Shadow waves for pressureless gas balance laws

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Abstract

The procedure used for finding delta shock type solutions to some conservation laws known as Shadow Waves are now used for finding a solution to pressureless gas dynamics model with body force added as a source. The obtained solution in this paper resembles the one to pressureless gas dynamics model without a source. If the body force interpreted as the acceleration constant multiplied by the density the solution obtained here look physically reasonable since the velocities of waves are changed accordingly with that acceleration.

1 Introduction

The aim of this paper is to solve the pressureless gas dynamics model (PGD for short) with added the body force. That model can be derived from the well known isentropic gas dynamics model with added a force term on the right-hand side of momentum conservation law,

$$\begin{aligned} \partial_t\rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) &= b\rho \end{aligned}$$

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It is a model of gas dynamics in a gravitational field with entropy assumed to be a constant. The energy conservation law is now used as a selection criteria for admissible solutions: For all continuous solutions energy is conserved, while is should decrease for discontinuous ones. The body force source term is present if there is some external force acting on the fluid. The force assumed here is the gravity with b being the gravitational constant. By letting $p(\rho) \equiv 0$ we get the PGD conservation law system. It is known that admits a non-classical solution that contains the Dirac delta function (contrary the isentropic one). More precisely one can uniquely solve its Riemann problem only by using such singular solutions. There are a lot of nice, classical by now papers about the pressureless conservation laws system. One can look in [1] for definition of measure valued solutions, in [2] for sticky particles method, in [7] for variational method, in [6] for weak asymptotic method. All of them have detailed different methods of solving Riemann problem for PGD conservation laws system. One can look in [8] for a result about generalized pressureless system. In the paper [4] one can find a proof that passing from the isentropic to pressureless system by letting the pressure to vanish also transforms weak solution of one system to weak solutions of another one. The same was done for generalized pressureless system in [9].

Among a lot of different approaches in explaining such type of solutions, we will use the one from [10], so called shadow waves (SDW) in order to solve the balance law of pressureless gas with body force source term. Shadow waves are represented by nets of piecewise constant function for time variable t fixed parametrized by some small parameter $\varepsilon > 0$ and bounded in $L^1_{loc}(\mathbb{R})$. A use of such parameter enable us to include the Dirac delta function as a part of solution. A definition of a shadow wave is made to be as simple and robust as possible. Roughly speaking, we perturb a speed c of a wave from both sides by some small parameter ε so that left- and right-handed states are connected by a state that can be of order $1/\varepsilon$ is some components. The main advantage of their use is that one uses only Rankine-Hugoniot conditions for each ε. So we obtaining a net of classical weak solutions that satisfy the system in a distributional limit as $\varepsilon \to 0$. Also, the usual entropy inequality can be easily checked regardless of the form of entropy and entropy-flux functions. We shall use here a simpler condition - so called overcompressibility: All characteristics should run into the shock curve. Also, it is proved that entropy condition is not enough to exclude non-admissible waves for pressureless conservation law system in paper [7]. The next advantage is a simplicity of treating an interaction problem involving a shadow wave. We shall just give a comment at th end of the paper.

Our primary goal is to solve the following Riemann problem

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) &= b\rho, \end{aligned} \tag{1}$$

$$(\rho, u)(x, 0) = \begin{cases} (\rho_0, u_0), & x < 0 \\ (\rho_1, u_1), & x > 0. \end{cases}$$

It seems that the generalized pressure system can be treated in the same way. We left it for some future work.

2 Elementary waves

Let us first state some known fact about elementary waves of the given system. One can look in [3] or [5] for more details. Writing the system (1) into the evolutionary form by taking the new variable $m = \rho u$,

$$\begin{aligned} \partial_t \rho + \partial_x m &= 0 \\ \partial_t m + \partial_x \Big(\frac{m^2}{\rho} \Big) &= b\rho, \end{aligned}$$

one can easily see that it is a weakly hyperbolic with the double eigenvalue $\lambda_{1,2} = \frac{m}{\rho} = u$. Let us first look for a solution to (1) when initial data are constants, $(\rho(x, 0), u(x, 0)) = (\rho_0, u_0)$. For smooth solutions, one can substitute ρ_t from the first equation of (1) into the second one and eliminate ρ from it by division (provided that we are away from a vacuum state). So, we have now the following equation

