t)$, where for every $\varepsilon\in(0,1)$, $G_\varepsilon\in C^\infty_{\overline{b}}(\mathbb{R}^2_+)$ satisfies: {emn}
For every $(\alpha,\beta)\in\mathbb{N}^2_0$ and $T>0$, there exists $N\in\mathbb{N}$ such that

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |\partial^\alpha_x\partial^\beta_t G_\varepsilon(x,t)| = \mathcal{O}(\varepsilon^{-N}),\ \text{as } \varepsilon\to 0.$$

$\mathcal{E}_{M,g}(\mathbb{R}^2_+)$ is an multiplicative differential algebra, i.e. a ring of functions with the usual operations of addition and multiplication, and differentiation which satisfies Leibniz rule.

$\mathcal{N}_g(\mathbb{R}^2_+)$ is the set of all $G\in\mathcal{E}_{M,g}(\mathbb{R}^2_+)$, satisfying:
For every $(\alpha,\beta)\in\mathbb{N}^2_0$, $a\in\mathbb{R}$ and $T>0$

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |\partial^\alpha_x\partial^\beta_t G_\varepsilon(x,t)| = \mathcal{O}(\varepsilon^{a}),\ \text{as } \varepsilon\to 0.$$

□

Clearly, $\mathcal{N}_g(\mathbb{R}^2_+)$ is an ideal of the multiplicative differential algebra $\mathcal{E}_{M,g}(\mathbb{R}^2_+)$, i.e. if $G_\varepsilon \in \mathcal{N}_g(\mathbb{R}^2_+)$ and $H_\varepsilon \in \mathcal{E}_{M,g}(\mathbb{R}^2_+)$, then $G_\varepsilon H_\varepsilon \in \mathcal{N}_g(\mathbb{R}^2_+)$.

{g}

Definition 16. *The multiplicative differential algebra $\mathcal{G}_g(\mathbb{R}^2_+)$ of generalized functions is defined by $\mathcal{G}_g(\mathbb{R}^2_+) = \mathcal{E}_{M,g}(\mathbb{R}^2_+)/\mathcal{N}_g(\mathbb{R}^2_+)$. All operations in $\mathcal{G}_g(\mathbb{R}^2_+)$ are defined by the corresponding ones in $\mathcal{E}_{M,g}(\mathbb{R}^2_+)$.* □

If $C_b^\infty(\mathbb{R})$ is used instead of $C_b^\infty(\mathbb{R}^2_+)$ (i.e. drop the dependence on the t variable), then one obtains $\mathcal{E}_{M,g}(\mathbb{R})$, $\mathcal{N}_g(\mathbb{R})$, and consequently, the space of generalized functions on a real line, $\mathcal{G}_g(\mathbb{R})$.

In the sequel, G denotes an element (equivalence class) in $\mathcal{G}_g(\Omega)$ defined by its representative $G_\varepsilon \in \mathcal{E}_{M,g}(\Omega)$.

Since $C_{\overline{b}}^\infty(\mathbb{R}^2_+) = C_{\overline{b}}^\infty(\overline{\mathbb{R}^2_+})$, one can define a restriction of a generalized function to $\{t = 0\}$ in the following way.

For given $G \in \mathcal{G}_g(\mathbb{R}^2_+)$, its restriction $G|_{t=0} \in \mathcal{G}_g(\mathbb{R})$ is the class determined by a function $G_\varepsilon(x,0) \in \mathcal{E}_{M,g}(\mathbb{R})$. In the same way as above, $G(x-ct) \in \mathcal{G}_g(\mathbb{R})$ is defined by $G_\varepsilon(x-ct) \in \mathcal{E}_{M,g}(\mathbb{R})$.

If $G \in \mathcal{G}_g$ and $f \in C^\infty(\mathbb{R})$ is polynomially bounded together with all its derivatives, then one can easily show that the composition $f(G)$, defined by a representative $f(G_\varepsilon)$, $G \in \mathcal{G}_g$ makes sense. It means that $f(G_\varepsilon) \in \mathcal{E}_{M,g}$ if $G_\varepsilon \in \mathcal{E}_{M,g}$, and $f(G_\varepsilon) - f(H_\varepsilon) \in \mathcal{N}_g$ if $G_\varepsilon - H_\varepsilon \in \mathcal{N}_g$.

