DELTA AND SINGULAR DELTA LOCUS FOR ONE DIMENSIONAL SYSTEMS OF CONSERVATION LAWS

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This work gives a condition for existence of singular and delta shock wave solutions to Riemann problem for 2×2 systems of conservation laws. For a fixed left-hand side value of Riemann data, the condition obtained in the paper describes a set of possible right-hand side values. The procedure is similar to the standard one of finding the Hugoniot locus. Fluxes of the considered systems are globally Lipschitz with respect to one of the dependent variables. The association in a Colombeau-type algebra is used as a solution concept.

1. Introduction

The aim of this paper is to give a criterion for existence of, so called, delta and singular shock wave solutions to the Riemann problem

$$u_t + (f_1(u)v + f_2(u,v))_x = 0 \tag{1}$$

$$v_t + (g_1(u)v + g_2(u,v))_x = 0. \tag{2}$$

$$u(x,0) = \Theta_1(x) = \begin{cases} u_0, \, x < 0 \\ u_1, \, x > 0, \end{cases} \quad v(x,0) = \Theta_2(x) = \begin{cases} v_0, \, x < 0 \\ v_1, \, x > 0. \end{cases} \tag{3}$$

Here, f_i, g_i, $i = 1, 2$ are smooth functions, polynomially bounded together with all their derivatives, and f_2 and g_2 are sublinear with respect to v.

In order to have a well defined composition of functions, we use Colombeau-type generalized functions and solution concept defined in [16] (one can look in [1] or [15] for a general description of such spaces). In short, a generalized function is an object obtained by a factorization in an algebra of smooth function nets over an appropriate ideal. The representatives will be denoted by U_ε, where the letter ε denotes an element of the index set $I = (0,1)$.

Delta and singular shock waves are represented by pairs of nets $(U_\varepsilon, V_\varepsilon)$, converging to linear combinations of Dirac delta and step functions in $\mathcal{D}'$. Singular shock wave contains generalized functions which are zeros in the sense of distributions, but some of their powers converge to delta functions. Let us remark that such objects have some similarities to infinite narrow solitons which are introduced by Maslov and Omel'anov [13]. But all powers of a infinite narrow soliton are zero distributionally. Further results in this direction for systems of conservation laws can be found in paper [3] of Danilov, Maslov and Shelkovich.

For a given point $(u_0, v_0) \in \mathbb{R}^2$, the delta (singular delta) locus for (1-2) is a set of points $(u_1, v_1) \in \mathbb{R}^2$ for which there exists a delta (singular) shock wave connecting the Riemann data (u_0, v_0) and (u_1, v_1), i.e. (3) is satisfied. Thus, this definition is analogous to the one of the classic Hugoniot locus, in the case of shock waves.

If system (1-2) is a hyperbolic one (the hyperbolicity condition is not used in the construction of the solutions in this paper), admissibility condition for the delta or singular shock waves is usually taken to be

$$\lambda_2(u_0, v_0) > \lambda_1(u_0, v_0) \geq c \geq \lambda_2(u_1, v_1) > \lambda_1(u_1, v_1),$$

where c is a speed of the delta or singular shock wave, λ_1, λ_2 are the eigenvectors for the system, (u_0, v_0) and (u_1, v_1) denotes left- and right-hand side initial data, respectively. The waves which satisfy the above condition are said to be overcompressive (see [9], [17], [4] and [6]). Like in the classical case with Hugoniot locus, the delta singular locus can be used for construction of solutions containing also a rarefaction wave on the left or on the right-hand side of a singular or delta shock wave. But, there is a difference in using a delta locus for hyperbolic systems. Except in some particular systems, it is not possible to connect every delta shock wave with an another elementary wave with the above admissibility criterion.

Our investigations are motivated by Keyfitz and Kranzer ([9]) who found singular shock wave solutions to the system

$$\begin{aligned} &u_t + (u^2 - v)_x = 0 \\ &v_t + \left(\frac{1}{3}u^3 - u\right)_x = 0, \end{aligned} \tag{4}$$

in the form

$$u_\varepsilon(x,t) = G_\varepsilon(x-ct) + a\sqrt{\frac{t}{\varepsilon}}\rho\left(\frac{x-ct}{\varepsilon}\right),\ v_\varepsilon(x,t) = H_\varepsilon(x-ct) + \frac{a^2t}{\varepsilon}\rho^2\left(\frac{x-ct}{\varepsilon}\right),$$

where G_ε and H_ε converge to appropriate step functions defined by the Riemann initial data (3), $\rho_\varepsilon^2(\cdot) := \varepsilon^{-1}\rho^2(\cdot/\varepsilon)$, where $\rho \in C_0^\infty$, $\int \rho = 1$, converges to the delta distribution and ρ_ε^i converges to zero in $\mathcal{D}'$ as $\varepsilon \to 0$, $i = 1, 3$. System (4) is a special case of (1-2). Our paper shows that their approach is sufficiently general for solving (1-2) after a modification of a definition for the singular shock wave. This solution is fully recovered with our approach with Colombeau generalized functions (as already predicted in [9]), as one can see in Corollary 1.

The following examples of conservation law systems contain delta shock wave solutions. Notice that these systems consist of single equation pairs, that is the first one contains only one dependent variable. After Theorem 1 it will be clear that this in not an essential assumption on existence of solutions of this type.

An example of a system with a delta shock wave solution is

$$\begin{aligned} u_t + (u^2)_x &= 0 \\ v_t + (uv)_x &= 0, \end{aligned} \tag{5}$$

which was obtained from a zero pressure gas dynamics model

$$\begin{aligned} u_t + (uv)_x &= 0 \\ (uv)_t + (uv^2)_x &= 0, \end{aligned} \tag{5'}$$

after some smooth transformations and elimination of u_t from (5'). The Riemann problem for system (5) is solved in [17]. One of possible solutions is

$$u_\varepsilon(x,t) = G_\varepsilon(x-ct),\ v_\varepsilon(x,t) = H_\varepsilon(x-ct) + st\delta_\varepsilon(x-ct),$$

where G_ε and H_ε are the same as above and δ_ε is a delta net. Also, Tan, Zhang and Zheng ([17]) proved the existence of the vanishing viscosity solution. Contrary to the form of solution to (4), there is no additional "singular" term in u_ε. Clearly, (5) and (5') are not equivalent in the class of weak solutions. At the end of the paper one can find some discussion concerning (5').

Also, the following systems have the similar form of solutions as (5).

Joseph ([7]) and Oberguggenberger ([15]) proved that for

$$\begin{aligned} u_t + (u^2/2)_x &= 0 \\ v_t + (uv)_x &= 0 \end{aligned} \tag{6}$$

the viscosity limit is a delta shock wave.

By using Le Flock and Vol'pert definition of the product, Hayes and Le Flock ([6]) found a delta shock wave solution to

$$\begin{aligned} u_t + (u^2)_x &= 0 \\ v_t + ((u-1)v)_x &= 0. \end{aligned} \tag{7}$$

They also proved that the vanishing viscosity limit exists and it is equal to the solution of (7).

Finally, Ercole ([4]) proved that the system

$$\begin{aligned} u_t + f(u)_x &= 0 \\ v_t + (g(u)v)_x &= 0, \end{aligned} \tag{8}$$

with some mild assumptions on f and g, has a delta shock solution considered as a limit of smooth solutions obtained by the vanishing viscosity method.

All the systems considered in the paper have a flux which is of the linear growth with respect to one of the variables. An interpretation of a limit of ϕ_ε^2, where ϕ_ε is a real valued delta net, is quite vogue. Namely, Colombeau proved in [1] that ϕ_ε^2 defines an element in the space of the generalized functions which is not associated with any classical distribution. Of course, this is not always an unsolvable problem, but considering a fairly general form of such a systems is a difficult task.

Each of systems (5,6-8) contains a delta locus which is a subset of $\mathbb{R}^2$ with the non-zero Lebesgue measure, because $f_1 \not\equiv 0$ from (1). In each of the cases when a delta shock wave solution is found in the cited papers, Theorem 1 gives also the same solution and vice versa. This is also true for each singular shock wave solution of (4) found in [9] and Theorem 2.

Before looking for a generalized solution of (1-3), initial data (3) is "regularized", i.e. substituted by generalized functions. The regularization procedure is not a priori known. (One only knows that the generalized initial data are functions bounded with respect to a small parameter ε and equal to constants out of $[-\varepsilon, \varepsilon]$.) Thus, a generalized solution do not depend on a behavior of the initial data regularization (its "microstructure"). Also, we avoid an approximation of L^∞-functions on a discontinuity line.

System (5') has a solution having the form of delta shock wave (see [18]) for a Riemann data

$$u(x,0) = \begin{cases} u_0, & x < 0 \\ u_1, & x > 0 \end{cases}, \; v(x,0) = \begin{cases} v_0, & x < 0 \\ v_1, & x > 0 \end{cases}, \; v_0 > v_1.$$

The approach used in Theorem 1 does not permit solutions having this form. But for the above data there exists a solution which has the form of singular shock wave (see Example 1). Both solutions are the same in the distributional sense (association procedure for generalized functions).

The paper is organized in the following way. In Section 2 we give the definitions of the algebra of generalized functions, delta and singular shock wave solutions. In Section 3 we give a form of approximate solutions which will be used in finding delta locus for (1-2). Roughly speaking, the main property of approximations is a splitting of a singular part "mass" with respect to the left- and right-hand side of a discontinuity line. Properly placed masses determine the delta locus. Theorem 1 applied to systems (5,6-8) gives the results previously obtained in the quoted papers. System (4) can not be solved by this procedure.