$$\partial_t u + u \partial_x u = b,$$

that can be solved by a method of characteristics,

$$\frac{du}{dt} = 0, \ \frac{dx}{dt} = u, \ x(0) = x_0, \ u(0) = u_0.$$

A solution for constant initial data is given by

$$u = bt + u_0, \ x = x_0 + \frac{1}{2} b t^2 + u_0 t.$$

The first equation then becomes

$$\partial_t \rho + (bt + u_0) \partial_x \rho = 0$$

with a solution $\rho = \rho_0$ on each curve $x = x_0 + \frac{1}{2}bt^2 + u_0 t$. So, a "constant state" solution is given by

$$(\rho, u) = (\rho_0, bt + u_0).$$

It will be used in the rest of the paper.

Let us now look at a Riemann problem

$$(\rho, u)(x, 0) = \begin{cases} (\rho_0, u_0), & x < 0 \\ (\rho_1, u_1), & x > 0. \end{cases}$$

In the case $u_0 = u_1$ there is a contact discontinuity solution (CD) given by

$$(\rho, u)(x, t) = \begin{cases} (\rho_0, u_0 + bt), & x < \frac{1}{2}bt^2 + u_0 t \\ (\rho_1, u_0 + bt), & x > \frac{1}{2}bt^2 + u_0 t. \end{cases}$$

Also , one can see that the vacuum state is always a solution. Thus, in a general case when $u_0 < u_1$ we have a solution of the form CD+Vacuum+CD, two contact discontinuities connected by the vacuum:

$$(\rho, u)(x, t) = \begin{cases} (\rho_0, u_0 + bt), & x < \frac{1}{2}bt^2 + u_0 t \\ (0, u), & \frac{1}{2}bt^2 + u_0 t < x < \frac{1}{2}bt^2 + u_1 t \\ (\rho_1, u_1 + bt), & x > \frac{1}{2}bt^2 + u_1 t \end{cases} \tag{2}$$

where u is an arbitrary function satisfying

$$u(\frac{1}{2}bt^2 + u_0 t, t) = bt + u_0 t \text{ and } u(\frac{1}{2}bt^2 + u_1 t, t) = bt + u_1 t.$$

3 Shadow waves

In the case $u_0 > u_1$, there is not elementary wave solutions to the Riemann problem. One can try to substitute a SDW solution (see [10])

$$(\rho, u)(x, t) = \begin{cases} (\rho_0, u_0 + bt), & x < c(t) - \varepsilon t \\ (\rho_\varepsilon(t), u_\varepsilon(t)), & c(t) - \varepsilon t < x < c(t) + \varepsilon t \\ (\rho_1, u_1 + bt), & x > c(t) + \varepsilon t \end{cases} \tag{3}$$

in both equations of the system. The classical solution in the case $u_0 \leq u_1$ satisfies all the usual admissibility criteria (entropy inequalities). As an admissibility criteria for SDWs we will use the overcompressibility condition. That is the most frequent admissibility condition for all delta shock type nonstandard solutions of conservation law systems in the literature.

Definition 1. A shadow wave of the form (3) is called overcompressive if

$$\lambda_2(\rho_0, u_0+bt) \geq \lambda_1(\rho_0, u_0+bt) \geq c'(t) \geq \lambda_2(\rho_1, u_1+bt) \geq \lambda_1(\rho_1, u_1+bt), \quad (4)$$

i.e. all characteristics run into a shock. One can look in [2] or [7] for a detailed explanation of that admissibility condition.

Now we can formulate the following theorem.

Theorem 1. *The Riemann problem (1) has a unique solution in a set of elementary and shadow waves. If $u_0 \leq u_1$ a solution consists of two contact discontinuities connected with the vacuum state (2). In the case $u_0 > u_1$, there exists an overcompressive SDW solution of the form (3).*