The equality in the space of the generalized functions $\mathcal{G}_g$ is to strong for our purpose (see [16]), so we need to define a weaker relation, so called, association.

{ass}

Definition 17. A generalized function $G \in \mathcal{G}_g(\Omega)$ is said to be *associated with* $u \in \mathcal{D}'(\Omega)$, $G \approx u$, if for some (and hence every) representative G_ε of G, $G_\varepsilon \to u$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0$. Two generalized functions G and H are said to be associated, $G \approx H$, if $G - H \approx 0$. The rate of convergence in $\mathcal{D}'$ with respect to ε is called the order of association. □

A generalized function G is said to be *of a bounded type* if

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |G_\varepsilon(x,t)| = \mathcal{O}(1) \text{ as } \varepsilon \to 0,$$

for every $T > 0$.

Let $u \in \mathcal{D}'_{L^\infty}(\mathbb{R})$. Let $\mathcal{A}_0$ be the set of all functions $\phi \in C_0^\infty(\mathbb{R})$ satisfying $\phi(x) \geq 0$, $x \in \mathbb{R}$, $\int \phi(x)dx = 1$ and $\operatorname{supp}\phi \subset [-1,1]$, i.e.

$$\mathcal{A}_0 = \{\phi \in C_0^\infty : \ (\forall x \in \mathbb{R})\phi(x) \geq 0, \ \int \phi(x)dx = 1, \ \operatorname{supp}\phi \subset [-1,1]\}.$$

Let $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$, $x \in \mathbb{R}$. Then

$$\iota_\phi : u \mapsto u * \phi_\varepsilon/\mathcal{N}_g,$$

where $u * \phi_\varepsilon/\mathcal{N}_g$ denotes the equivalence class with respect to the ideal $\mathcal{N}_g$, defines a mapping of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ into $\mathcal{G}_g(\mathbb{R})$, where $*$ denotes the usual convolution in $\mathcal{D}'$. It is clear that ι_ϕ commutes with the derivation, i.e.

$$\partial_x \iota_\phi(u) = \iota_\phi(\partial_x u).$$

21. Generalized solutions to conservation law systems

We shall restrict ourselves on a 2×2 system

$$\begin{aligned} &u_t + (f_1(u)v + f_2(u))_x = 0\\ &v_t + (g_1(u)v + g_2(u))_x = 0, \end{aligned} \tag{40}$$

{dss1}

with the Riemann initial data

$$u(x,0) = \begin{cases} u_0, & x<0\\ u_1, & x>0 \end{cases} \text{ and } v(x,0) = \begin{cases} v_0, & x<0\\ v_1, & x>0. \end{cases} \tag{41}$$

{dssid}

Before a construction of solutions, let us give definitions of its building blocks.

{s-d}

Definition 18. (a) $G \in \mathcal{G}_g(\mathbb{R})$ is said to be *a generalized step function* with value (y_0, y_1) if it is of bounded type and

$$G_\varepsilon(y) = \begin{cases} y_0, & y < -\varepsilon\\ y_1, & y > \varepsilon \end{cases}$$

Denote $[G] := y_1 - y_0$.

(b) $D \in \mathcal{G}_g(\mathbb{R})$ is said to be *generalized split delta function* (*Sδ-function*, for short) with value (α_0, α_1) if $D = \alpha_0 D^- + \alpha_1 D^+$, where $\alpha_0 + \alpha_1 = 1$ and $D^\pm \in \mathcal{G}_g(\mathbb{R})$ are given by the representatives

$$D_\varepsilon^\pm(y) := \frac{1}{\varepsilon}\phi\Big(\frac{y - (\pm 2\varepsilon)}{\varepsilon}\Big),\ \phi \in \mathcal{A}_0.$$

(c) Let m be an odd positive integer. A generalized function $d \in \mathcal{G}_g(\mathbb{R})$ is said to be m'*-singular delta function* (m'*SD-function*, for short) with value (β_0, β_1) if $d = \beta_0 d^- + \beta_1 d^+$, $\beta_0^{m-1} + \beta_1^{m-1} = 1$, $d^\pm \in \mathcal{G}_g(\mathbb{R})$, $\operatorname{supp} d_\varepsilon^- \subset (-\infty, -\varepsilon)$, $\operatorname{supp} d_\varepsilon^+ \subset (\varepsilon, \infty)$, $(d^\pm)^i \approx 0$, $i \in \{1, \ldots, m-2, m\}$, $(d^\pm)^{m-1} \approx \delta$.