In Section 4 we are dealing with systems with polynomial fluxes. There exists a broad class of polynomials for which we can find singular delta locus having non-zero Lebesgue measure in $\mathbb{R}^2$, modifying u_ε by adding a net which converge to zero, and some of its powers converge to the delta distribution. In fact we use the idea of [9] adopted for arbitrary polynomials. In contrast to the singular shock wave solution obtained in [9], we have to assume that singular parts of approximations u_ε and v_ε have disjoint supports. This assumption can be omitted if f_1 and g_1 are constants, or linear functions with respect to u.

In Section 5, one can find a remark concerning the range of variables, as well as an example of singular shock wave solution to system (5'). The case when the system (1-2) is hyperbolic is also considered in this section. In this case the well known criteria for entropy solutions already exists. The classical ones for shock, rarefaction waves and contact discontinuities (see for example [12]) are combined with the overcompressive condition for delta (singular) waves (see [6], [9] or [17]).

There are many open problems concerning system (1-2). For example one can try to obtain a limit of viscosity self-similar solutions to the system like it was done in Dafermos and DiPerna's paper [2]. Also, one can try to describe singular shock wave solution formally obtained as a net of approximate solutions by using a weighted measure space as it was done by Keyfitz and Kranzer in [9]. Some results on interaction of delta (but not singular) shock waves with shock or rarefaction waves will be given in [14].

This could be starting point for investigation of Cauchy problems for (1-2) with a fairly general initial data, which is an important open problem.

Also, there is a need for a result about existence or non-existence of delta or singular shock waves in situation when they are not wanted, as well as any kind of uniqueness result.

2. Definitions

We shall briefly repeat some definitions of Colombeau algebra given in [16]. Denote $\mathbb{R}^2_+ := \mathbb{R} \times (0,\infty)$, $\overline{\mathbb{R}^2_+} := \mathbb{R} \times [0,\infty)$ and let $C^\infty_b(\Omega)$ be the algebra of smooth functions on Ω bounded together with all their derivatives. Let $C^\infty_{\overline{b}}(\mathbb{R}^2_+)$ be a set of all functions $u \in C^\infty(\mathbb{R}^2_+)$ satisfying $u|_{\mathbb{R}\times(0,T)} \in C^\infty_b(\mathbb{R} \times (0,T))$ for every $T > 0$. Let us remark that every element of $C^\infty_b(\mathbb{R}^2_+)$ has a smooth extension up to the line $\{t = 0\}$, i.e. $C^\infty_b(\mathbb{R}^2_+) = C^\infty_b(\overline{\mathbb{R}^2_+})$. This is also true for $C^\infty_{\overline{b}}(\mathbb{R}^2_+)$.

$\mathcal{E}_{M,g}(\mathbb{R}^2_+)$ is the set of all maps $G : (0,1) \times \mathbb{R}^2_+ \to \mathbb{R}$, $(\varepsilon, x, t) \mapsto G_\varepsilon(x,t)$, $G_\varepsilon \in C^\infty_{\overline{b}}(\mathbb{R}^2_+)$ for every $\varepsilon \in (0,1)$ satisfying:
For every $(\alpha,\beta) \in \mathbb{N}_0^2$ and $T > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |\partial_x^\alpha \partial_t^\beta G_\varepsilon(x,t)| = \mathcal{O}(\varepsilon^{-N}), \text{ as } \varepsilon \to 0.$$

$\mathcal{N}_g(\mathbb{R}^2_+)$ is the set of all $G_\varepsilon \in \mathcal{E}_{M,g}(\mathbb{R}^2_+)$ satisfying:
For every $(\alpha,\beta) \in \mathbb{N}_0^2$, $a \in \mathbb{R}$ and $T > 0$

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |\partial_x^\alpha \partial_t^\beta G_\varepsilon(x,t)| = \mathcal{O}(\varepsilon^{a}), \text{ as } \varepsilon \to 0.$$

Clearly, $\mathcal{N}_g(\mathbb{R}^2_+)$ is an ideal of the multiplicative differential algebra $\mathcal{E}_{M,g}(\mathbb{R}^2_+)$. Thus one defines the multiplicative differential algebra $\mathcal{G}_g(\mathbb{R}^2_+)$ of generalized functions by $\mathcal{G}_g(\mathbb{R}^2_+) = \mathcal{E}_{M,g}(\mathbb{R}^2_+)/\mathcal{N}_g(\mathbb{R}^2_+)$. All operations in $\mathcal{G}_g(\mathbb{R}^2_+)$ are defined by the corresponding ones in $\mathcal{E}_{M,g}(\mathbb{R}^2_+)$.

If one uses $C_b^\infty(\mathbb{R})$ instead of $C_b^\infty(\mathbb{R}_+^2)$ (i.e. drop the dependence on the t variable), one obtains $\mathcal{E}_{M,g}(\mathbb{R})$, $\mathcal{N}_g(\mathbb{R})$ and consequently, the space of generalized functions on a real line, $\mathcal{G}_g(\mathbb{R})$.

Additionaly, if functions from $\mathcal{E}_{M,g}(\mathbb{R})$ and $\mathcal{N}_g(\mathbb{R})$ are substituded with reals, one obtains the ring $\mathcal{E}_{M,0}$ and it ideal $\mathcal{N}_0$, respectively. Thus, the ring of generalized real numbers is defined by $\overline{\mathbb{R}} = \mathcal{E}_{M,0}/\mathcal{N}_0$.

In the sequel, G denotes an element (equivalence class) in $\mathcal{G}_g(\Omega)$ defined by $G_\varepsilon \in \mathcal{E}_{M,g}(\Omega)$.

Since $C_{\bar{b}}^\infty(\mathbb{R}_+^2) = C_{\bar{b}}^\infty(\overline{\mathbb{R}_+^2})$, a restriction of a generalized function to $\{t = 0\}$ is defined in the following way. For given $G \in \mathcal{G}_g(\mathbb{R}_+^2)$, its restriction $G|_{t=0} \in \mathcal{G}_g(\mathbb{R})$ is the class determinated by a function $G_\varepsilon(x, 0) \in \mathcal{E}_{M,g}(\mathbb{R})$. In the same way as above, $G(x - ct) \in \mathcal{G}_g(\mathbb{R})$ is defined by $G_\varepsilon(x - ct) \in \mathcal{E}_{M,g}(\mathbb{R})$.

If $G \in \mathcal{G}_g$ and f is a smooth function polynomially bounded together with all its derivatives, then one can easily show that the composition $f(G)$, defined by a representative $f(G_\varepsilon)$, $G \in \mathcal{G}_g$ makes sense. It means that $f(G_\varepsilon) \in \mathcal{E}_{M,g}$ if $G_\varepsilon \in \mathcal{E}_{M,g}$, and $f(G_\varepsilon) - f(H_\varepsilon) \in \mathcal{N}_g$ if $G_\varepsilon - H_\varepsilon \in \mathcal{N}_g$.

The equality in the space of the generalized functions $\mathcal{G}_g$ is not appropriate for conservation laws as one can see in [15]. A generalized function $G \in \mathcal{G}_g(\Omega)$ is said to be associated with $u \in \mathcal{D}'(\Omega)$, $G \approx u$, if for some (and hence every) representative G_ε of G, $G_\varepsilon \to u$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0$. Two generalized functions G and H are said to be associated, $G \approx H$, if $G - H \approx 0$. One can easily verify that the association is linear and an equivalence relation.

A generalized function $G \in \mathcal{G}_g(\Omega)$ is pointwisely non-negative if for every $x \in \Omega$, $G(x) \geq 0$, i.e. there exists $Z_\varepsilon \in \mathcal{N}_0$ such that $G_\varepsilon(x) \geq Z_\varepsilon$, for ε small enough.

A generalized function $G \in \mathcal{G}_g(\Omega)$ is distributionally non-negative if for every $\psi \in C_0^\infty(\Omega)$, $\int_\Omega G_\varepsilon(x)\psi(x) \geq 0$, for ε small enough.

Let $u \in \mathcal{D}'_{L^\infty}(\mathbb{R})$. Let $\mathcal{A}_0$ be the set of all functions $\phi \in \mathcal{D}(\mathbb{R})$ satisfying $\phi(x) \geq 0$, $x \in \mathbb{R}$, $\int \phi(x)dx = 1$ and $\operatorname{supp} \phi \subset [-1, 1]$. Let $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$, $x \in \mathbb{R}$. Then

$$\iota_\phi : u \mapsto \text{ class of } u * \phi_\varepsilon$$

defines a mapping of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ into $\mathcal{G}_g(\mathbb{R})$. It is clear that ι_ϕ commutes with the derivation. Also, $\iota_\phi(\delta)$ is a class defined by a delta net ϕ_ε.

Lemma 1. *The generalized function defined by the representative $\phi_\varepsilon(x - ct) \in \mathcal{E}_{M,g}(\mathbb{R}_+^2)$, $\phi \in \mathcal{A}_0$, $c \in \mathbb{R}$, is associated with $\delta(x - ct) \in \mathcal{D}'(\mathbb{R}_+^2)$.*

Proof. Let $\psi \in C_0^\infty(\mathbb{R}_+^\infty)$ and

$$I_\varepsilon := \iint \varepsilon^{-1}\phi((x - ct)/\varepsilon)\psi(x, t)dxdt.$$

Changing the variables $(x - ct)/\varepsilon \mapsto y$, $t \mapsto s$, using the Lebesgue dominated convergence theorem and the properties of the functions from $\mathcal{A}_0$ gives

$$\begin{aligned} I_\varepsilon &= \iint \phi(y)\psi(\varepsilon y + cs, s)dyds \\ &\to \int \Big(\int \phi(y)dy\Big)\psi(cs, s)ds = \int \psi(cs, s)ds, \text{ as } \varepsilon \to 0. \Delta \end{aligned}$$

The step functions, mapped by ι into $\mathcal{G}_g(\mathbb{R})$, belong to the following important class of generalized functions. $G \in \mathcal{G}_g(\Omega)$ is said to be of a bounded type if

$$\sup_{x\in\Omega} |G_\varepsilon(x)| = \mathcal{O}(1) \text{ as } \varepsilon \to 0,$$

for every $T > 0$.