Proof. The first, elementary waves case, $u_0 \leq u_1$ is explained in the previous section. Suppose $u_0 > u_1$ and substitute a function of the form (3) into system (1). For the first equation we have

$$\begin{aligned} I_1 := &-\int_0^\infty \int_{-\infty}^{c(t)-\varepsilon t} \rho_0 \partial_t \varphi(x,t) + \rho_0(u_0+bt)\partial_x \varphi(x,t) dxdt \\ &- \int_0^\infty \int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t} \rho_\varepsilon \partial_t \varphi(x,t) + \rho_\varepsilon u_\varepsilon \partial_x \varphi(x,t) dxdt \\ &- \int_0^\infty \int_{c(t)+\varepsilon t}^{\infty} \rho_1 \partial_t \varphi(x,t) + \rho_1(u_1+bt)\partial_x \varphi(x,t) dxdt = 0. \end{aligned}$$

where $\varphi \in C_0^\infty(\mathbb{R}^2)$. The first relation is obtained from δ terms and the other one is from δ' terms. Integration by parts gives

$$\begin{aligned} I_1 \approx &\int_0^\infty \rho_0(c'(t)-\varepsilon)\varphi(c(t)-\varepsilon t, t)dt \\ &+ \int_{-\infty}^0 \rho_0 \varphi(x,0)dx - \int_0^\infty \rho_0(u_0+bt)\varphi(c(t)-\varepsilon t,t)dt \\ &+ \int_0^\infty \rho_\varepsilon(t)\varphi(c(t)+\varepsilon t,t)(c'(t)+\varepsilon)dt - \int_0^\infty \rho_\varepsilon(t)\varphi(c(t)-\varepsilon t,t) \\ &\cdot (c'(t)-\varepsilon)dt + \int_0^\infty \int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t} \partial_t \rho_\varepsilon(t)\varphi(x,t)dxdt \\ &- \int_0^\infty \rho_\varepsilon(t)u_\varepsilon(t)(\varphi(c(t)+\varepsilon t,t) - \varphi(c(t)-\varepsilon t,t))dt \\ &- \int_0^\infty \rho_1(c'(t)+\varepsilon)\varphi(c(t)+\varepsilon t,t)dt + \int_0^\infty \rho_1\varphi(x,0)dx \\ &+ \int_0^\infty \rho_1(u_1+bt)\varphi(c(t)+\varepsilon t,t)dt. \end{aligned}$$

The sign "$\approx$" simply means a convergence to zero as $\varepsilon \to 0$. Note that

$$\int_{-\infty}^{0} \rho_0 \varphi(x,0)dx + \int_{0}^{\infty} \rho_1 \varphi(x,0)dx = \langle \rho|_{t=0}, \varphi \rangle$$

that cancels with the initial data and we will drop it in the rest of calculations. We will use the fact that

$$\varphi(c(t) \pm \varepsilon t, t) = \varphi(c(t), t) \pm \partial_x \varphi(c(t), t)\varepsilon t + \mathcal{O}(\varepsilon^2),$$

and

$$\rho_\varepsilon \sim \frac{1}{\varepsilon},\ u_\varepsilon \sim \text{const.}$$

Then (assuming that the initial conditions are satisfied) we get the following equation

$$\begin{aligned} &-\int_0^\infty ([\rho]c'(t) - [\rho(u+bt)] - 2(t\partial_t \rho_\varepsilon(t) + \rho_\varepsilon(t))\varepsilon \varphi(c(t),t)dt \\ &+ 2\int_0^\infty (\rho_\varepsilon(t)c'(t) - \rho_\varepsilon(t)u_\varepsilon(t))\varepsilon t \partial_x \varphi(c(t),t)dt \approx 0, \end{aligned}$$

where $[x] := x_1 - x_0$. Note that we have abused the usual notation since here $[\rho u]$ means $\rho_1 u_1 - \rho_0 u_0$ and not the real jump $\rho_1(u_1 + bt) - \rho_0(u_0 + bt)$, that is denoted by $[\rho(u+bt)]$. One could see that the above relation is true if and only if

$$\lim_{\varepsilon\to 0} 2\varepsilon(\rho_\varepsilon + t\partial_t \rho_\varepsilon) = k_1 := c'(t)[\rho] - [\rho(u+bt)] \tag{5}$$

$$\lim_{\varepsilon\to 0} \rho_\varepsilon(c'(t) - u_\varepsilon)\varepsilon = 0. \tag{6}$$