(d) Let m be an positive integer. A generalized function $d \in \mathcal{G}_g(\mathbb{R})$ is said to be m*-singular delta function* (m*SD-function*, for short) with value (β_0, β_1) if $d = \beta_0 d^- + \beta_1 d^+$, $\beta_0^m + \beta_1^m = 1$, $d^\pm \in$

$\mathcal{G}_g(\mathbb{R})$, $\operatorname{supp} d_\varepsilon^- \subset (-\infty,-\varepsilon)$, $\operatorname{supp} d_\varepsilon^+ \subset (\varepsilon,\infty)$, $(d^\pm)^i \approx 0$, $i \in \{1,\dots,m-1\}$, $(d^\pm)^m \approx \delta$.

□

Let us note that $D_\varepsilon^\pm$ are in fact model delta nets (for this notion one can look in [16], for example) shifted by $\pm\varepsilon$.

In the sequel we shall suppose that one of the following is true:

$$\operatorname{supp} d_\varepsilon^+ \cap \operatorname{supp} D_\varepsilon^+ = \operatorname{supp} d_\varepsilon^- \cap \operatorname{supp} D_\varepsilon^- = \emptyset$$

(compatibility condition), or a assumption

$$D^+d^+ = D^-d^- = 0 \text{ (as it was done in [9]).}$$

{singsh}

Definition 19. *Singular shock wave* is an associated solution to (40) of the form

$$\begin{aligned} u(x,t) &= G(x-ct) + s_1(t)(\alpha_0 d^-(x-ct) + \alpha_1 d^+(x-ct)) \\ v(x,t) &= H(x-ct) + s_2(t)(\beta_0 D^-(x-ct) + \beta_1 D^+(x-ct)), \end{aligned} \tag{42}$$

where

(i) $c \in \mathbb{R}$ is the speed of the wave,
(ii) $s_i(t)$, $t \geq 0$ are smooth functions, $s_i(0) = 0$, $i = 1,2$.
(iii) G and H are generalized step functions with values (u_0,u_1) and (v_0,v_1) respectively,
(iv) $d = \alpha_0 d^- + \alpha_1 d^+$ is an m'SD-function or mSD-function,
(v) $D = \beta_0 D^- + \beta_1 D^+$ is an Sδ function compatible with d.

The wave is *overcompressive* if its speed is less or equal to the left- and greater or equal to the right-hand side of the characteristics i.e.

$$\lambda_2(u_0,v_0) > \lambda_1(u_0,v_0) \geq c \geq \lambda_2(u_1,v_1) > \lambda_1(u_1,v_1).$$

Delta shock wave is singular shock wave with $s_1 \equiv 0$.

22. Results on delta and singular shock existence

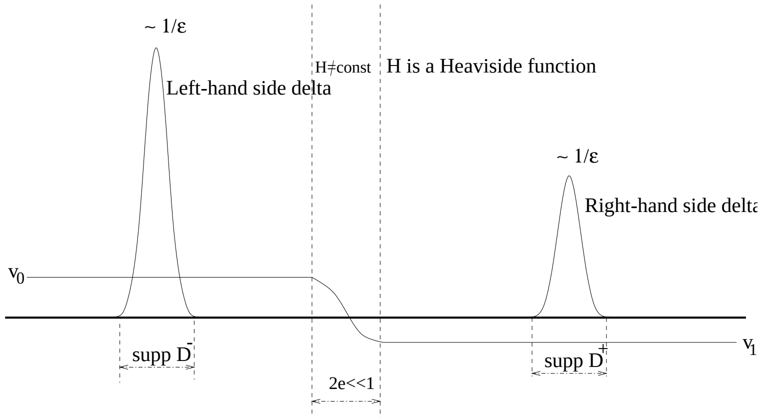
We shall begin this section with few assertion which helped us in actual construction of solutions.