Definition 1. (a) $G \in \mathcal{G}(\mathbb{R})$ is said to be a generalized step function with value (y_0, y_1) if it is of bounded type and

$$G_\varepsilon(y) = \begin{cases} y_0, \, y < -\varepsilon \\ y_1, \, y > \varepsilon \end{cases}$$

Denote $[G] := y_1 - y_0$.
(b) $D \in \mathcal{G}_g(\mathbb{R})$ is said to be generalized splitted delta function (Sδ-function for short) with value (α_0, α_1) if $D = \alpha_0 D^- + \alpha_1 D^+$, where $\alpha_0 + \alpha_1 = 1$ and $D^\pm \in \mathcal{G}_g(\mathbb{R})$ are given by the representatives

$$D_\varepsilon^\pm(y) := \frac{1}{\varepsilon}\phi\Big(\frac{y-(\pm 2\varepsilon)}{\varepsilon}\Big), \; \phi \in \mathcal{A}_0.$$

Let us note that $D_\varepsilon^\pm$ are in fact shifted model delta nets (for the notion of the model delta net one can look in [15]). Δ

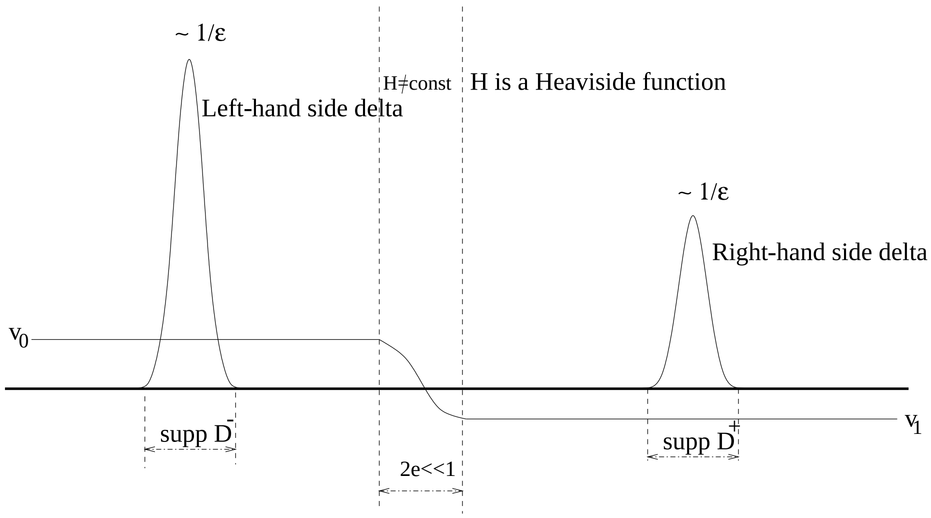

Figure 1. Delta shock wave

Let us note that the definition of Sδ-function implies supp $D_\varepsilon^- \subset (-\infty, -\varepsilon)$ and supp $D_\varepsilon^+ \subset (\varepsilon, \infty)$.

Lemma 2. *If G is a generalized step function with value (y_0, y_1) and D is an Sδ-function with value (α_0, α_1), then the following hold.*
(i) $f(G)$ is a generalized step function with value $(f(y_0), f(y_1))$, where f is a smooth function.
(ii)

$$G \cdot D \approx (y_0\alpha_0 + y_1\alpha_1)\delta.$$

Proof. The proof is a straightforward consequence of the definitions. Δ

Remark. The support property of Sδ-function ensures the uniqueness in the association sense of its product with a generalized step function.

The generalized initial data for (1-2) are now generalized step functions G and H with values (u_0, u_1) and (v_0, v_1) instead of Θ_1 and Θ_2, respectively. One can see that the inclusion by ι_ϕ of a classical step function gives a generalized step function in the sense of Definition 1(a) for every $\phi \in \mathcal{A}_0$.

Definition 2. $(U, V) \in (\mathcal{G}(\mathbb{R}^2_+))^2$ is called delta shock wave solution to Riemann problem (1-3) if a) and b) hold:
a)

$$U_t + (f_1(U)V + f_2(U,V))_x \approx 0 \tag{9}$$

$$V_t + (g_1(U)V + g_2(U,V))_x \approx 0. \tag{10}$$

$$U|_{t=0} = G,\ V|_{t=0} = H. \tag{11}$$

b) $U(x,t) = G(x-ct)$, and $V(x,t) = H(x-ct) + s(t)D(x-ct)$,
where G and H are generalized step functions, f_i, g_i, $i = 1, 2$, are smooth functions polynomially bounded together with all derivatives, f_2 and g_2 are also sublinearly bounded with respect to V, $c \in \mathbb{R}$ is a speed of the shock, $s \in C^1([0,\infty))$, $s(0) = 0$ and D is an Sδ-function. Δ

3. The generalized functions and the delta locus

Before the main theorem, we shall give two technical lemmas.

Lemma 3. *Let $g \in C^\infty(\mathbb{R}^2)$ satisfies*

$$|g(x,y_1) - g(x,y_2)| \le c(x)|y_1 - y_2|^a,\ 0 \le a < 1,$$

for every $x, y_1, y_2 \in \mathbb{R}$. Suppose that G and H are generalized step functions of bounded type, s is a smooth function on $[0,\infty)$, $c \in \mathbb{R}$ and let D be an Sδ-function. Then

$$g(G(x-ct), H(x-ct)) \approx g(G(x-ct), H(x-ct) + s(t)D(x-ct)).$$

Proof. First, suppose that $\psi \in C_0^\infty(\mathbb{R}^2_+)$ has a support contained in $[-X, X] \times [0, T]$, for some $X, T > 0$, and let $\max_{y\in\mathbb{R}}\{|G_\varepsilon(y)|, |H_\varepsilon(y)|\} \le C$. Then

$$||D^\pm_\varepsilon(x-ct)||_{L^1(\mathbb{R}^2_T)} \le 2\int_0^T \int_{-\infty}^{\infty} |\frac{1}{\varepsilon}\phi(\frac{x-ct-\xi}{\varepsilon})|dxdt \le 2\int_0^T\int_{-\infty}^{\infty}\phi(y)dydt \le T,$$

where $\mathbb{R}^2_T = \{(x,t) \in \mathbb{R}^2_+ : t \le T\}$ and ξ is either 2ε or -2ε. Put

$$\begin{aligned} I_\varepsilon := \int_0^\infty \int_{-\infty}^{\infty} \Big(& g(G_\varepsilon(x-ct), H_\varepsilon(x-ct)) \\ & - g(G_\varepsilon(x-ct), H_\varepsilon(x-ct) + s(t)D_\varepsilon(x-ct))\Big)\psi(x,t)dxdt. \end{aligned}$$

Using the assumptions on g gives

$$|I_\varepsilon| \leq C \sup_{t\in[0,T]} |s(t)|^a \sup_{(x,t)\in\mathbb{R}^2_+} |\psi(x,t)| \int_0^T \int_{-X}^X \varepsilon^{-a}\phi^a\Big(\frac{x-ct-\xi}{\varepsilon}\Big)dxdt$$
$$\leq C_1 \int_0^T \int_{-\infty}^{\infty} \varepsilon^{1-a}\phi^a(y)dydt \to 0, \text{ as } \varepsilon \to 0,$$

since

$$\|\phi^a\|_{L^1(\mathbb{R})} \leq \|1\|_{L^{1/b}(\operatorname{supp}\phi)} \|\phi^a\|_{L^{1/a}(\operatorname{supp}\phi)} \leq (2\operatorname{diam}\phi)^b \|\phi\|^a_{L^1(\mathbb{R})},$$

where b is chosen in a way that $a+b=1$. Δ

Lemma 4. *Let $f, g \in C^\infty(\mathbb{R}^2)$, $c \in \mathbb{R}$, and let G and H be the generalized step functions of the bounded type with values (y_0, y_1) and (ξ_0, ξ_1), respectively. Then*

$$\partial_t f(G(x-ct), H(x-ct)) + \partial_x g(G(x-ct), H(x-ct))$$
$$\approx\big(-c(f(y_1,\xi_1)-f(y_0,\xi_0)) + g(y_1,\xi_1) - g(y_0,\xi_0)\big)\delta(x-ct).$$

Proof. Using the fact that $\mathcal{G}_g(\mathbb{R}^2_+)$ is an multiplicative algebra, we have

$$\partial_t f(G(x-ct), H(x-ct)) = -c\partial_x f(G(x-ct), H(x-ct)).$$

Suppose that $\operatorname{supp}\psi \subset [-X, X]\times[0,T]$, for some $X, T > 0$. Then

$$\iint \big(-c\partial_x f(G_\varepsilon(x-ct), H_\varepsilon(x-ct)) + \partial_x g(G_\varepsilon(x-ct), H_\varepsilon(x-ct))\big)\psi(x,t)dxdt$$
$$=\iint \big(cf(G_\varepsilon(x-ct), H_\varepsilon(x-ct)) - g(G_\varepsilon(x-ct), H_\varepsilon(x-ct))\big)\partial_x\psi(x,t)dxdt$$
$$=\int_0^T \int_{-X}^{ct-\varepsilon} \big(cf(y_0,\xi_0) - g(y_0,\xi_0)\big)\partial_x\psi(x,t)dxdt$$
$$+\int_0^T \int_{ct-\varepsilon}^{ct+\varepsilon} \big(cf(G_\varepsilon(x-ct), H_\varepsilon(x-ct)) - g(G_\varepsilon(x-ct), H_\varepsilon(x-ct))\big)\partial_x\psi(x,t)dxdt$$
$$+\int_0^T \int_{ct+\varepsilon}^{X} \big(cf(y_1,\xi_1) - g(y_1,\xi_1)\big)\partial_x\psi(x,t)dxdt$$
$$=I_1 + I_2 + I_3.$$