One sees immediately that

$$u_s(t) := \lim_{\varepsilon\to 0} u_\varepsilon(t) = c'(t)$$

Using the notation $\xi = \xi(t) := \lim_{\varepsilon\to 0} 2\varepsilon\rho_\varepsilon$ equation (5) becomes

$$t\xi'(t) + \xi(t) = k_1(t) = [\rho]c'(t) - [\rho u] - b[\rho]t \tag{7}$$

with $\xi(0) = 0$ because we do not have a delta function in the initial data. With the same method, and with the substitution

$$\rho_\varepsilon \to \rho_\varepsilon u_\varepsilon,\ \rho_\varepsilon u_\varepsilon \to \rho_\varepsilon u_\varepsilon^2.$$

from the second equation we have

$$\begin{aligned}
&\int_0^\infty \rho_0(u_0+bt)(c'(t)-\varepsilon)\varphi(c(t)-\varepsilon t,t)dt\\
&+\int_0^\infty\int_{-\infty}^{c(t)-\varepsilon t}\rho_0 b\varphi(x,t)dxdt\\
&+\int_{-\infty}^0 \rho_0u_0\varphi(x,0)dx-\int_0^\infty \rho_0(u_0+bt)^2\varphi(c(t)-\varepsilon t,t)dt\\
&+\int_0^\infty \rho_\varepsilon(t)u_\varepsilon(t)\varphi(c(t)+\varepsilon t,t)(c'(t)+\varepsilon)dt\\
&-\int_0^\infty \rho_\varepsilon(t)u_\varepsilon(t)\varphi(c(t)-\varepsilon t,t)(c'(t)-\varepsilon)dt\\
&+\int_0^\infty\int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t}\partial_t(\rho_\varepsilon(t)u_\varepsilon(t))\varphi(x,t)dxdt\\
&-\int_0^\infty \rho_\varepsilon(t)u_\varepsilon^2(t)(\varphi(c(t)+\varepsilon t,t)-\varphi(c(t)-\varepsilon t,t))dt\\
&-\int_0^\infty \rho_1(u_1+bt)(c'(t)+\varepsilon)\varphi(c(t)+\varepsilon t,t)dt\\
&+\int_0^\infty\int_{c(t)-\varepsilon t}^{\infty}\rho_1 b\varphi(x,t)dxdt\\
&+\int_0^\infty \rho_1u_1\varphi(x,0)dx+\int_0^\infty \rho_1(u_1+bt)^2\varphi(c(t)+\varepsilon t,t)dt\\
&-b\int_0^\infty\Big(\int_{-\infty}^{c(t)-\varepsilon t}\rho_0\varphi(x,t)dx+\int_{c(t)-\varepsilon t}^{c(t)+\varepsilon t}\rho_\varepsilon\varphi(x,t)dx.\\
&.+\int_{c(t)+\varepsilon t}^\infty \rho_1\varphi(x,t)dx\Big)dt\approx 0.
\end{aligned}$$

Like in the previous case, one can see that the above relation holds if the following relations are satisfied,

$$\begin{aligned}
&t(\xi(t)u_s(t))'+\xi(t)u_s(t)-bt\xi(t)\\
=&c'(t)(\rho_1(u_1+bt)-\rho_0(u_0+bt))-(\rho_1(u_1+bt)^2-\rho_0(u_0+bt)^2)=:k_2(t)
\end{aligned}\tag{8}$$

$$\xi(t)u_s(t)c'(t)=\xi(t)u_s(t)^2. \tag{9}$$

One can see that (6) and (9) are equivalent and satisfied if and only if $c'(t)=u_s(t)$.

Next, let us write (7) in the following form:

$$(t\xi(t))'=\Big(c(t)-\frac{b}{2}t^2\Big)'[\rho]-[\rho u].$$

Then

$$t\xi(t) = (c(t) - \frac{b}{2}t^2)[\rho] - [\rho u]t,$$

due to the initial data. Its substitution into (8) gives:

$$\begin{aligned}
&\Big(\big((c(t) - \frac{b}{2}t^2)[\rho] - [\rho u]t\big)(c'(t) - bt)\Big)' \\
&+ \underbrace{\Big(\big((c(t) - \frac{b}{2}t^2)[\rho] - [\rho u]t\big)bt\Big)'}_{=\Big((c'(t)-bt)[\rho]-[\rho u]\Big)bt+\Big((c(t)-\frac{b}{2}t^2)[\rho]-[\rho u]t\Big)b} \quad -b\Big(\big(c(t) - \frac{b}{2}t^2\big)[\rho] - [\rho u]t\Big) \\
=&(c'(t) - bt)([\rho u] + b[\rho]t) + bt([\rho u] + b[\rho]t) - [\rho u^2] - 2b[\rho u]t - b^2[\rho]t^2
\end{aligned}$$