{lema1}

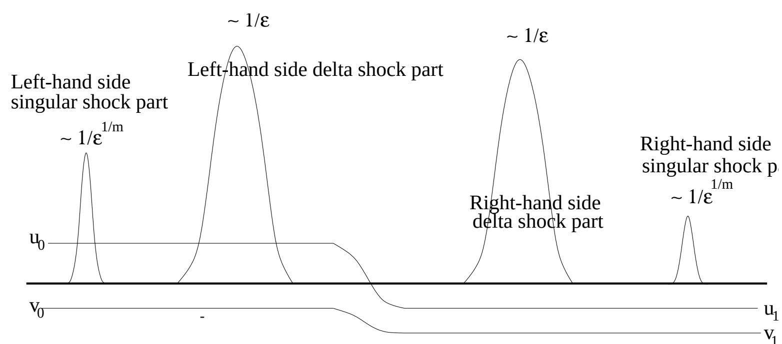
Lemma 2. *The generalized function defined by the representative $\phi_\varepsilon(x-ct) \in \mathcal{E}_{M,g}(\mathbb{R}^2_+)$, $\phi \in \mathcal{A}_0$, $c \in \mathbb{R}$, is associated with $\delta(x-ct) \in \mathcal{D}'(\mathbb{R}^2_+)$.*

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^\infty_+)$ and

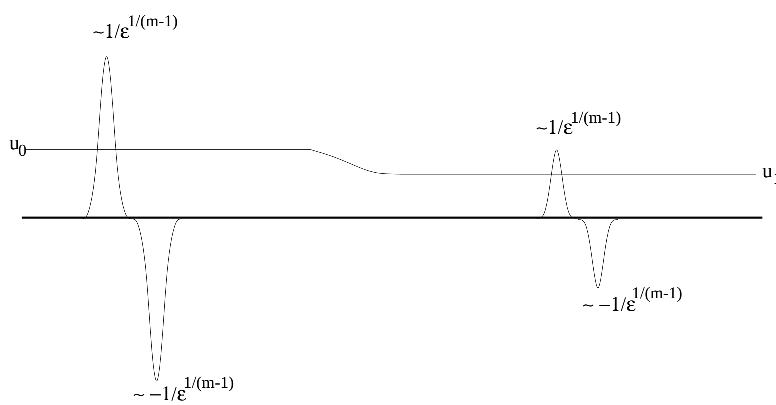
$$I_\varepsilon := \iint \varepsilon^{-1}\phi((x-ct)/\varepsilon)\psi(x,t)\,\mathrm{dx}\,\mathrm{dt}\,.$$



{dsw} FIGURE 14. Delta shock wave



{dssw} FIGURE 15. Singular shock wave with mSD-function



{dssw2} FIGURE 16. m'SD-function

Changing the variables $(x-ct)/\varepsilon \mapsto y$, $t \mapsto s$, using the Lebesgue dominated convergence theorem and the properties of the functions from $\mathcal{A}_0$ gives

$$I_\varepsilon = \iint \phi(y)\psi(\varepsilon y + cs, s)\,\mathrm{dy}\,\mathrm{ds}$$
$$\to \int \Big(\int \phi(y)dy\Big)\psi(cs,s)\,\mathrm{ds} = \int \psi(cs,s)\,\mathrm{ds}, \text{ as } \varepsilon \to 0.$$

{lema2}

Lemma 3. *If G is a generalized step function with value (y_0, y_1) and D is an $S\delta$-function with value (α_0, α_1), then the following hold.*

(i) *$f(G)$ is a generalized step function with value $(f(y_0), f(y_1))$, where f is a smooth function.*

(ii)
$$G \cdot D \approx (y_0\alpha_0 + y_1\alpha_1)\delta.$$

Proof. The proof is a straightforward consequence of the definitions.

22.1. **Singular shock wave solution.** In order to work more easy with system (40), we shall suppose that he fluxes f_1, f_2, g_1 and g_2 are polynomials depending only on u. Suppose that the maximal degree of all polynomials in the fluxes equals m. Let

$$f_1(y) = \sum_{i=0}^m a_{1,i}y^i,\; f_2(y) = \sum_{i=0}^m a_{2,i}y^i,\; g_1(y) = \sum_{i=0}^m b_{1,i}y^i,\; g_2(y) = \sum_{i=0}^m b_{2,i}y^i.$$

Before the final theorem we shall give examples how the building blocks for singular shock waves can be constructed.

One can construct mSD and m'SD functions in the following way by using powers of the model delta nets. Let $\phi \in \mathcal{A}_0$.