First, note that

$$|I_2| \leq \int_0^T 2\varepsilon C C_\psi dt \to 0, \text{ as } \varepsilon \to 0,$$

because G_ε, H_ε and $\partial_x\psi$ are bounded with respect to ε. Then,

$$I_1 + I_3 = (cf(y_0,\xi_0) - g(y_0,\xi_0)) \int_{-T}^T \psi(ct-\varepsilon, t)dt$$
$$-(cf(y_1,\xi_1) - g(y_1,\xi_1)) \int_{-T}^T \psi(ct+\varepsilon, t)dt$$
$$\to (-c(f(y_1,\xi_1) - f(y_0,\xi_0)) + g(y_1,\xi_1) - g(y_0,\xi_0))\langle\delta|_{x=ct}, \psi\rangle, \text{ as } \varepsilon \to 0.$$

Δ

Theorem 1. *a) Let $f_1 \not\equiv \mathrm{const}$. Then a delta shock wave solution to (9-11) exists if $u_0 \neq u_1$, $f_1(u_0) \neq f_1(u_1)$ and*

$$c = \frac{f_1(u_1)v_1 + f_2(u_1,v_1) - f_1(u_0)v_0 - f_2(u_0,v_0)}{u_1 - u_0} = \frac{g_1(u_0)f_1(u_1) - g_1(u_1)f_1(u_0)}{f_1(u_1) - f_1(u_0)}, \tag{12}$$

where c is the velocity of the delta shock. The set of all points (u_1, v_1) such that (12) holds is the delta locus for (1-3) (for the point (u_0, v_0)).
b) If $f_1(u_0) = f_1(u_1) = 0$ (specially, if $f_1 \equiv 0$) and $g_1 \not\equiv \mathrm{const}$, then the delta locus is the set of all points (u_1, v_1) such that $g_1(u_0) \neq g_1(u_1)$.
c) If $f_1 \equiv 0$ and $g_1 \equiv b \in \mathbb{R}$, then the delta locus is the set of all points (u_1, v_1) such that $b(u_1 - u_0) = f_2(u_1) - f_2(u_0)$.

Proof. We look for a solution in a form of the delta shock wave

$$U(x,t) = G(x-ct),\ V(x,t) = H(x-ct) + s(t)D(x-ct), \tag{13}$$

where D is the Sδ-function with value (α_0, α_1) and s is the function from Definition 2.

Substitution of (13) into (9) and Lemma 3 give

$$\begin{aligned}&\partial_t G(x-ct) + \partial_x\big(f_1(G(x-ct))H(x-ct) + f_2(G(x-ct), H(x-ct))\big)\\ &+s(t)\partial_x(f_1\big(G(x-ct)D(x-ct))\big) \approx 0.\end{aligned} \tag{14}$$

Lemma 4 implies that the sum of first two members in (14) is associated with δ multiplied by a constant, i.e.

$$\begin{aligned}&\partial_t G(x-ct) + \partial_x\big(f_1(G(x-ct))H(x-ct) + f_2(G(x-ct), H(x-ct))\big)\\ &\approx -(c[G] - [f_1(G)H + f_2(G,H)])\delta(x-ct).\end{aligned}$$

By Lemma 2,

$$\partial_x(f_1(G(x-ct))D(x-ct)) \approx (f_1(u_0)\alpha_0 + f_1(u_1)\alpha_1)\delta'(x-ct).$$

Therefore,

$$\begin{aligned}&c(u_1 - u_0) - (f_1(u_1)v_1 + f_2(u_1,v_1) - (f_1(u_1)v_1 + f_2(u_1,v_1))) = 0\\ &f_1(u_0)\alpha_0 + f_1(u_1)\alpha_1 = 0.\end{aligned}$$

The first equation in the above system equation determines the speed of the wave

$$c = \frac{f_1(u_1)v_1 + f_2(u_1,v_1) - f_1(u_0)v_0 - f_2(u_0,v_0)}{u_1 - u_0}.$$

From the definition of the Sδ-function we have $\alpha_0 + \alpha_1 = 1$. Together with the second equation in the above system, it implies

$$\alpha_0 = \frac{f_1(u_1)}{f_1(u_1) - f_1(u_0)},\ \alpha_1 = \frac{-f_1(u_0)}{f_1(u_1) - f_1(u_0)},$$

if $f_1(u_1) \neq f_1(u_0)$.

If $f_1 \equiv \text{const} \neq 0$, the delta locus is the empty set. In this case it is possible to find singular shock solutions of system (9-10), but only if f_2, g_1 and g_2 are polynomials as one will see in the next section. If $f_1 \equiv 0$, then α_0 and α_1 can be chosen in a way that

$$cD' \approx \partial_x(g_1(G)D),$$

with the single condition $g_1(u_0) \neq g_1(u_1)$. The set of all (u_1, v_1) such that u_1 satisfies the above condition, and v_1 is arbitrary, constitutes the delta locus. If g_1 is also a constant then c has to be the same constant. The set of all u_1 for which this is true determines the delta locus. This concludes the proofs of b) and c).

Substituting U and V into (10) gives

$$\begin{aligned}
&\partial_t H(x-ct) + s'(t)D(x-ct) - cs(t)\partial_t D(x-ct) + \partial_x\big(g_1(G(x-ct))H(x-ct)\\
&\quad + g_2(G(x-ct), H(x-ct)) + g_1(G(x-ct))s(t)D(x-ct)\big)\\
=&\partial_t H(x-ct) + \partial_x\big(g_1(G(x-ct))H(x-ct) + g_2(G(x-ct), H(x-ct))\big)\\
&\quad + s'(t)D(x-ct) - cs(t)\partial_t D(x-ct) + s(t)\partial_x\big(g_1(G(x-ct))D(x-ct)\big),
\end{aligned}$$

where we again have used Lemma 3. Since G and H are generalized step functions,

$$\partial_t H(x-ct) + \partial_x\big(g_1(G(x-ct))H(x-ct) + g_2(G(x-ct), H(x-ct))\big) \approx -\alpha\delta(x-ct),$$

where α is Rankine-Hugoniot deficit defined by

$$\alpha := c(v_1 - v_0) - (g_1(u_1)v_1 + g_2(u_1, v_1) - g_1(u_0)v_0 - g_2(u_0, v_0)).$$

If $\alpha = 0$, then (u_1, v_1) has to belong to the standard Rankine-Hugoniot locus, i.e. $s = 0$. If $\alpha \neq 0$, then $s'(t) = \alpha$, and consequently $s(t) = \alpha t$, i.e.

$$cD' \approx \partial_x(g_1(G)D).$$

Therefore

$$c = \frac{g_1(u_0)f_1(u_1) - g_1(u_1)f_1(u_0)}{f_1(u_1) - f_1(u_0)}. \tag{15}$$

That means that the delta locus is the set of all (u_1, v_1) such that (15) holds, since c is already determinated. Δ

4. Singular shock waves

If the fluxes f_1, f_2, g_1 and g_2 are polynomials depending only on u, we are in a position to find a singular shock solution to system (9-11) for a larger set of initial data. This will be done by adding some terms in u_ε. When the functions in the fluxes are the polynomials, the behavior of these additional terms in u_ε can be controlled. Suppose that the maximal degree of all polynomials in the fluxes equals m. Let

$$f_1(y) = \sum_{i=0}^m a_{1,i}y^i,\ f_2(y) = \sum_{i=0}^m a_{2,i}y^i,\ g_1(y) = \sum_{i=0}^m b_{1,i}y^i,\ g_2(y) = \sum_{i=0}^m b_{2,i}y^i.$$

We will give the definitions of these new building blocks for singular shock waves.

Definition 3. A generalized function $d \in \mathcal{G}_g(\mathbb{R})$ is said to be m-singular delta function (mSD-function for short) with value (β_0, β_1) if $d = \beta_0 d^- + \beta_1 d^+$, $d^\pm \in \mathcal{G}_g(\mathbb{R})$, $\operatorname{supp} d_\varepsilon^- \subset (-\infty, \varepsilon)$, $\operatorname{supp} d_\varepsilon^+ \subset (\varepsilon, \infty)$, $(d^\pm)^i \approx 0$, $i \in \{1, \dots, m-1\}$, $(d^\pm)^m \approx \delta$.

Let m be an odd positive integer. A generalized function $d \in \mathcal{G}_g(\mathbb{R})$ is said to be m'-singular delta function (m'SD-function for short) with value (β_0, β_1) if $d = \beta_0 d^- + \beta_1 d^+$, $d^\pm \in \mathcal{G}_g(\mathbb{R})$, $\operatorname{supp} d_\varepsilon^- \subset (-\infty, \varepsilon)$, $\operatorname{supp} d_\varepsilon^+ \subset (\varepsilon, \infty)$, $(d^\pm)^i \approx 0$, $i \in \{1, \dots, m-2, m\}$, $(d^\pm)^{m-1} \approx \delta$.

An Sδ-function and an mSD-function (or an m'SD-function) are said to be compatible if their representatives have disjoint supports for ε small enough. Δ

Remark. The definition of m'SD-function d implies $Gd^m \approx 0$ if G is a generalized step function.