With the change of variables $s(t) = c(t) - \frac{b}{2}t^2$ we have the equation

$$((s(t)[\rho] - [\rho u]t)s'(t))' - s'(t)[\rho u] + [\rho u^2] = 0$$

that can be integrated again, so we get

$$(s(t)[\rho] - [\rho u]t)s'(t) - s(t)[\rho u] + [\rho u^2]t = \text{const} = 0,$$

using $s(0) = 0$.

The above equation can be written as

$$\frac{1}{2}[\rho](s(t)^2)' - [\rho u](ts(t))' + [\rho u^2]t = 0,$$

and integrated, so

$$\frac{1}{2}[\rho]s(t)^2 - [\rho u]ts(t) + \frac{1}{2}[\rho u^2]t^2 = 0, \tag{10}$$

where we again have used that $s(0) = 0$. Suppose that $\rho_0 \neq \rho_1$. Thus, one can find an explicit formula for s

$$s(t) = \frac{[\rho u]t \pm t\sqrt{([\rho u]^2 - [\rho][\rho u^2])}}{[\rho]}.$$

Then

$$\begin{aligned}
c(t) &= \frac{[\rho u]t \pm t\sqrt{([\rho u]^2 - [\rho][\rho u^2])}}{[\rho]} + \frac{b}{2}t^2 \text{and} \\
c'(t) &= bt + \frac{\rho_1 u_1 - \rho_0 u_0 \pm (u_0 - u_1)\sqrt{\rho_0\rho_1}}{\rho_1 - \rho_0}
\end{aligned}$$

We have to find which sigh one has to use such that the obtained SDW satisfies the overcompressibility condition: $u_0 + bt \geq c'(t) \geq u_1 + bt$. It will suffice to prove that

$$u_0 \geq c'(0) = \frac{[\rho u] \pm \sqrt{[\rho u]^2 - [\rho][\rho u^2]}}{[\rho]} \geq u_1,$$

since $c'(t) = c(0) + bt$. Thus,

$$c'(0) = \frac{u_0(\sqrt{\rho_0\rho_1} - \rho_0) \pm u_1(\rho_1 - \sqrt{\rho_0\rho_1})}{\rho_1 - \rho_0} = \frac{\sqrt{\rho_0\rho_1} - \rho_0}{\rho_1 - \rho_0} u_0 \pm \frac{\rho_1 - \sqrt{\rho_0\rho_1}}{\rho_1 - \rho_0} u_1,$$

i.e. $c'(0) = \alpha u_0 \pm \beta u_1$,with $\alpha + \beta = 1$. One can check that in both cases $\rho_0 < \rho_1$ or $\rho_0 > \rho_1$, we have $\alpha, \beta \geq 0$. That implies

$$u_0 \geq \alpha u_0 + \beta u_1 = c'(0) \geq u_1$$

if one uses the plus sign above. Thus, if $u_0 > u_1$ the weak solution (3) to (1) is always admissible. It is unique with respect to a limit in the distributional sense. One can easily see that there are no unwanted SDWs in the case $u_0 \leq u_1$ since it contradicts (4).

Let us now check the case $\rho_0 = \rho_1$. Then (10) reduces to

$$s(t) = \frac{1}{2}(u_0 + u_1)t \text{ and } c(t) = s'(t) + bt = \frac{1}{2}(u_0 + u_1) + bt.$$

Then the solution is always overcompressive since $u_0 > \frac{1}{2}(u_0 + u_1) > u_1$. That concludes the proof. □

Remark 1. One could say that we have proved that shadow waves follow the physical intuitions as well as all other elementary waves in the given balance law system: If b denotes the gravity acceleration, an SDW speed is increased exactly by bt (or decreased if $b < 0$) as expected.

Remark 2. In the above proof, we have exploited a special form of pressureless system. In general, system (7,8) is a singular ODE system, since the second equation is of the form $\xi(t)u_s'(t) = ...$ with the initial data $\xi(0) = 0$ end the usual existence-uniqueness theorems are not applicable immediately.