(i) Let m be an odd positive integer. Put

$$d_\varepsilon^+(y) = \left(\frac{1}{2}\phi\left(\frac{y-4\varepsilon}{\varepsilon}\right) - \phi\left(\frac{y-6\varepsilon}{\varepsilon}\right)\right)^{1/(m-1)},\ \text{for } y \geq 0,$$
$$d_\varepsilon^-(y) = -d_\varepsilon^+(-y),\ \text{for } y < 0.$$

Now $\beta_0 d_\varepsilon^- + \beta_1 d_\varepsilon^+$ is a possible represent for an mSD-function with value (β_0, β_1).

(ii) Put

$$d_\varepsilon^\pm(y) = \left(\phi\left(\frac{y-(\pm 4\varepsilon)}{\varepsilon}\right)\right)^{1/m}.$$

Then $\beta_0 d_\varepsilon^- + \beta_1 d_\varepsilon^+$ is a represent for an mSD-function with value (β_0, β_1).

The constructed mSD- and m'SD-functions are compatible with the Sδ-functions defined in the example after Definition 1.

Theorem 7. *Let $G(x-ct)$ and $H(x-ct)$ be generalized step functions with the speed c and values (u_0, u_1) and (v_0, v_1), respectively.*

Let

$$\begin{aligned}\alpha &= c(v_1 - v_0) - (g_1(u_1)v_1 + g_2(u_1) - g_1(u_0)v_0 - g_2(u_0)) \\ &= c[H] - [g_1(G)H + g_2(G)]\end{aligned}$$

to be Rankine-Hugoniot deficit (defined in [9]*).*

A singular shock wave solution to (dss1) exists if one of the following two assertions are true:
(i) There exists a solution $(\alpha_0, y_0) \in \mathbb{R}^2$ to the system

$$\begin{aligned}\alpha[f_1(G)]\alpha_0 + \sigma[H]a_{1,m}y_0 &= \sigma(v_1 a_{1,m} + a_{2,m}) + \alpha f_1(u_1) \\ \alpha[g_1(G)]\alpha_0 + \sigma[H]b_{1,m}y_0 &= \sigma(v_1 b_{1,m} + b_{2,m}) + \alpha(g_1(u_1) - c),\end{aligned}$$

for some $\sigma \in \mathbb{R} \setminus \{0\}$. If m is an even number, then σ also has to be positive and $y_0 \in [0,1]$.
(ii) m is an odd number and there exists a solution $(\alpha_0, y_0) \in \mathbb{R} \times \mathbb{R}_+$ to the system

$$\begin{aligned}&\alpha[f_1(G)]\alpha_0 + \sigma(a_{1,m-1}[G] + ma_{1,m}[GH] + ma_{2,m}[G])y_0 \\ =&\sigma(a_{1,m-1}v_1 + ma_{1,m}u_1v_1 + ma_{2,m}u_1 + a_{2,m-1}) + \alpha f_1(u_1) \\ &\alpha[g_1(G)]\alpha_0 + \sigma(b_{1,m-1}[G] + mb_{1,m}[GH] + mb_{2,m}[G])y_0 \\ =&\sigma(b_{1,m-1}v_1 + mb_{1,m}u_1v_1 + mb_{2,m}u_1 + b_{2,m-1}) + \alpha(g_1(u_1) - c)\end{aligned}$$

for some $\sigma \in \mathbb{R}_+$.

The speed of the singular shock waves is always given by

$$c = \frac{f_1(u_1)v_1 + f_2(u_1) - f_1(u_0)v_0 - f_2(u_0)}{u_1 - u_0} = \frac{[f_1(G)H + f_2(G)]}{[G]}$$

Part 6. **Introduction into numerical methods**

23. Conservative schemes

As one could see, weak solutions of conservation law systems are not unique in general and that produces a lot of numerical problems. But the situation for nonlinear problems could be even worse, as one can see in the following example.

Example 8. Take Burgers' equation

$$u_t + uu_x = 0,$$

with the initial data

$$U_j^0 = \begin{cases} 1, & j < 0 \\ 0, & j \geq 0. \end{cases}$$

One can take simple scheme for the above equation under the hypothesis $U_j^n \geq 0$, for every j, n:

$$U_j^{n+1} = U_j^n - \frac{k}{h} U_j^n (U_j^n - U_{j-1}^n).$$

That gives $U_j^1 = U_j^0$ for every j. Thus, $U_j^n = U_j^0$ for every j, n, and approximate solution converges to $u(x,t) = u_0(x)$, which is not even solution to the given equation.