Example. One can construct mSD and m'SD functions in the following way by using powers of the model delta nets. Let $\phi \in \mathcal{A}_0$.
(i) Put

$$d_\varepsilon^\pm(y) = \left(\phi\left(\frac{y - (\pm 4\varepsilon)}{\varepsilon}\right)\right)^{1/m}.$$

Then $\beta_0 d_\varepsilon^- + \beta_1 d_\varepsilon^+$ is a represent for an mSD-function with value (β_0, β_1).
(ii) Let m be an odd positive integer. Put

$$\begin{aligned} d_\varepsilon^+(y) &= \left(\frac{1}{2}\phi\left(\frac{y-4\varepsilon}{\varepsilon}\right) - \phi\left(\frac{y-6\varepsilon}{\varepsilon}\right)\right)^{1/(m-1)}, \text{ for } y \geq 0, \\ d_\varepsilon^-(y) &= -d_\varepsilon^+(-y), \text{ for } y < 0. \end{aligned}$$

Now $\beta_0 d_\varepsilon^- + \beta_1 d_\varepsilon^+$ is a possible represent for an mSD-function with value (β_0, β_1).

The constructed mSD- and m'SD-functions are compatible with the Sδ-functions defined in the example after Definition 1.

Definition 4. $(U, V) \in (\mathcal{G}(\mathbb{R}_+^2))^2$ is called singular shock wave solution to Riemann problem (1-3) if a) and b) hold:
a)

$$U_t + (f_1(U)V + f_2(U,V))_x \approx 0 \tag{9}$$

$$V_t + (g_1(U)V + g_2(U,V))_x \approx 0. \tag{10}$$

$$U|_{t=0} = G, \; V|_{t=0} = H. \tag{11}$$

b) $U(x,t) = G(x-ct) + s_1(t)d_1(x-ct)$, and $V(x,t) = H(x-ct) + s_2(t)D(x-ct) + s_3(t)d_2(x-ct)$,
where G and H are generalized step functions, f_i, g_i, $i = 1, 2$, are polynomials of the degree at most m $c \in \mathbb{R}$ is a speed of the shock, $s, s_1, s_2 \in C^1([0,\infty))$, $s_1(0) = s_2(0) = s_3(0) = 0$, D is an Sδ-function, and d_j are mSD or m'SD-function, $j = 1, 2$. Δ

Lemma 5. *a) Let $d \in \mathcal{G}_g(\mathbb{R})$ be an mSD-function with value (β_0, β_1), $\beta_0^m + \beta_1^m = 1$, $G \in \mathcal{G}_g(\mathbb{R})$ generalized step function with value (y_0, y_1), $s \in C^1(\mathbb{R}_+)$, $s(0) = 0$, and $\Gamma(y) = \sum_{i=0}^m a_i y^i$ be a real valued polynomial. Then*

$$\begin{aligned} &\Gamma(G(x-ct)) + s(t)d(x-ct)) \\ \approx &\Gamma(G(x-ct)) + a_m s^m(t)\big(\beta_0^m (d^-)^m(x-ct) + \beta_1^m (d^+)^m(x-ct)\big) \\ \approx &\Gamma(G(x-ct)) + a_m s^m(t)\delta(x-ct). \end{aligned}$$

b) Let $d \in \mathcal{G}_g(\mathbb{R})$ be an m'SD-function with value (β_0, β_1), $\beta_0^{m-1} + \beta_1^{m-1} = 1$, while G, s and Γ are as above. Then

$$\begin{aligned}
&\Gamma(G(x-ct) + s(t)d(x-ct)) \\
\approx&\Gamma(G(x-ct)) + a_{m-1}s^{m-1}(t)\big(\beta_0^{m-1}(d^-)^{m-1}(x-ct) + \beta_1^{m-1}(d^+)^{m-1}(x-ct)\big) \\
&+ m a_m s^{m-1}(t)\big(\beta_0^{m-1}y_0(d^-)^{m-1}(x-ct) + \beta_1^{m-1}y_1(d^+)^{m-1}(x-ct)\big). \\
\approx&\Gamma(G(x-ct)) + a_{m-1}s^{m-1}(t)\delta(x-ct) \\
&+ m a_m s^{m-1}(t)(\beta_0^{m-1}y_0 + \beta_1^{m-1}y_1)\delta(x-ct).
\end{aligned}$$

Proof. a) Let $\psi \in C_0^\infty(\mathbb{R}_+^2)$ with $\operatorname{supp}\psi \subset [-X,X] \times [0,T]$, for some $X, T > 0$. Then

$$\begin{aligned}
&\iint G_\varepsilon(x-ct)d_\varepsilon^j(x-ct)\psi(x,t)dxdt \\
=&y_0\beta_0^j \int_0^T \int_{-X-cT}^{-\varepsilon} (d_\varepsilon^-)^j(y)\psi(y+ct,t)dydt + \int_0^T \int_{-\varepsilon}^{\varepsilon} G_\varepsilon(y)\psi(y+ct,t)dydt \\
&+y_1\beta_1^j \int_0^T \int_{\varepsilon}^{X-cT} (d_\varepsilon^+)^j(y)\psi(y+ct,t)dydt = I_1 + I_2 + I_3.
\end{aligned}$$

Assumptions on d implies that I_1 and I_3 converge to zero as $\varepsilon \to 0$, for $j < m$. Also, $I_2 \to 0$, as $\varepsilon \to 0$, because G_ε is of bounded type. The association of m-th order of d is given in Lemma 1, due to definition of mSD-functions. Since the multiplication by a smooth function preserves association,

$$\begin{aligned}
&(G(x-ct) + s(t)d(x-ct))^j \approx G^j(x-ct),\ j < m \\
&(G(x-ct) + s(t)d(x-ct))^m \approx G^m(x-ct) + s^m(t)d^m(x-ct).
\end{aligned}$$

Collecting all the terms together,

$$\Gamma(G(x-ct) + s(t)d(x-ct)) \approx \Gamma(G(x-ct)) + s^m(t)a_m d^m(x-ct).$$

b) Like in the previous case,

$$\begin{aligned}
&(G(x-ct) + s(t)d(x-ct))^j \approx G^j(x-ct),\ j < m-1 \\
&(G(x-ct) + s(t)d(x-ct))^{m-1} \approx G^{m-1}(x-ct) + s^{m-1}(t)\delta(x-ct).
\end{aligned}$$

Also

$$\begin{aligned}
&(G(x-ct) + s(t)d(x-ct))^m = \sum_{i=0}^m \binom{m}{i} G^i(x-ct)s^{m-i}(t)d^{m-i}(x-ct) \\
\approx&G(x-ct) + ms^{m-1}(t)(y_0\beta_0^{m-1}d^-(x-ct) + y_1\beta_1^{m-1}d^+(x-ct)) \\
\approx&G(x-ct) + ms^{m-1}(t)(y_0\beta_0^{m-1} + y_1\beta_1^{m-1})\delta(x-ct),
\end{aligned}$$

where we have used

$$\begin{aligned}
&G(x-ct)d^{m-1}(x-ct) \approx y_0\beta_0^{m-1}(d^-)^{m-1}(x-ct) + y_1\beta_1^{m-1}(d^+)^{m-1}(x-ct) \\
\approx&(y_0\beta_0^{j-1} + y_1\beta_1^j)\delta(x-ct),
\end{aligned}$$

following from the properties of the generalized function d. This proves the lemma. Δ

It will be enough to look for solutions to (9-11) which has the form

$$
\begin{aligned}
U(x,t) &= G(x-ct) + s_1(t)d(x-ct) \\
V(x,t) &= H(x-ct) + s_2(t)D(x-ct),
\end{aligned} \tag{23}
$$

where the notation is from Definition 4.

For an even m, d is an mSD-function, and the shapes of functions U and V is illustrated in the Figure 2.

If m is odd, U can be of the same shape as above (when d is an mSD-function), or in the shape illustrated in the Figure 3 below (when d is an m'SD-function). The particular choice will depend on the system in question as one can see in the proof of Theorem 2.

Definition 5. The set of all points $(u_1, v_1) \in \mathbb{R}^2$ for which there exists a singular shock wave solution is called the singular delta locus.

Admissible singular delta locus is its part where the singular shock wave is overcompressive. Δ

Theorem 2. *Let $G(x-ct)$ and $H(x-ct)$ be generalized step functions with the speed c and values (u_0, u_1) and (v_0, v_1), respectively.*

Let

$$
\alpha = c(v_1 - v_0) - (g_1(u_1)v_1 + g_2(u_1) - g_1(u_0)v_0 - g_2(u_0)) = c[H] - [g_1(G)H + g_2(G)]
$$

to be Rankine-Hugoniot deficit (defined in [9]*).*

A singular shock wave solution to (9-11) which has the form (23) exists if one of the following two assertions are true:

(i) There exists a solution $(\alpha_0, y_0) \in \mathbb{R}^2$ to the system

$$
\begin{aligned}
&\alpha[f_1(G)]\alpha_0 + \sigma[H]a_{1,m}y_0 = \sigma(v_1 a_{1,m} + a_{2,m}) + \alpha f_1(u_1) \\
&\alpha[g_1(G)]\alpha_0 + \sigma[H]b_{1,m}y_0 = \sigma(v_1 b_{1,m} + b_{2,m}) + \alpha(g_1(u_1) - c),
\end{aligned} \tag{24}
$$

for some $\sigma \in \mathbb{R} \setminus \{0\}$. If m is an even number, then σ also has to be positive and $y_0 \in [0,1]$.