4 Further possibilities

Suppose that an interaction involving a split delta shock happens at a time $t = T$ in a point $x = X$. Then one have to solve a new initial data that contains a delta function, say

$$(\rho,u)|_{t=T} = \begin{cases} (\rho_0,u_0), & x-X<0 \\ (\rho_1,u_1), & x-X>0 \end{cases} + \gamma_0\delta(X,T).$$

Note that (ρ_i,u_i). $i=0,1$ are not necessary the initial values for the above Riemann problem. Solution of any Riemann problem fond above has values of (ρ,u) that depends only on t, so (ρ_i,u_i). $i=0,1$ are obtained by freezing $t=T$.

We will try to find a solution to (1) and the above initial data in the form of SDW,

$$(\rho,u)|_{t=T} = \begin{cases} (\rho_0,u_0+b(t-T)), & x-X<c(t)-\varepsilon(t-T)-x_0\varepsilon \\ (\rho_\varepsilon(t),u_\varepsilon(t)), & c(t)-\varepsilon(t-T)-x_0\varepsilon<x-X \\ & <c(t)+\varepsilon(t-T)+x_0\varepsilon \\ (\rho_1,u_1+b(t-T)), & c(t)+\varepsilon(t-T)+x_0\varepsilon<x-X \end{cases}$$

Due to Theorem 7.1 in [10] about infraction of SDW's one can see that the value of x_0 has to be chosen in a way that one has a continuity of delta function across the interaction line $t=T$. We shall see that bellow.

Using the same change of variables and arguments as in the Riemann case, we get the same equations (7) and (8),

$$\begin{aligned} t\xi'(t)+\xi(t) &= [\rho]u_s'(t)-[\rho u]-b[\rho]t \\ t(\xi(t)u_s(t))'-b\xi(t) &= u_s(t)([\rho u](1-2bt)+bt[\rho](1-bt)-[\rho u^2]) \end{aligned}$$

but now with the initial data

$$\xi(T)=2x_0\varepsilon\rho_\varepsilon=\gamma_0>0,\ u_s(T)=\zeta_0.$$

The values for initial data are chosen in order to preserve mass of delta functions before and rather the interaction (see Theorem 7.1 in [10]), Thus, if there is one incoming SDW with $\overline{\xi}(t),\overline{u}_s(t)$ determined from appropriate equations (5–9), then $\gamma_0=\overline{\xi}(T),\zeta_0=\overline{u}_s(T)$. If there are two of them, with $\overline{\xi}_1(t),\overline{u}_{s,1}(t)$ and $\overline{\xi}_2(t),\overline{u}_{s,2}(t)$ determined, then $\gamma_0=\overline{\xi}_0(T)+\overline{\xi}_1(T)$ and ζ_0 can be found from the relation

$$\zeta_0\gamma_0=\overline{u}_{s,1}(T)\overline{\xi}_1(T)+\overline{u}_{s,2}(T)\overline{\xi}_2(T).$$

Concerning a solution to an interaction problem, it can be solved like in [10]. One just has to check overcompressibility conditions once when a solution to the above problem is found. Note that the system is not singular (the initial data are not given at zero anymore):

$$
\begin{aligned}
\xi'(t) =& \frac{[\rho]u_s(t) - b[\rho]t - [\rho u]}{t\xi(t)}, \ \xi(T) = \gamma_0 \\
u_s'(t) =& \frac{1}{t\xi^2(t)}((u_s(t) - b)\xi^2(t) + (b^2[\rho]t^2 + b[\rho]t + [\rho u^2] - [\rho u])u_s(t)\xi(t) \\
& + [\rho]u_s^2(t) - (b[\rho]t + [\rho u])u_s(t)), \ u_s(T) = \zeta_0.
\end{aligned}
$$

Contrary to the Riemann case, we do not have to use manipulation using special properties of (7) and (8). Now, at least in some small enough time interval after $t > T$, the above initial data problem always has a solution due to the usual existence-uniqueness theorems for ordinary differential equations (Picard-Lindelöf Theorem, for example). That is possible since $\xi(t) > 0$, at least for some small time interval $t > T$ since $\gamma_0 > 0$,

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