Because of this one can use more appropriate procedure. One good class are so called *conservative procedures (schemes)*.

Definition 20. Numerical procedure is *conservative* if it can be written in the following form

(43)

$$U_j^{n+1} = U_j^n - \frac{k}{h}[F(U_{j-p}^n, U_{j-p+1}^n, ..., U_{j+q}^n) - F(U_{j-p-1}^n, U_{j-p}^n, ..., U_{j+q-1}^n)].$$

Function F is called *numerical flux function.*

In the simplest case, for $p = 0$ and $q = 1$, relation (43) is

$$U_j^{n+1} = U_j^n - \frac{k}{h}[F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)]. \tag{44}$$

Let U_j^n be an average value of u in $[x_{j-1/2}, x_{j+1/2}]$ defined by

$$\overline{u}_j^n = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n) \,\mathrm{dx}\,.$$

Since weak solution $u(x,t)$ satisfies the integral form of conservation law, we have

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_{n+1}) \,\mathrm{dx} = \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n) \,\mathrm{dx}$$

$$- [\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t)) \,\mathrm{dt} - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t)) \,\mathrm{dt}].$$

Dividing it by h gives

$$\overline{u}_j^{n+1} = \overline{u}_j^n - \frac{1}{h}[\int\limits_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))\,\mathrm{dt} - \int\limits_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))\,\mathrm{dt}].$$

One can see that

$$F(U_j, U_{j+1}) \sim \frac{1}{k}\int\limits_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))\,\mathrm{dt}\,.$$

For the simplicity of notation we shall use

$$F(U^n; j) = F(U^n_{j-p}, U^n_{j-p+1}, ..., U^n_{j+q}),$$

so (43) can be written in the form

$$U_j^{n+1} = U_j^n - \frac{k}{h}[F(U^n; j) - F(U^n, j-1)]. \tag{45}$$

{konzlaki}

Definition 21. Numerical procedure (44) is *consistent* with an original conservation law if for $u(x,t) \equiv \overline{u}$ it holds

$$F(\overline{u}, \overline{u}) = f(\overline{u}),$$

for every $\overline{u} \in \mathbb{R}$.

For the consistency one finds that F should be Lipschitz continuous with respect to all its variables.

In general, if F is a function of more than two variables, consistency condition reads

$$F(\overline{u}, \overline{u}, ..., \overline{u}) = f(\overline{u}),$$

and for Lipschitz condition there has to exist a constant K such that

$$|F(U_{j-p}, ..., U_{j+q}) - f(\overline{u})| \leq K \max_{-p \leq i \leq q} |U_{j+i} - \overline{u}|,$$

holds true for all U_{j+i} close enough to $\overline{u}$.

The following theorem is of the crucial importance for numerical solving of conservation law systems.

Theorem 8 (Lax-Wendorff)**.** *Let a sequence of schemes indexed by $l = 1, 2, \ldots$ with parameters $k_l, h_l \to 0$, as $l \to \infty$. Let $U_l(x,t)$ be a numeric approximation obtained by a consistent and conservative procedure at l-th scheme. Suppose that $U_l \to u$, as $l \to \infty$. Then, a function $u(x,t)$ is a weak solution to conservation law system.*

In order to prove that a weak solution $u(x,t)$, obtained by a conservative procedure, satisfy entropy condition, it is enough to prove that it satisfies so called *discrete entropy condition* (see [12])

(46) {diskretni uslov}
$$\eta(U_j^{n+1}) \leq \eta(U_j^n) - \frac{k}{h}[\Psi(U^n;j) - \Psi(U^n;j-1)], \tag{46}$$

where Ψ is appropriate *numerical entropy flux* consistent with a entropy flux ψ in the same sense as F with f is.