(ii) m is an odd number and there exists a solution $(\alpha_0, y_0) \in \mathbb{R} \times \mathbb{R}_+$ to the system

$$
\begin{aligned}
&\alpha[f_1(G)]\alpha_0 + \sigma(a_{1,m-1}[G] + ma_{1,m}[GH] + ma_{2,m}[G])y_0 \\
=&\sigma(a_{1,m-1}v_1 + ma_{1,m}u_1v_1 + ma_{2,m}u_1 + a_{2,m-1}) + \alpha f_1(u_1) \\
&\alpha[g_1(G)]\alpha_0 + \sigma(b_{1,m-1}[G] + mb_{1,m}[GH] + mb_{2,m}[G])y_0 \\
=&\sigma(b_{1,m-1}v_1 + mb_{1,m}u_1v_1 + mb_{2,m}u_1 + b_{2,m-1}) + \alpha(g_1(u_1) - c)
\end{aligned} \tag{25}
$$

for some $\sigma \in \mathbb{R}_+$.

The speed of the singular shock waves is always given by

$$
c = \frac{f_1(u_1)v_1 + f_2(u_1) - f_1(u_0)v_0 - f_2(u_0)}{u_1 - u_0} = \frac{[f_1(G)H + f_2(G)]}{[G]}
$$

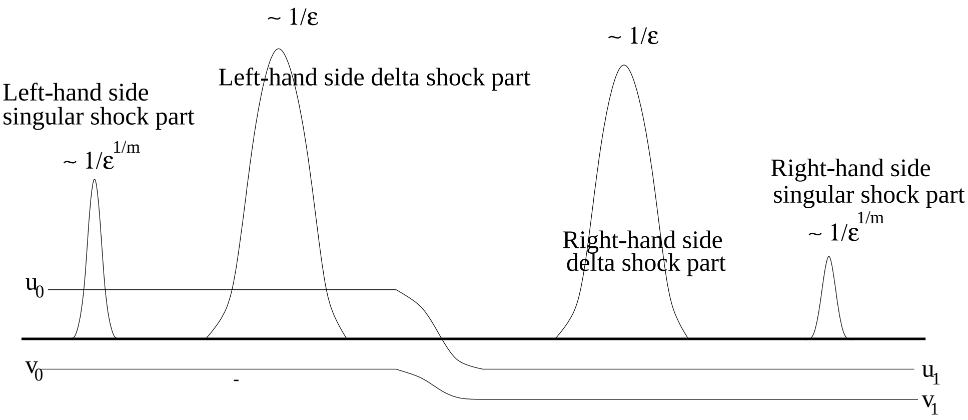

Figure 2. A generalized step function, an Sδ- and an mSD-functions

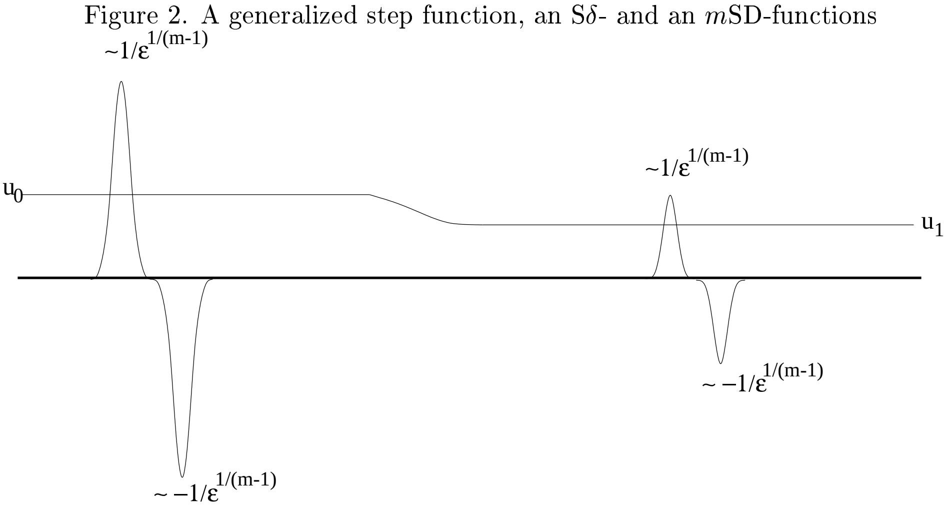

Figure 3. A generalized step function with an m'SD-function

Proof. (i) Without a loss in generality, we can take $\beta_0^m + \beta_1^m = 1$, Let U and V be given by (23), where d is an mSD-function with values (β_0, β_1) and D is an Sδ-function with values (α_0, α_1) such that d and D are compatible. Let $s_j \in C^\infty[0,\infty)$, $s_j(0) = 0$, $j = 1, 2$.

Lemma 5 b) implies

$$\begin{aligned}
&f_j(G(x-ct) + s_1(t)d(x-ct)) \approx f_j(G(x-ct)) + a_{j,m}s_1^m(t)d^m(x-ct)\\
&g_j(G(x-ct) + s_1(t)d(x-ct)) \approx g_j(G(x-ct)) + b_{j,m}s_1^m(t)d^m(x-ct),\ j = 1, 2.
\end{aligned}$$

Compatibility of d and D implies

$$\begin{aligned}
f_1(U(x,t)) \cdot s_2(t)D(x-ct) &\approx s_2(t)(f_1(u_0)\alpha_0 + f_1(u_1)\alpha_1)\delta(x-ct)\\
g_1(U(x,t)) \cdot s_2(t)D(x-ct) &\approx s_2(t)(g_1(u_0)\alpha_0 + g_1(u_1)\alpha_1)\delta(x-ct).
\end{aligned}$$

Therefore, substitution of U and V into (9) gives

$$\begin{aligned}
&\partial_t U(x,t) + \partial_x (f_1(U(x,t))V(x,t) + f_2(U(x,t))\\
\approx&\partial_t G(x-ct) + \partial_x \Big(f_1(G(x-ct))H(x-ct) + a_{1,m} s_1^m(t)(v_0\beta_0^m + v_1\beta_1^m)\delta(x-ct)\\
&+ s_2(t)(f_1(u_0)\alpha_0 + f_1(u_1)\alpha_1)\delta(x-ct) + f_2(G(x-ct)) + a_{2,m}s_1^m(t)\delta(x-ct)\Big)\\
\approx&\Big(-c(u_1-u_0) + f_1(u_1)v_1 + f_2(u_1) - f_1(u_0)v_0 - f_2(u_0)\Big)\delta(x-ct)\\
&+ \Big(a_{1,m}s_1^m(t)(v_0\beta_0^m + v_1\beta_1^m) + s_2(t)(f_1(u_0)\alpha_0 + f_1(u_1)\alpha_1)\\
&+ a_{2,m}s_1^m(t)\Big)\delta'(x-ct) \approx 0
\end{aligned}$$

So, the speed c is determinated by

$$c = \frac{f_1(u_1)v_1 + f_2(u_1) - f_1(u_0)v_0 - f_2(u_0)}{u_1 - u_0}$$

and

$$a_{1,m}s_1^m(t)(v_0\beta_0^m + v_1\beta_1^m) + s_2(t)(f_1(u_0)\alpha_0 + f_1(u_1)\alpha_1) + a_{2,m}s_1^m(t) = 0 \tag{26}$$

has to be true.

Substitution of U and V into (10) gives

$$\begin{aligned}
&\partial_t V(x,t) + \partial_x (g_1(U(x,t))V(x,t) + g_2(U(x,t)))\\
\approx&\partial_t H(x-ct) + s_2'(t)D(x-ct) + s_2(t)\partial_t D(x-ct)\\
&+ \partial_x \Big(g_1(G(x-ct))H(x-ct) + s_2(t)(g_1(u_0)\alpha_0 + g_1(u_1)\alpha_1)\delta(x-ct)\\
&+ b_{1,m}s_1^m(t)(v_0\beta_0^m + v_1\beta_1^m)\delta(x-ct) + g_2(G(x-ct)) + b_{2,m}s_1^m(t)\delta(x-ct)\Big)\\
\approx&\Big(-c(v_1-v_0) + s_2'(t) + g_1(u_1)v_1 + g_2(u_1) - g_1(u_0)v_0 - g_2(u_0)\Big)\delta(x-ct)\\
&+ \Big(s_2(t)(-c + g_1(u_0)\alpha_0 + g_1(u_1)\alpha_1)\\
&+ s_1^m(t)(b_{1,m}(v_0\beta_0^m + v_1\beta_1^m) + b_{2,m})\Big)\delta'(x-ct) \approx 0.
\end{aligned}$$

From the first term we have

$$s_2'(t) = c(v_1 - v_0) + g_1(u_1)v_1 + g_2(u_1) - g_1(u_0)v_0 - g_2(u_0) = \alpha,$$

i.e. $s_2(t) = \alpha t$, because $s_2(0) = 0$. Now, from the second term we have

$$s_1 = (\sigma t)^{1/m} \tag{27}$$

$$\sigma(b_{1,m}(v_0\beta_0^m + v_1\beta_1^m) + b_{2,m}) = \alpha(c - g_1(u_0)\alpha_0 - g_1(u_1)\alpha_1). \tag{28}$$

Using (27), equation (26) is now

$$\alpha(f_1(u_0)\alpha_0 + f_1(u_1)\alpha_1) + \sigma(a_{1,m}(v_0\beta_0^m + v_1\beta_1^m) + a_{2,m}) = 0. \tag{29}$$

Since $\alpha_0 + \alpha_1 = 1$ and $\beta_0^m + \beta_1^m = 1$, equations (28) and (29) become

(30)
$$\alpha(g_1(u_1) - g_1(u_0))\alpha_0 + \sigma b_{1,m}(v_1 - v_0)\beta_0^m = \sigma(b_{1,m}v_1 + b_{2,m}) - \alpha(c - g_1(u_1))$$
(31)
$$\alpha(f_1(u_1) - f_1(u_0))\alpha_0 + \sigma a_{1,m}(v_1 - v_0)\beta_0^m = \alpha f_1(u_1) + \sigma(a_{2,m} + a_{1,m}v_1).$$

If m is an even number then a singular shock wave solution connecting initial states (u_0, v_0) and (u_1, v_1) exists if system (30-31) has a solution $(\alpha_0, \beta_0^m) \in \mathbb{R}\times\mathbb{R}^+$ for some $\sigma > 0$.

If m is an odd number then a singular shock wave solution connecting initial states (u_0, v_0) and (u_1, v_1) exists if system (30-31) has a solution $(\alpha_0, \beta_0^m) \in \mathbb{R}^2$ for some $\sigma \neq 0$. ($\sigma = 0$ means that (u_1, v_1) belongs to the classical Hugoniot locus.)

(ii) Without a loss in generality, we can take $\beta_0^{m-1} + \beta_1^{m-1} = 1$, Using the same arguments as in (i), Lemma 5 and substituting U and V into (9), we have

$$\begin{aligned}
&\partial_t U(x,t) + \partial_x(f_1(U(x,t))V(x,t) + f_2(U(x,t)))\\
\approx&\partial_t G(x-ct) + \partial_x\Big(\big(f_1(G(x-ct)) + a_{1,m-1}s_1^{m-1}(t)d^{m-1}(x-ct)\\
&+ ma_{1,m}s_1^{m-1}(t)(u_0\beta_0^{m-1}(d^-)^{m-1}(x-ct) + u_1\beta_1^{m-1}(d^+)^{m-1}(x-ct))\big)\\
&(H(x-ct) + s_2(t)D(x-ct)) + f_2(G(x-ct))\\
&+ a_{2,m-1}s_1^{m-1}(t)d^{m-1}(x-ct)\\
&+ ma_{2,m}s_1^{m-1}(t)(u_0\beta_0^{m-1} + u_1\beta_1^{m-1})d^{m-1}(x-ct)\Big)\\
\approx&\Big(-c(u_1-u_0) + f_1(u_1)v_1 + f_2(u_1) - f_1(u_0)v_0 - f_2(u_0)\Big)\delta(x-ct)\\
&+\Big(s_1^{m-1}(t)\big(a_{1,m-1}(v_0\beta_0^{m-1} + v_1\beta_1^{m-1}) + ma_{1,m}(u_0v_0\beta_0^{m-1} + u_1v_1\beta_1^{m-1})\\
&+ a_{2,m-1} + ma_{2,m}(u_0\beta_0^{m-1} + u_1\beta_1^{m-1})\big)\\
&+ s_2(t)(f_1(u_1)\alpha_0 + f_1(u_0)\alpha_1)\Big)\delta'(x-ct) \approx 0.
\end{aligned}$$

The speed of the wave is determinated from the term multiplied by δ,

$$c = \frac{f_1(u_1)v_1 + f_2(u_1) - f_1(u_0)v_0 - f_2(u_0)}{u_1 - u_0}.$$

From the one multiplied by δ', we have

(32)
$$\begin{aligned}
&s_1^{m-1}(t)\big(a_{1,m-1}(v_0\beta_0^{m-1} + v_1\beta_1^{m-1}) + ma_{1,m}(u_0v_0\beta_0^{m-1} + u_1v_1\beta_1^{m-1})\\
&+a_{2,m-1} + ma_{2,m}(u_0\beta_0^{m-1} + u_1\beta_1^{m-1})\big) + s_2(t)(f_1(u_0)\alpha_0 + f_1(u_1)\alpha_1) = 0.
\end{aligned}$$

Substitution of U and V into (10) gives

$$\begin{aligned}
&\partial_t V(x,t) + \partial_x (g_1(U(x,t))V(x,t) + g_2(U(x,t)) \\
\approx &\partial_t H(x-ct) + s_2'(t)D(x-ct) + s_2(t)\partial_t D(x-ct) \\
&+ \partial_x \Big(\big(g_1(G(x-ct)) + b_{1,m-1}s_1^{m-1}(t)d^{m-1}(x-ct) \\
&+ mb_{1,m}s_1^{m-1}(t)(u_0\beta_0^{m-1}(\delta^-)^{m-1}(x-ct) + u_1\beta_1^{m-1}(\delta^+)^{m-1}(x-ct)\big) \\
&(H(x-ct) + s_2(t)D(x-ct)) + g_2(G(x-ct)) + b_{2,m-1}s_1^{m-1}(t)d^{m-1}(x-ct) \\
&+ mb_{2,m}s_1^{m-1}(t)(u_0\beta_0^{m-1} + u_1\beta_1^{m-1})d^{m-1}(x-ct)\Big) \\
\approx &\Big(-c(v_1-v_0) + s_2'(t) + g_1(u_1)v_1 + g_2(u_1) - g_1(u_0)v_0 - g_2(u_0)\Big)\delta(x-ct) \\
&+ \Big(s_1^{m-1}(t)(b_{1,m-1}(v_0\beta_0^{m-1} + v_1\beta_1^{m-1}) + mb_{1,m}(u_0v_0\beta_0^{m-1} + u_1v_1\beta_1^{m-1}) \\
&+ b_{2,m-1} + mb_{2,m}(u_0\beta_0^{m-1} + u_1\beta_1^{m-1})) \\
&+ s_2(t)(-c + g_1(u_0)\alpha_0 + g_1(u_1)\alpha_1)\Big)\delta'(x-ct) \approx 0.
\end{aligned}$$

Then,

$$s_2'(t) = c(v_1-v_0) - g_1(u_1)v_1 - g_2(u_1) + g_1(u_0)v_0 + g_2(u_0) = \alpha,$$

i.e. $s_2(t) = \alpha t$. Immediately, one can see that this gives $s_1(t) = (\sigma t)^{1/(m-1)}$, for some $\sigma > 0$, and consequently

(33)
$$\begin{aligned}
&\sigma(b_{1,m-1}(v_0\beta_0^{m-1} + v_1\beta_1^{m-1}) + mb_{1,m}(u_0v_0\beta_0^{m-1} + u_1v_1\beta_1^{m-1}) \\
&+ b_{2,m-1} + mb_{2,m}(u_0\beta_0^{m-1} + u_1\beta_1^{m-1})) + \alpha(-c + g_1(u_0)\alpha_0 + g_1(u_1)\alpha_1) = 0.
\end{aligned}$$

Using this and $\alpha_0 + \alpha_1 = 1$, $\beta_0^{m-1} + \beta_1^{m-1} = 1$, one gets

$$\begin{aligned}
&\alpha(f_1(u_1) - f_1(u_0))\alpha_0 + \sigma\big(a_{1,m-1}(v_1-v_0) \\
&\quad + ma_{1,m}(u_1v_1 - u_0v_0) + ma_{2,m}(u_1-u_0)\big)\beta_0^{m-1} \\
&= \alpha f_1(u_1) + \sigma(a_{1,m-1}v_1 + ma_{1,m}u_1v_1 + a_{2,m-1} + ma_{2,m}u_1)
\end{aligned} \tag{34}$$

instead of (32), and

$$\begin{aligned}
&\alpha(g_1(u_1) - g_1(u_0))\alpha_0 + \sigma\big(b_{1,m-1}(v_1-v_0) + mb_{1,m}(u_1-u_0) \\
&\quad + mb_{2,m}(u_1-u_0)\big)\beta_0^{m-1} \\
&= \alpha(g_1(u_1) - c) + \sigma(b_{1,m-1}v_1 + mb_{1,m}u_1v_1 + b_{2,m-1} + mb_{2,m}u_1)
\end{aligned} \tag{35}$$

instead of (33).

Since $m-1$ is an even number, one will obtain the solution in the form of singular shock wave connecting states (u_0, v_0) and (u_1, v_1) if the system (34-35) has a solution $(\alpha_0, \beta_0^{m-1})$ for some $\sigma > 0$. This concludes the proof. Δ

In the following corollary, one will see that the singular shock wave solution to system (4) constructed in [9] is recovered by the procedure from this paper.

Corollary 1. *In all cases given in* [9] *when a singular shock wave solution to the Riemann problem (4),(3) exists, the same type of solution can be constructed by the mean of Theorem 2.*

Proof. Let (u_0, v_0) be a given point. The set of all points (u_1, v_1) which can be joined with (u_0, v_0) by a singular shock wave is denoted by Q_7 in [9]. This area lies below the line $D = \{(u,v): \ v = v_0 + u^2 + (1-u_0)u - u_0, \ u \le u_0 - 3\}$, above the line $E = \{(u,v): \ v = v_0 + (u-u_0)(u_0-1), \ u \le u_0 - 3\}$ and on the left-hand side of Hugoniot locus.