24. Godunov method

The basic idea of this procedure is the following: Numerical solution U^n is used for defining piecewise constant function $\widetilde{u}^n(x,t_n)$ which equals U_j^n in a cell $x_{j-1/2} < x < x_{j+1/2}$. Given function is not a constant in $t_n \leq t < t_{n+1}$. Because of that we use $\widetilde{u}^n(x,t_n)$ as an initial data for conservation law, which we analytically solve in order to get $\widetilde{u}^n(x,t)$ for $t_n \leq t \leq t_{n+1}$. After that we define the approximate solution U^{n+1} at time t_{n+1} as a mean value of the exact solution at time t_{n+1},

$$U_j^{n+1} = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \widetilde{u}^n(x,t_{n+1})\,\mathrm{dx}\,. \tag{47}$$

So, we have values for a piecewise constant function $\widetilde{u}^{n+1}(x,t_{n+1})$ and procedure continues. One can easily obtain (47) from integral form of the conservation law. Namely, since $\widetilde{u}$ is a weak solution of the conservation law, there holds

$$\begin{aligned}\int_{x_{j-1/2}}^{x_{j+1/2}} \widetilde{u}^n(x,t_{n+1})\,\mathrm{dx} = \int_{x_{j-1/2}}^{x_{j+1/2}} \widetilde{u}^n(x,t_n)\,\mathrm{dx} + \int_{t_n}^{t_{n+1}} f(\widetilde{u}^n(x_{j-1/2},t))\,\mathrm{dt}\\ - \int_{t_n}^{t_{n+1}} f(\widetilde{u}^n(x_{j+1/2},t))\,\mathrm{dt}\,.\end{aligned} \tag{48}$$

After division of the above expression by h, one uses (47) and the fact that $\widetilde{u}^n(x,t_n) \equiv U_j^n$ in the interval $(x_{j-1/2}, x_{j+1/2})$ to transform (48) to

$$U_j^{n+1} = U_j^n - \frac{k}{h}[F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)].$$

Here, the numerical flux function F is given by

$$F(U_j^n, U_{j+1}^n) = \frac{1}{k}\int_{t_n}^{t_{n+1}} f(\widetilde{u}^n(x_{j+1/2}, t))\,\mathrm{dt}\,. \tag{49}$$

That proves that Godunov procedure is conservative (it can be written in the form (44)). Additionally, calculation of integral 49 is very simple, since $\widetilde{u}^n$ in constant in (t_n, t_{n+1}) at the point $x_{j+1/2}$. That follows from the fact that a solution to Riemann problem is a constant along a characteristic curve

$$(x - x_{j+1/2})/t = \text{const}\,.$$

Since $\widetilde{u}^n$ depends only on U_j^n i U_{j+1}^n along the line $x = x_{j+1/2}$, $\widetilde{u}^n$ can be denoted by $u^*(U_j^n, U_{j+1}^n)$. Then numerical flux (49) becomes

(50) {fluksGodunov} $$F(U_j^n, U_{j+1}^n) = f(u^*(U_j^n, U_{j+1}^n)),$$

and Godunov procedure is now given by

$$U_j^{n+1} = U_j^n - \frac{k}{h}[f(u^*(U_j^n, U_{j+1}^n)) - f(u^*(U_{j-1}^n, U_j^n))].$$

Obviously, (50) is consistent with f because

$$U_j^n = U_{j+1}^n \equiv \overline{u}$$

implies

$$u^*(U_j^n, U_{j+1}^n) = \overline{u}.$$

Lipschitz continuity follows from smoothness of f.

But constancy of $\widetilde{u}^n$ in interval (t_n, t_{n+1}) at the point $x_{j+1/2}$ depends on a length of the interval. If a time interval is to long, then interaction of waves obtained by solving the closest Riemann problems may occur. Since speeds of these waves are bounded by characteristic values of the matrix $f'(u)$ and since sequential discontinuity points (origins of appropriate Riemann problems) are separated by h, $\widetilde{u}^n(x_{j+1/2}, t)$ is constant in the interval $[t_n, t_{n+1}]$ for k small enough. So, in order to avoid interactions, one introduces the condition

(51) {cfl} $$\left|\frac{k}{h}\lambda_p(U_j^n)\right| \leq 1,$$

for every λ_p and U_j^n.

Definition 22. The number

$$\text{CFL} = \max_{j,p}\left|\frac{k}{h}\lambda_p(U_j^n)\right|$$

is called *Courant number* or CFL (Courant-Friedrichs-Levy) for short. The condition

$$\text{CFL} \leq 1$$

is called *CFL condition.*

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