If $f_1 \equiv \gamma_1 \in \mathbb{R} \setminus \{0\}$, $g_1 \equiv \gamma_2 \in \mathbb{R}$, then system (25) becomes

$$\begin{aligned} \sigma m a_{2,m}[G]y_0 &= \sigma(m a_{2,m} u_1 + a_{2,m-1}) + \alpha\gamma_1 \\ \sigma m b_{2,m}[G]y_0 &= \sigma(m b_{2,m} u_1 + b_{2,m-1}) + \alpha(\gamma_2 - c). \end{aligned}$$

Therefore, the condition for the existence of a singular shock wave solution is

$$y_0 = \frac{(c-\gamma_2)(m a_{2,m} u_1 + a_{2,m-1}) + \gamma_1(m b_{2,m} u_1 + b_{2,m-1})}{(\gamma_1 m b_{2,m} + (c-\gamma_2) m a_{2,m})[G]} \in [0,1].$$

Substituting values $m = 3$, $\gamma_1 = -1$, $\gamma_2 = 0$, $a_{2,m-1} = 1$, $a_{2,m} = 0$, $b_{2,m-1} = 0$ and $b_{2,m} = 1/3$ given in (4), we obtain

$$\begin{aligned} \alpha &= \sigma > 0 \\ c &= u_0 + u_1 - \frac{v_1 - v_0}{u_1 - u_0} \\ y_0 &= \frac{u_1 - c}{u_1 - u_0} \in [0,1]. \end{aligned}$$

The last condition is equivalent to

$$v_1 \ge v_0 + u_0 u_1 - u_0^2 \text{ and } v_1 \le v_0 - u_0 u_1 + u_1^2.$$

With $u_1 \le u_0 - 3$ these inequalities are

$$v_0 + u_0 u_1 - u_0^2 c \le v_0 + (u-u_0)(u_0-1) \text{ and } v_0 - u_0 u_1 + u_1^2 \ge v_0 + u^2 + (1-u_0)u - u_0.$$

Therefore, each point in Q_7 is contained in delta singular locus, given in Theorem 2.

The points in the complement of Q_7 are not valid right-hand sides in Riemann data since the singular shock wave is not overcompressive. Δ

Remark. In [9] the singular parts of functions U and V have nonempty intersection (i.e. $d_\varepsilon^2 = D_\varepsilon \approx \delta$). This is possible because in the system (4) there are no multiplication of d_ε and D_ε, i.e. $f_1 = -1$, $g_1 = 0$ are the constants.

5. Discussion

There are three main obstacles for using of delta or singular delta locus.

1) The general assumption used herre is that all values for U and V are possible. But, often this is not a case in a particular system. If there exists a set $\mathcal{S} \subset \mathbb{R}^2$

where (U,V) has to belong, and if $\mathcal{D}$ is a delta (singular) locus, a real locus is actually $\mathcal{S}\cap\mathcal{D}$, provided that the range of (U,V) lies in $\mathcal{S}$.

2) The type of system (1-2) is not relevant for the constructions of solutions in the paper. For hyperbolic systems one can state that admissible part of the delta or singular delta locus is the set of all points (u_1,v_1) such that $\lambda_1(u_1,v_1)\leq c\leq \lambda_2(u_0,v_0)$ where λ_1,λ_2 are the eigenvalues of (1-2) and c is the speed of the wave connecting initial states (u_0,v_0) and (u_1,v_1). For example, this condition is used in papers [6], [9] and [17]. With the above condition, the admissible part of the delta locus is just a part of a curve, while singular delta locus is a part of an area. Except for the above particular cases, there is no any kind of uniqueness result for delta or singular shock wave solutions.

3) If one wants to use a delta locus (a curve in general) for connecting different states in a usual way like for Hugoniot locus or rarefaction curve, the problems may occur. The only two possibilities of connecting delta shock are rarefaction waves (RW_1 from the left, and RW_2 from the right-hand side of the delta shock) and it is necessary that the speed c of the delta shock equals $\lambda_1(u_0,v_0)$ or $\lambda_2(u_1,v_1)$. This means that a part of a delta locus where the points which can be joined with some other states is a discrete set, in general. An attempt of giving some possibilities to overcome this problem will be given in [14] by investigating delta shock and shock wave interactions.

For a delta singular locus (or in a "degenerate" case of delta locus when it has non-zero Lebesge measure) the general situation is quite different. An intersection of admissible part of the delta singular locus (or delta locus) with the curve obtained by solving equation $c=\lambda_1(u_0,v_0)$ and equation $c=\lambda_2(u_1,v_1)$ is now a curve in a general case. Therefore, there is a possibility to connect singular delta shock wave with a rarefaction wave, as it was done in [6] or [17].

Some of the above remarks will be treated in the following example - presureless gas dynamic model.

Example 1. The Rieman problem for presureless gas dynamic model is given by the system

$$u_t+(uv)_x=0 \tag{36}$$

$$(uv)_t+(uv^2)_x=0 \tag{37}$$

and the initial data (3), where u is a density and v is a velocity. Two eigenvalues of the system are the same, $\lambda_1=\lambda_2=v$, thus the system is weakly hyperbolic.

In the case $v_0\leq v_1$. the problem has a classical weak entropy solution consisting of combinations of contact discontinuities and vacuum states (when $u=0$), see [18]. So, we shall suppose that $v_0>v_1$.

The fact that the system is not in the evolutive form makes no problem with applying procedures described in the proof of Theorem 2. One can substitute the generalized functions

$$\begin{aligned}
&U(x,t)=G(x-ct)+s_1(t)(\alpha_0D^-+\alpha_1D^+)+s_2(t)(\beta_0d^-+\beta_1d^+),\ \alpha_0+\alpha_1=1\\
&V(x,t)=H(x-ct)+s_3(t)(\gamma_0d^-+\gamma_1d^+),
\end{aligned}$$

where $\alpha_0+\alpha_1=1$, D and d, $d^3\approx\delta$, are compatible Sδ- and 3SD-functions, respectively. Due to physical reasons, G and D has to be nonnegative generalized functions. This imply that U is pointwisely or distributionally non-negative. But, as one will see during calculation procedure, the representative of U, U_ε, is non non-negative. In fact U is non non-negative in all generalized points (see [11]) for this notion.

By the exactly same procedure and arguments as in the proof of Theorem 2, (36) and association procedure gives

$$\begin{aligned} &-c[G]\delta(x-ct)+s_1'(t)\delta(x-ct)-cs_1(t)\delta'(x-ct)\\ &+[GH]\delta(x-ct)+s_1(t)(\alpha_0 v_0+\alpha_1 v_1)\delta'(x-ct)\approx 0, \end{aligned}$$

where we have used the compatibility condition for d and D, and the fact that $(d^{\pm})^2 \approx 0$. The above equation yields

(38) $$s_1(t)=\sigma_1 t,\ \sigma_1=c[G]-[GH],\ \sigma_1>0 \text{ (physical condition)}$$

(39) $$\alpha_0=\frac{v_1-c}{v_1-v_0},\ \alpha_1=\frac{c-v_0}{v_1-v_0}.$$

Similarly, from (37), we have

$$\begin{aligned} &-c[GH]\delta(x-ct)+s_1'(t)(\alpha_0 v_0+\alpha_1 v_1)\delta(x-ct)\\ &-cs_1'(t)(\alpha_0 v_0+\alpha_1 v_1)\delta'(x-ct)+[GH^2]\delta(x-ct)\\ &+s_1(t)(\alpha_0 v_0^2+\alpha_1 v_1^2)\delta'(x-ct)+s_2(t)s_3^2(t)(u_0\beta_0\gamma_0^2+u_1\beta_1\gamma_1)\delta'(x-ct)\approx 0. \end{aligned}$$

From the above equation we have

(40) $$\sigma_1(\alpha_0 v_0+\alpha_1 v_1)=\sigma_1 c=c[GH]-[GH^2]$$

(41) $$-s_2(t)s_3^2(t)=s_1(t)$$

(42) $$\alpha_0(v_0^2-cv_0)+\alpha_1(v_1^2-cv_1)=u_0\beta_0\gamma_0^2+u_1\beta_1\gamma_1^2.$$

Equations (38) and (40) imply

(41) $$(u_1-u_0)c^2-2(u_1v_1-u_0v_0)c+(u_1v_1^2-u_0v_0^2)=0.$$

Obviously, (42) can be exchanged with

(42') $$\alpha_0(v_0^2-cv_0)+\alpha_1(v_1^2-cv_1)>0$$

and one can take

$$s_2(t)=-(\sigma_1 t)^{1/3},\ s_3(t)=(\sigma_1 t)^{1/3}$$

without a loss in generality.

Thus, there are two possibilities

(i) $u_0=u_1$. Then

$$\alpha_0=\alpha_1=\frac{1}{2},\ c=\frac{v_0+v_1}{2}.$$

Inequality (42') is now

$$\frac{1}{4}(v_0-v_1)^2>0$$

obviously true. Since $\lambda_l = v_0 > c > v_1 = \lambda_r$, the wave is overcompressive.

(ii) $u_0 \neq u_1$. Then

$$c = \frac{u_1 v_1 - u_0 v_0 \pm (v_0 - v_1)\sqrt{u_0 u_1}}{u_1 - u_0} \tag{44}$$

and (42') is equivalent to

$$(v_0 - c)(c - v_1)(v_0 - v_1) > 0,$$

which is a condition to wave be overcompressive, in fact. One can easily check that

$$v_0 > c = \frac{u_1 v_1 - u_0 v_0 + (v_0 - v_1)\sqrt{u_0 u_1}}{u_1 - u_0} > v_1$$

(we had to take + sign in (44)).

Now

$$\sigma_1 = (v_0 - v_1)\sqrt{u_0 u_1} > 0,$$

i.e. the strenght of the singular shock wave at time t is $\sigma_1 = (v_0 - v_1)\sqrt{u_0 u_1}t$ in both cases. Thus,

$$\begin{aligned} U(x,t) &\approx \begin{cases} u_0, & x < ct \\ u_1, & x > ct \end{cases} + (v_0 - v_1)\sqrt{u_0 u_1} t \delta(x - ct), \\ V(x,t) &\approx \begin{cases} v_0, & x < ct \\ v_1, & x > ct, \end{cases} \end{aligned} \tag{45}$$

where

$$c = \frac{[GH] - [H]\sqrt{u_0 u_1}}{[G]} \text{ or } c = \frac{v_0 + v_1}{2} \text{ if } [G] = 0.$$

Remark. Let us note that (45) is in fact the solution to (5') (with u and v interchanged, and $f(u) = u$) given in [18,(3,13),(3.21)]

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